Chapter 2  Signal Processing Transforms

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2.1 Introduction

In a broader scope, a transformation in mathematics can broadly be defined as the operation which takes its input and “represents” it in a different form. Such a definition immediately implies preserving the essential characteristics of the input through several conservation rules or laws. In order to outline the complete input–output relationship in an abstract transformation, input to and output of the transformation should be characterized as well. Therefore, mathematically speaking, a transformation can be viewed as a special function (or a correspondence) whose input and output can be a single value or another function. In this context, signals are considered to be both input and output functions of the transformation of interest.

Depending both on the transform and the input signal characteristics, different types of transforms exist in the literature. For instance, there are transforms which assume deterministic input (e.g., Fourier transform), whereas there are transforms operating on stochastic input (e.g., Karhunen–Loève transform). A different way of classifying transforms is to consider the nature of the input signal such as continuous and discrete transforms, which assume continuous and discrete signals as input, respectively.

It is worth mentioning at this point that some of the transformations yield output in a different domain – which is sometimes called “transform domain”– from that of the input. A very well-known example of this is a time–frequency transformation, which operates on a signal in the time domain and yields an output signal in the frequency domain or vice versa. However, such a domain change is not always observed for every transformation.

2.2 Basic Transformations

When an independent variable is considered, several transformations can be defined in the context given in Section 2.1. Basic transformations are listed as follows for any function \(f(t)\), which can be imagined to be one that varies in time \((t \in \mathbb{R})\):

\[\begin{align*}
{\text{(A.I)}} & \quad f(t) \xrightarrow{\text{Scale}} f(a \cdot t), \quad \forall a \in \mathbb{R} \quad \text{(Linear scale)} \quad (2.1) \\
{\text{(A.II)}} & \quad f(t) \xrightarrow{\text{Reflection}} f(-t) \quad \text{(Reflection (Time reversal))} \quad (2.2) \\
{\text{(A.III)}} & \quad f(t) \xrightarrow{\text{Shift}} f(t - t_0), \quad \forall t_0 \in \mathbb{R} \quad \text{(Shift)} \quad (2.3)
\end{align*}\]

Note that the basic transformations listed above can be used separately as well as in combinations. Such combinations help form the basic concepts such as odd–even functions and periodic functions. Also, such combinations, when allowed and appropriately applied, are very useful to simplify the analysis especially for linear systems.

\(^{1}\)Note that, depending on the type and characteristic of the transformation, it is not obligatory that a transformation is a bijection. As will be shown subsequently, there are some transformations which take a single value as their input and yield a set of values as their output.
2.3 Fourier Series and Transform

2.3.1 Basic Definitions

Fourier series and transform are two fundamental topics in many branches of science and mathematics. They are defined as follows.

**Definition 2.3.1 (Periodicity).** A function is said to be periodic with a period $T$ if:

$$f(t) = f(t + T)$$ (2.4)

is satisfied for all $t$.

A direct consequence of the Definition 2.3.1 is the periodicity as $f(t) = f(t + nT)$ for all $t$ and $n \in \mathbb{Z}$. This reasoning necessitates the following remark.

**Definition 2.3.2 (Fundamental Period).** The fundamental period $T_0$ of a periodic function $f(\cdot)$ is the smallest positive value of $T$ for which (2.4) holds.

It is noteworthy to state that if $f(\cdot)$ is constant, then the fundamental period is “undefined,” since there is no “smallest” positive value. Also, a function is said to be “aperiodic” if (2.4) does not hold.

2.3.2 Fourier Series

**Theorem 2.3.1 (Fourier Series Expansion).** If a function $f(t)$ satisfying Dirichlet conditions is periodic with a fundamental period $T_0$, then it can be represented in the following Fourier series expansion:

$$f(t) = \sum_{k=\infty}^{\infty} a_k e^{jk\omega_0 t}$$ (2.5)

where $j = \sqrt{-1}$ and $T_0 = 2\pi/\omega_0$. In (2.5), the term obtained when $k = 1$ is called “fundamental mode” or “first harmonic” and the constant obtained when $k = 0$ is called “dc component” or “average value”.

Note that, since (2.5) in Theorem 2.3.1 is an expansion, the convergence of its right-hand side expansion must be questioned. The convergence is guaranteed when $f(t)$ has finite energy over one period. However, the Dirichlet conditions, if satisfied, offer a set of sufficient conditions for the existence of “equality” rather than a “convergence” except for the isolated values causing discontinuity [1].

**Theorem 2.3.2 (Dirichlet Conditions).** Any function $f(t)$ is equal to its Fourier series representation at the values where the function $f(t)$ is continuous and converges to the mean of the discontinuity values (average of the left- and right-hand limits of $f(t)$ at values where $f(t)$ is discontinuous) if the following conditions are satisfied:

- Function $f(t)$ is bounded periodic,
- Function $f(t)$ has a finite number of discontinuities,
- Function $f(t)$ has a finite number of extrema,
- Function $f(t)$ is absolutely integrable $\int_{T_0} |f(t)| \, dt < \infty$

As stated earlier, the conditions given in Theorem 2.3.2 are “sufficient” to guarantee that the Fourier series expansion representation of $f(t)$ exists. This should not be interpreted as when conditions in Theorem 2.3.2 are not satisfied, Fourier series expansion representation does not exist. For the existence of Fourier series expansion of any function, “necessary” conditions are not yet known. However, in real-world applications, almost all of the signals encountered satisfy the conditions of Theorem 2.3.2 implying the existence of the Fourier series expansion. Therefore, from the practical perspective, seeking necessary conditions is not critical.

In conjunction with the statements above, it might be necessary to contemplate the behavior of Fourier series expansion at points where the function of interest exhibits discontinuities especially for a finite number of terms included (partial sums). Let one define $f_N(t)$ to be:

$$f_N(t) = \sum_{k=-N}^{N} a_k e^{jk\omega_0 t}$$ (2.6)
in parallel with (2.5). In this case, one might want to minimize the energy of the difference:

\[ e(t) = f_N(t) - f(t) \quad (2.7) \]

i.e., \( E_N = \int_0^T |e(t)|^2 \, dt \). It is clear that \( \lim_{N \to \infty} E_N = 0 \). This implies that the difference in terms of energy would be negligible as the number of terms in the series increases \(^2\). However, at the values where discontinuities occur, as \( N \) increases a strange phenomenon takes place and overshoot/undershoot effects are observed. This is known as “Gibbs phenomenon” in the honor of Josiah Willard Gibbs who clarified the Albert Michelson’s concerns about a possible deficiency in his mechanical harmonic analyzer calculating the “truncated” Fourier series approximation \(^3\). An illustration of the Gibbs phenomenon is plotted in Figure 2.1.

Gibbs phenomenon implies spectral growth. In other words, it implies that if a time signal has discontinuities, its Fourier representation tends to have more terms (\( N \to \infty \)) with respect to a signal that does not have any discontinuities. In practical applications, truncation due to finite support causes ringing (ripple) effects that are referred to as spectral leakage.

### 2.3.3 Fourier Transform

Periodic signals can be represented via Fourier series expansions as explained in Section 2.3.2. It is of interest to represent aperiodic signals in a similar way. The concept of Fourier series expansion can be utilized to represent aperiodic signals as well.

An aperiodic signal does not repeat itself over time. Mathematically speaking, one can view an aperiodic signal as a periodic signal that repeats itself with a period of infinity. It might be helpful to look at the same reasoning from the perspective of frequencies. Recall that Fourier series expansion actually yields a set of frequencies of discrete values each
of which is a multiple integer of the fundamental frequency:

\[
\omega_0 = \frac{2\pi}{T_0}
\]  

(2.8)

Then the difference between successive frequency components of the output of Fourier expansion is given by

\[
\Delta \omega = \frac{2(k + 1)\pi}{T_0} - \frac{2k\pi}{T_0} = \omega_0
\]  

(2.9)

via (2.5). Therefore, in case the fundamental period \(T_0\) gets larger, the gap between discrete frequency components becomes smaller, i.e., if \(T_0 \to \infty\), then \(\Delta \omega \to 0\). Of course, in the limiting case the summation in the Fourier series expansion converts into an integral that is referred to as the Fourier integral. Therefore, the Fourier transform integral can be introduced as:

\[
F(j\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} \, dt,
\]  

(2.10)

where \(F(j\omega)\) is called the “Fourier transform of \(f(t)\)” and is equivalently denoted in terms of notation with \(\mathfrak{F} \{f(t)\}\).

Thus, \(\mathfrak{F} \{f(t)\} = F(j\omega)\). The inverse Fourier transform is given by:

\[
f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(j\omega) e^{j\omega t} \, d\omega
\]  

(2.11)

and it can also be represented as \(\mathfrak{F}^{-1} \{f(t)\}\). Both (2.10) and (2.11) are referred to as a Fourier transform pair. A direct consequence of both (2.10) and (2.11) is the following relationship:

\[
\mathfrak{F}^{-1} \{\mathfrak{F} \{f(t)\}\} = f(t)
\]  

(2.12a)

\[
\mathfrak{F} \{\mathfrak{F}^{-1} \{F(j\omega)\}\} = F(j\omega).
\]  

(2.12b)

**Properties of Fourier Transform**

**Linearity.** If \(F_1(j\omega) = \mathfrak{F} \{f_1(t)\}\) and \(F_2(j\omega) = \mathfrak{F} \{f_2(t)\}\), then

\[
\mathfrak{F} \{a_1 f_1(t) + a_2 f_2(t)\} = a_1 \mathfrak{F} \{f_1(t)\} + a_2 \mathfrak{F} \{f_2(t)\}
\]  

(2.13)

As a result of both (2.10) and (2.11), linearity applies also to \(\mathfrak{F}^{-1} \{\cdot\}\).

**Symmetry.** If \(F(j\omega) = \mathfrak{F} \{f(t)\}\) for \(f(t) \in \mathbb{R}\), then:

\[
F(-j\omega) = F^*(j\omega).
\]  

(2.14)

It is important to state that even though \(f(t) \in \mathbb{R}\), the Fourier transform in (2.10) can easily be extended to complex valued functions because of the linearity property since \(f_x(t) = f_x(t) + j(f_y(t))\) for any \(f_x(t) \in \mathbb{C}\), where \(f_x(t) = \Re(f_x(t))\) and \(f_y(t) = \Im(f_x(t))\) with \(\Re(\cdot)\) and \(\Im(\cdot)\) denoting the real and imaginary parts, respectively.

**Scaling.** If \(F(j\omega) = \mathfrak{F} \{f(t)\}\), then

\[
\mathfrak{F} \{f(at)\} = \frac{1}{|a|} F \left( \frac{\omega}{a} \right).
\]  

(2.15)

**Shifting.** If \(F(j\omega) = \mathfrak{F} \{f(t)\}\), then

\[
\mathfrak{F} \{f(t - t_0)\} = e^{-j\omega t_0} F(j\omega).
\]  

(2.16)
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**Differentiation.** If $F(j\omega) = \mathfrak{F}\{f(t)\}$, then

$$
\mathfrak{F}\left\{ \frac{d}{dt}f(t) \right\} = j\omega F(j\omega).
$$

(2.17)

**Integration.** If $F(j\omega) = \mathfrak{F}\{f(t)\}$, then

$$
\mathfrak{F}\left\{ \int_{-\infty}^{t} f(x) \, dx \right\} = \frac{1}{j\omega} F(j\omega) + \pi F(0)\delta(j\omega),
$$

(2.18)

where $\delta(\cdot)$ denotes the Dirac delta function.

**Duality.** If $F(j\omega) = g(\omega) = \mathfrak{F}\{f(t)\}$, then

$$
\mathfrak{F}\{g(t)\} = 2\pi f(-\omega).
$$

(2.19)

Even though duality seems to be an insignificant property, actually, it is quite powerful, especially, when considered along with the aforementioned properties.

**Properties of Fourier Transform in Linear Systems**

In the preceding subsection, very important fundamental properties of Fourier transform have been reviewed. However, the power of Fourier transform is better understood when linear systems are analyzed. As will be shown subsequently, the fundamental properties of Fourier transform will be of great help in the analysis of linear time-invariant (LTI) systems yielding very important results.

**Parseval’s Equality.** If $F_1(j\omega) = \mathfrak{F}\{f_1(t)\}$ and $F_2(j\omega) = \mathfrak{F}\{f_2(t)\}$, then

$$
\int_{-\infty}^{\infty} f_1(t)f_2^*(t) \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} F_1(j\omega)F_2^*(j\omega) \, d\omega.
$$

(2.20)

As a special case of Parseval’s equality is obtained for $f_1(t) = f_2(t)$, which is known to be “Bessel’s equality,” or simply energy conservation property.

**Convolution.** It is known that the output of a LTI system that is fed by an input signal $x(t)$ is characterized by the convolution of the “impulse response” of the LTI system, say $h(t)$, with the input signal $x(t)$. Formally, this input–output relationship can be expressed as follows:

$$
y(t) = h(t) \ast x(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau) \, d\tau,
$$

(2.21)

where $y(t)$ is the output of the LTI system at the time instant $t$. If the frequency domain representation of this output is desired, then the following very important result is obtained:

$$
\mathfrak{F}\{y(t)\} = Y(j\omega) = H(j\omega)X(j\omega),
$$

(2.22)

where $H(j\omega)$ is known to be the “frequency response” of the LTI system of interest. As shown in (2.22), the Fourier transform allows one to investigate directly the output of LTI systems in the frequency domain. Note that, considering the duality property mentioned earlier, convolution property implies a very important simplicity in the analyses. For instance, because of the convolution property of Fourier transform, ordering is not important in cascaded systems from the perspective of overall system response.
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Modulation. Through the use of duality, if convolution is applied in frequency domain, then the result in time domain should be characterized by multiplication. Formally, this is stated by:

\[ y(t) = x_1(t)x_2(t) = \mathcal{F}^{-1}\{X_1(j\omega) \ast X_2(j\omega)\}. \]  

(2.23)

This property of Fourier transform is called “modulation” because the frequency domain convolution operation leads to a scaling one of the signals with the other signal in the time domain. Since multiplication of two signals is referred to as “amplitude modulation” (i.e., scale of the amplitude) in theory of communications, (2.23) is known as “modulation property.”

2.4 Sampling

As stated in Section 2.1, transformation is an operation that allows one to represent its input in another way. From this perspective, as will be shown subsequently, sampling can be considered to be a type of transformation under specific conditions. Moreover, sampling is the bridge between continuous- and discrete-time signals. Considering the fact that dealing with discrete values (samples) is more flexible and preferable in terms of saving and processing, sampling actually lies in the heart of many engineering applications.

Before giving the details of sampling theorem, it is appropriate first to show how sampling is mathematically established.

2.4.1 Impulse-Train Sampling

In order to collect samples from a continuous-time signal, one needs to have a mathematical object that allows to pick the value of the signal at a specific time and to repeat this process in a periodic manner. Both of these can be established by applying a Dirac delta function (impulse) in a periodic way. Assume that \( p(t) \) is the sequence of impulses (impulse train or sampling function) defined as follows:

\[ p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s) \]  

(2.24)

The period of \( p(t) \) is \( T_s \) and it represents sampling period. Then, the fundamental frequency of \( p(t) \) is \( \omega_s = \frac{2\pi}{T_s} \). Hence, the sampling operation can be expressed as:

\[ f_p(t) = f(t)p(t). \]  

(2.25)

Therefore, (2.25) can be rewritten as:

\[ f_p(t) = \sum_{n=-\infty}^{\infty} f(nT_s)\delta(t - nT_s). \]  

(2.26)

Note that (2.25) (and also (2.26)) represents a multiplication in the time domain between two sequences. Therefore, the output can be expressed as the convolution of the Fourier transforms of \( p(t) \) and \( f(t) \). Since \( \mathcal{F}\{p(t)\} \) is another impulse train in frequency domain, the output is actually the sum of shifted replicas of \( \mathcal{F}\{f(t)\} \) (see Exercise 2.12.9).

Note that the property mentioned above points out shifted replicas of the signal and their summation along the frequency axis. This raises some concerns about replicas of the Fourier transform overlapping in the frequency domain because if an overlap occurs in the frequency domain, the recovery of the signal becomes impossible. In order to guarantee that no overlap occurs in the frequency domain, impulses forming \( \mathcal{F}\{p(t)\} \) should be separated sufficiently apart from each other. This condition leads to the sampling theorem:

**Theorem 2.4.1** (Sampling Theorem). Let a bandlimited function \( f(t) \) be defined as \( F(j\omega) = 0 \) for \( |\omega| > \omega_B \). Then \( f(t) \) can be uniquely represented by its samples \( f(nT) \), \( n \in \mathbb{Z} \), provided that:

\[ 2\omega_B < \omega_s, \]  

(2.27)

where

\[ \omega_s = \frac{2\pi}{T_s}. \]  

(2.28)

It is important to emphasize that reconstruction of the signal \( f(t) \) in Theorem 2.4.1 is actually established by filtering the samples, which are sufficiently close to each other, with an ideal low-pass filter. This process is known to be “bandlimited interpolation.” However, due to practical considerations, it is difficult to implement an ideal low-pass filter. Therefore, different interpolation methods are applied in real-world applications.
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2.4.2 Aliasing

One of the most important requirements in Theorem 2.4.1 is to sample the signal with a sufficiently high frequency such that the condition $2\omega_B < \omega_s$ is met. At this point, one might ask about why such a condition is imposed on the theorem.

In order to better understand what happens if the condition $2\omega_B < \omega_s$ is not satisfied, a frequency domain representation of a bandlimited baseband signal $f(t)$ is considered. Since the bandlimited signal in the frequency domain, i.e., $F(j\omega)$, has no frequency component beyond $\omega_B$, after the sampling operation, the replicas of $F(j\omega)$ are formed by shifting them across the $\omega$–axis at multiple integers of $\omega_s$. This implies that, for instance, the first shifted replica of $F(j\omega)$ is centered at $\omega_s$, which is the alias term $F(j(\omega-\omega_s))$. Therefore, the lowest frequency component of $F(j(\omega-\omega_s))$ is located at $\omega_L = \omega_s - \omega_B$. Now, in case the lowest frequency component of the original spectrum $F(j\omega)$, then information in the frequency domain is lost and the analog signal cannot be reconstructed perfectly from the samples of discrete–time signal. This phenomenon is referred to as “aliasing”. Note that the highest frequency component of $F(j\omega)$ is denoted with $\omega_B$ by definition and it is also referred to as the signal bandwidth. Thus, in order not to have aliasing:

$$\omega_B < \omega_L \iff \omega_B < \omega_s - \omega_B \iff 2\omega_B < \omega_s$$ (2.29)

must be satisfied, as stated in Theorem 2.4.1.

2.5 Cosine and Sine Transforms

Even though they are not as tractable as Fourier transform, cosine and sine transforms exhibit very interesting and nice properties in different areas of signal processing such as spectral analysis of real sequences and data compression. Therefore, it is worth reviewing these transforms.

2.5.1 Cosine Transform

For any function $f(t) \in \mathbb{C}$ that is defined over $t \geq 0$, the cosine transform is defined by:

$$\mathfrak{F}_c \{ f(t) \} = F_c(\omega) = \int_0^\infty f(t) \cos(\omega t) \, dt.$$ (2.30)

where $\mathfrak{F}_c \{ \cdot \}$ represents the cosine transform operator and $F_c(\omega)$ is the cosine transform of $f(t)$. Note that the integration begins from zero and the transformation kernel is $\cos(\omega t)$. These details imply that $\mathfrak{F}_c \{ \cdot \}$ does not necessarily exist.

The inverse cosine transform is given by:

$$\mathfrak{F}_c^{-1} \{ F_c(\omega) \} = \frac{2}{\pi} \int_0^\infty F_c(\omega) \cos(\omega t) \, d\omega$$ (2.31)

for $t \geq 0$.

Properties of Cosine Transform

Let $f(t)$ and $g(t)$ be defined as in Section 2.5.1. Some of the important properties of cosine transform are listed below.

**Linearity.** If $F_c(\omega) = \mathfrak{F}_c \{ f(t) \}$ and $G_c(\omega) = \mathfrak{F}_c \{ g(t) \}$, then

$$\mathfrak{F}_c \{ a_1 f(t) + a_2 g(t) \} = a_1 \mathfrak{F}_c \{ f(t) \} + a_2 \mathfrak{F}_c \{ g(t) \} = a_1 F_c(\omega) + a_2 G_c(\omega).$$ (2.32)

for two arbitrary scalars $a_1$ and $a_2$.

**Scaling.** If $F_c(\omega) = \mathfrak{F}_c \{ f(t) \}$, then

$$\mathfrak{F}_c \{ f(at) \} = \frac{1}{a} F_c \left( \frac{\omega}{a} \right)$$ (2.33)
Shifting. Since the cosine transform is defined over $t \geq 0$, shifting the signal to the left requires further investigation. If $f_E(t) = f(|t|)$, then:

$$\mathcal{F}_c \{ f(t + a) + f(|t - a|) \} = 2F_c \cos (a\omega). \quad (2.34)$$

With the same argument, if the shift is considered to be over the frequency domain, then:

$$F_c (\omega \pm a) = \mathcal{F}_c \{ f(t) \cos (at) \} \mp \mathcal{F}_c \{ f(t) \sin (at) \} \quad (2.35)$$

where $\mathcal{F}_s \{ \cdot \}$ is the sine transform in \[(2.45)\], which will be introduced subsequently in this chapter. If these properties are extended, the following relations are obtained:

$$\mathcal{F}_c \{ f(at) \cos (at) \} = \frac{1}{2a} \left( F_c \left( \frac{\omega + a}{a} \right) + F_c \left( \frac{\omega - a}{a} \right) \right), \quad (2.36)$$

and

$$\mathcal{F}_c \{ f(at) \sin (at) \} = \frac{1}{2a} \left( F_s \left( \frac{\omega + a}{a} \right) - F_s \left( \frac{\omega - a}{a} \right) \right), \quad (2.37)$$

where $F_s (\cdot)$ is the sine transform of $f(t)$.

Differentiation. As in shifting, due to the special structure of cosine transform, odd– and even–order differentiations exhibit different characteristics. The cosine transform of first–order derivative is given by:

$$\mathcal{F}_c \left\{ \frac{d}{dt} f(t) \right\} = \omega F_s (\omega) - f(0) - D \cos (\omega t_D) \quad (2.38)$$

where $D$ is the magnitude of the discontinuity (discontinuity jump) at the time instant $t_D$ given by $D = f(t_D^+)^{-} - f(t_D^-)^{+}$. Similarly, the second–order differentiation is given by:

$$\mathcal{F}_c \left\{ \frac{d^2}{dt^2} f(t) \right\} = -\omega^2 F_c (\omega) - f'(0) - \omega D \sin (\omega t_D) - D' \cos (\omega t_D) \quad (2.39)$$

where $D' = f'(t_D^+)^{-} - f'(t_D^-)^{+}$. It must be noted here that in order for all of the formulae given above to hold, all–order derivatives (including the zeroth–order derivative, i.e., the function itself) are assumed to vanish as $t \to \infty$.

When differentiation in frequency domain considered, the odd–even structure is preserved. However, they manifest in a different way. Odd–order differentiation is given by:

$$\frac{d^{(2n+1)} f(t)}{dt^{(2n+1)}} = \mathcal{F}_c \left\{ (-1)^{n+1} t^{2n+1} f(t) \right\}, \quad (2.40)$$

whereas for the even–order differentiation is given by:

$$\frac{d^{(2n)} f(t)}{dt^{(2n)}} = \mathcal{F}_c \left\{ (-1)^n t^{2n} f(t) \right\}, \quad (2.41)$$

where $n \in \mathbb{Z}^+$. It must be emphasized that the existence of the integrals is assumed a fact, which can only be guaranteed by imposing some extra restrictions on $f(t)$ such as being piecewise continuous and etc.

Integration. Integration in the time domain for cosine transform is expressed via:

$$\mathcal{F}_c \left\{ \int_{t}^{\infty} f(\tau) \, d\tau \right\} = \frac{1}{\omega} F_s (\omega), \quad (2.42)$$

whereas the integration in the frequency domain is given by:

$$\mathcal{F}_s^{-1} \left\{ \int_{\omega}^{\infty} f(\alpha) \, d\alpha \right\} = -\frac{1}{t} f(t). \quad (2.43)$$

These results are valid under general conditions such as the function being absolutely integrable.
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Convolution. The cosine transform of the convolution is related to the sine transform and is expressed as:

\[
\mathcal{F}_c \left\{ \int_0^\infty f(\tau) [g(t + \tau) + g_o(t - \tau)] \, d\tau \right\} = 2F_s(\omega)G_s(\omega)
\]  
(2.44)

where \( g_o(\cdot) \) denotes the odd part of \( g(t) \).

2.5.2 Sine Transform

For any function \( f(t) \in \mathbb{C} \) that is defined over \( t \geq 0 \), the sine transform is defined by:

\[
\tilde{\mathcal{F}}_s \{ f(t) \} = F_s(\omega) = \int_0^\infty f(t) \sin(\omega t) \, dt,
\]  
(2.45)

where \( \tilde{\mathcal{F}}_s \{ \cdot \} \) represents the sine transform operator and \( F_s(\omega) \) is the sine transform of \( f(t) \), which assumes the domain \( \omega \geq 0 \). Note that the integration begins from zero and the transformation kernel is \( \sin(\omega t) \). These details imply that \( \tilde{\mathcal{F}}_s \{ \cdot \} \) does not necessarily exist.

The inverse sine transform is then given by:

\[
\tilde{\mathcal{F}}^{-1}_s \{ F_s(\omega) \} = \frac{2}{\pi} \int_0^\infty F_s(\omega) \sin(\omega t) \, d\omega,
\]  
(2.46)

and assumed the domain \( t \geq 0 \).

Properties of Sine Transform

Let \( f(t) \) and \( g(t) \) be defined as in Section 2.5.2. Then some of the important properties of sine transform are listed below.

Linearity. If \( F_s(\omega) = \tilde{\mathcal{F}}_s \{ f(t) \} \) and \( G_s(\omega) = \tilde{\mathcal{F}}_s \{ g(t) \} \), then

\[
\tilde{\mathcal{F}}_s \{ a_1f(t) + a_2g(t) \} = a_1\tilde{\mathcal{F}}_s \{ f(t) \} + a_2\tilde{\mathcal{F}}_s \{ g(t) \} = a_1F_s(\omega) + a_2G_s(\omega).
\]  
(2.47)

for arbitrary scalars \( a_1 \) and \( a_2 \).

Scaling. If \( F_s(\omega) = \tilde{\mathcal{F}}_s \{ f(t) \} \), then

\[
\tilde{\mathcal{F}}_s \{ f(at) \} = \frac{1}{a} F_s \left( \frac{\omega}{a} \right)
\]  
(2.48)

with \( 0 < a \).

Shifting. Since sine transform is defined over \( t \geq 0 \), shifting the signal to the left requires further investigation. If \( f_E(t) = f(|t|) \), then:

\[
\tilde{\mathcal{F}}_s \{ f(t + a) + f(|t - a|) \} = 2F_s \sin(aw).
\]  
(2.49)

With the same argument, if the shift is considered to be over the frequency domain, then:

\[
F_s(\omega + a) = \tilde{\mathcal{F}}_s \{ f(t) \cos(\omega t) \} + \tilde{\mathcal{F}}_s \{ f(t) \sin(\omega t) \},
\]  
(2.50)

where \( \tilde{\mathcal{F}}_s \{ \cdot \} \) is the sine transform. The following more general relationships hold:

\[
\tilde{\mathcal{F}}_s \{ f(\omega t) \cos(\omega t) \} = \frac{1}{2a} \left( F_s \left( \frac{\omega + a}{a} \right) + F_s \left( \frac{\omega - a}{a} \right) \right),
\]  
(2.51)

and

\[
\tilde{\mathcal{F}}_s \{ f(\omega t) \sin(\omega t) \} = \frac{1}{2a} \left( F_c \left( \frac{\omega + a}{a} \right) + F_c \left( \frac{\omega - a}{a} \right) \right),
\]  
(2.52)

where \( F_s(\cdot) \) is the sine transform and \( F_c(\cdot) \) is the cosine transform of \( f(t) \), respectively.
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Differentiation. As in shifting, due to the special structure of cosine transform, odd- and even-order differentiations present different characteristics. The sine transform of the first-order derivative is given by:

\[ \mathfrak{F}_s \left\{ \frac{d}{dt} f(t) \right\} = -\omega F_c(\omega) + D \sin(\omega t_D), \]

(2.53)

where \( D \) is the magnitude of the discontinuity (discontinuity jump) at the time instant \( t_D \): \( D = f(t_D^+) - f(t_D^-) \). Similarly, the second-order differentiation is given by:

\[ \mathfrak{F}_s \left\{ \frac{d^2}{dt^2} f(t) \right\} = -\omega^2 F_s(\omega) - \omega f(0) \]

(2.54)

where \( f(t) \) is assumed to be continuous to the first-order. For all of the formulae given above to hold, all-order differentiation (including the zeroth-order, that is, the function itself) functions must vanish as \( t \to \infty \). In order to obtain higher order derivatives, operational rule must be applied.

When differentiation in the frequency domain considered, the odd-even structure is preserved; however, they are presented in a different way. Odd-order derivative is given by:

\[ \frac{d^{(2n+1)}}{dt^{(2n+1)}} F_s(\omega) = \mathfrak{F}_c \left\{ (-1)^n t^{2n+1} f(t) \right\} \]

(2.55)

with the use of \( \mathfrak{F}_c \{ \cdot \} \), whereas for the even-order differentiation is:

\[ \frac{d^{(2n)}}{dt^{(2n)}} F_s(\omega) = \mathfrak{F}_s \left\{ (-1)^n t^{2n} f(t) \right\} \]

(2.56)

where \( n \in \mathbb{Z}^+ \). Again, it must be emphasized that the existence of the integrals are assumed to exist, which can only be guaranteed by some extra restrictions on \( f(t) \) such as being piece-wise continuous.

Integration. The integration in the time domain for sine transform is given by:

\[ \mathfrak{F}_s \left\{ \int_0^t f(\tau) d\tau \right\} = \frac{1}{\omega} F_c(\omega) \]

(2.57)

whereas the integration in the frequency domain is expressed as:

\[ \mathfrak{F}_c^{-1} \left\{ \int_0^\infty F_s(x) dx \right\} = \frac{1}{t} f(t). \]

(2.58)

where the result exists under general sufficient conditions such as being absolutely integrable.

Convolution. The sine transform of the convolution is related to the cosine transform and is expressed as:

\[ \mathfrak{F}_s \left\{ \int_0^\infty g(\tau) [f(t+\tau) + f_o(t-\tau)] d\tau \right\} = 2F_s(\omega)G_c(\omega) \]

(2.59)

where \( f_o(\cdot) \) denotes the odd part of \( f(t) \) and \( G_c(\omega) \) is \( \mathfrak{F}_c \{ g(t) \} \).

2.6 Laplace Transform

In Section 2.3 recall that the Fourier transformation yields an output in the frequency domain by calculating the time domain signal \( f(t) \) with the complex exponential \( e^{j\omega t} \). Recall also that, because of this property, \( \mathfrak{F} \{ f(t) \} \) is represented as \( F(j\omega) \). This is a direct consequence of the transform kernel employed. Since the Fourier transform employs a complex exponential of pure imaginary form (i.e., \( e^{-j\omega t} \) in \( \mathfrak{F} \{ \cdot \} \)), one might wonder what happens if the transform kernel is
generalized in such a way that it employs a complex exponential of a general complex form \( e^{st} = e^{(\sigma+j\omega)t} \) where \( s \) denotes the “complex frequency in generalized form” rather than being in pure imaginary form. It is easier to follow that \( e^{st} \) degenerates to Fourier transform kernel \( e^{j\omega t} \) when \( \sigma = 0 \).

In the light of the generalized transform kernel \( e^{st} \), if Fourier transform integral in (2.10) can be expressed as:

\[
F(\sigma + j\omega) = \int_{-\infty}^{\infty} f(t)e^{-(\sigma+j\omega)t} \, dt = \int_{-\infty}^{\infty} \left[ f(t)e^{-\sigma t} \right] e^{-j\omega t} \, dt.
\] (2.60)

Note that (2.60) is regarded as the Laplace transform of signal \( f(t) \). One can easily conclude that the Laplace transform of a function \( f(t) \) is the Fourier transform of \( g(t) = f(t)e^{-\sigma t} \). Thus, the Laplace transform is defined as:

\[
F(s) = \int_{-\infty}^{\infty} f(t)e^{-st} \, dt,
\] (2.61)

where \( s = \sigma + j\omega \). At this point, it is important to make the following observations.

First of all, although the lower limit of (2.61) is stretched back to negative infinity, it is clear that integration actually disregards the interval \([-\infty, 0)\) because of the multiplication of the real exponential \( e^{-\sigma t} \). Therefore, sometimes it is said that Laplace transform deals with “causal” functions even though the function \( f(t) \) can be defined all over the real axis. This is because of the behavior of the transform kernel.

Second, since (2.61) is an improper integral, its convergence must be carefully investigated:

\[
\int_{-\infty}^{\infty} f(t)e^{-st} \, dt = \lim_{T \to 0} \int_{-\infty}^{T} f(t)e^{-st} \, dt + \lim_{T \to \infty} \int_{T}^{\infty} f(t)e^{-st} \, dt,
\] (2.62)

which forces one to consider the existence of both integrals on the right hand–side of (2.62). Note that if a function \( f(t) \) is defined on the interval \((-\infty, 0)\), then (2.61) can only yield an output for \( I^- \) in (2.62). Since the Laplace transform of different functions might give the same algebraic output, the concept of region of convergence (RoC) is introduced which consists of the complex frequency values, \( s = \sigma + j\omega \), for which the Fourier transform of \( g(t) \) in (2.60) exists. For this reason, in both cases \( I^- \) and \( I^+ \) exist and (2.61) is referred to as the “bilateral” or “two–sided” Laplace transform and denoted with \( \mathbb{L}_B \{ \cdot \} \). If (2.61) is the bilateral Laplace transform, then the “unilateral” or “one–sided” Laplace transform is given by:

\[
F(s) = \int_{0}^{\infty} f(t)e^{-st} \, dt,
\] (2.63)

where the convergence of the integral is assumed to be satisfied.

The inverse Laplace transform can be found by exploiting the relationship between Fourier transform and Laplace transform and it is given by:

\[
f(t) = \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} F(s)e^{st} \, ds
\] (2.64)

Computation of integral in (2.64) requires the use of contour integration. However, for the transforms that are of rational form, such an integration can easily be performed by using Cauchy’s residue formula [5].

### 2.6.1 Properties of Laplace Transform

Since the Laplace transform can be considered as an extension of the Fourier transform, similar properties are observed for the Laplace transform.
### Linearity

If \( F_1(s) = \mathcal{L}\{f_1(t)\} \) and \( F_2(s) = \mathcal{L}\{f_2(t)\} \) with \( \text{RoC} \) \( R_1 \) and \( R_2 \), respectively, then:

\[
\mathcal{L}\{a_1 f_1(t) + a_2 f_2(t)\} = a_1 \mathcal{L}\{f_1(t)\} + a_2 \mathcal{L}\{f_2(t)\}
\]

for arbitrary scalars \( a_1 \) and \( a_2 \) with a resulting \( \text{RoC} \) containing \( R_1 \cap R_2 \). Here, it is worth mentioning that if \( R_1 \cap R_2 = \emptyset \), then the linear combination of interest does not admit a Laplace transform.

### Conjugation

If \( F(s) = \mathcal{L}\{f(t)\} \), with a \( \text{RoC} \) \( R \), then:

\[
\mathcal{L}\{f^*(t)\} = F^*(s^*)
\]

with the same \( \text{RoC} \) \( R \).

### Scaling

If \( F(s) = \mathcal{L}\{f(t)\} \) admits a \( \text{RoC} \) \( R \), then:

\[
\mathcal{L}\{af(t)\} = \frac{1}{|a|} F\left(\frac{s}{a}\right),
\]

with a \( \text{RoC} \) \( R_1 \) satisfying \( R_1 = \frac{R}{a} \).

### Shifting

If \( F(s) = \mathcal{L}\{f(t)\} \) has the \( \text{RoC} \) \( R \), then:

\[
\mathcal{L}\{f(t-t_0)\} = e^{-st_0}F(s),
\]

with the same \( \text{RoC} \) \( R \). Similar to the time domain shifting, the shift in the complex frequency plane is given by:

\[
\mathcal{L}\{e^{st_0}f(t)\} = F(s-s_0)
\]

with a \( \text{RoC} \) \( R_1 = R + \sigma_0 \) where \( \sigma_0 = \Re(s_0) \).

### Differentiation

If \( F(s) = \mathcal{L}\{f(t)\} \) has the \( \text{RoC} \) \( R \), then:

\[
\mathcal{L}\left\{\frac{d}{dt} f(t)\right\} = sF(s) - f(0^+),
\]

with a \( \text{RoC} \) including \( R \). Similar to the differentiation in the time domain, differentiation in the complex frequency plane can be expressed as:

\[
\mathcal{L}\{-tf(t)\} = \frac{d}{ds} F(s)
\]

with the same \( \text{RoC} \) \( R \).

### Integration

If \( F(s) = \mathcal{L}\{f(t)\} \), with a \( \text{RoC} \) \( R \), then:

\[
\mathcal{L}\left\{\int_{-\infty}^{t} f(x) \, dx\right\} = \frac{1}{s} F(s) + \frac{1}{s} \left[ \int_{0^+} f(t) \, dt \right]_{t=0^+}
\]

with a \( \text{RoC} \) including \( R \cap \{\sigma > 0\} \).

### Convolution

Although the convolution property was reviewed previously in the Section 2.3.3 dealing with the Fourier transform, for Laplace transform it will be expressed under a general framework. If \( F_1(s) = \mathcal{L}\{f_1(t)\} \) and \( F_2(s) = \mathcal{L}\{f_2(t)\} \) with \( \text{RoC} \) \( R_1 \) and \( R_2 \), respectively, then:

\[
\mathcal{L}\{f_1(t) \ast f_2(t)\} = F_1(s)F_2(s)
\]

with a resulting \( \text{RoC} \) containing \( R_1 \cap R_2 \). Here, it is worth mentioning that if \( R_1 \cap R_2 = \emptyset \), then the linear combination does not admit a Laplace transform.
2.7 Hartley Transform

In Section 2.3.3, the transform kernel is defined to be a complex exponential, which implies a complex trigonometric series expansion via Euler’s formula: $e^{j\omega t} = \cos (\omega t) + j \sin (\omega t)$. It will be shown subsequently that the Hartley transform kernel drops $j$ and yields a real-valued kernel rather than a complex one. This implies that, especially when computational complexity is of concern, dealing only with real-valued functions seems to be an important alternative to the Fourier transform.

In the light of the discussion above, let the transform kernel be defined for Hartley transform as:

$$\text{cas} (\nu t) = \cos (\nu t) + \sin (\nu t),$$

(2.74)

where $\nu$ is the angular frequency and given by $\nu = 2\pi f$ with $f$ being expressed in Hertz. It is obvious that $\text{cas} (\cdot)$ can be represented in different forms via straightforward trigonometric conversions such as:

$$\text{cas} (\nu t) = \sqrt{2} \cos (\nu t - \pi/4) = \sqrt{2} \sin (\nu t + \pi/4)$$

(2.75)

Thus, the Hartley transform is given by:

$$F_H(\nu) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \text{cas} (\nu t) \, dt,$$

(2.76)

where $F_H(\nu)$ is called the “Hartley transform of $f(t)$” and equivalently represented as $\mathcal{H} \{ f(t) \}$.

As a direct consequence of (2.76), Hartley transform is an involutary transform. Therefore, the inverse Hartley transform is given by (2.76) as well. Hence, one may write the following:

$$f(t) = \mathcal{H}^{-1} \{ \mathcal{H} \{ f(t) \} \}$$

(2.77)

since $\mathcal{H}^{-1} \{ \cdot \}$ is the same as $\mathcal{H} \{ \cdot \}$.

2.7.1 Properties of Hartley Transform

Similarity to Fourier transform and cosine transform, the Hartley transform exhibits the following properties:

**Linearity.** If $F_{H_1}(\nu) = \mathcal{H} \{ f_1(t) \}$ and $F_{H_2}(\nu) = \mathcal{H} \{ f_2(t) \}$, then:

$$\mathcal{H} \{ a_1 f_1(t) + a_2 f_2(t) \} = a_1 \mathcal{H} \{ f_1(t) \} + a_2 \mathcal{H} \{ f_2(t) \}
\begin{align*}
&= a_1 F_{H_1}(\nu) + a_2 F_{H_2}(\nu).
\end{align*}$$

(2.78)

for some arbitrary scalars $a_1$ and $a_2$.

**Scaling.** If $F_H(\nu) = \mathcal{H} \{ f(t) \}$, then:

$$\mathcal{H} \{ f(at) \} = \frac{1}{|a|} F_H \left( \frac{\nu}{a} \right).$$

(2.79)

**Shifting.** If $F_H(\nu) = \mathcal{H} \{ f(t) \}$, then:

$$\mathcal{H} \{ f(t - t_0) \} = \cos (\nu t_0) F_H(\nu) + \sin (\nu t_0) F_H(-\nu).$$

(2.80)

Note that the transform domain shift can also be treated in the same way, since the Hartley transform is involutary.

**Differentiation.** If $F_H(\nu) = \mathcal{H} \{ f(t) \}$, then:

$$\mathcal{H} \left\{ \frac{d^n}{dt^n} f(t) \right\} = \text{cas}' \left( \frac{n\pi}{2} \right) \nu^n H_F ((-1)^n \nu)$$

(2.81)

where $n$ stands for the order of differentiation.
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Convolution. If \( F_{H_1}(\nu) = \mathcal{H}\{f_1(t)\} \) and \( F_{H_2}(\nu) = \mathcal{H}\{f_2(t)\} \), then:

\[
\mathcal{H}\{f_1(t) * f_2(t)\} = \frac{1}{2} [ F_{H_1}(\nu)F_{H_2}(\nu) + F_{H_1}(\nu)F_{H_2}(-\nu) + F_{H_1}(-\nu)F_{H_2}(\nu) - F_{H_1}(-\nu)F_{H_2}(-\nu) ]
\] (2.82)

Product. If \( F_{H_1}(\nu) = \mathcal{H}\{f_1(t)\} \) and \( F_{H_2}(\nu) = \mathcal{H}\{f_2(t)\} \), then:

\[
\mathcal{H}\{f_1(t)f_2(t)\} = \frac{1}{2} [ F_{H_1}(\nu) * F_{H_2}(\nu) + F_{H_1}(\nu) * F_{H_2}(-\nu) + F_{H_1}(-\nu) * F_{H_2}(\nu) - F_{H_1}(-\nu) * F_{H_2}(-\nu) ]
\] (2.83)

2.8 Hilbert Transform

The Hilbert transform for a one–dimensional real–valued signal \( f(t) \) is given by the following integral:

\[
\mathcal{H}\{f(t)\} = \Delta f(t) = -\frac{1}{\pi} \mathfrak{P} \int_{-\infty}^{\infty} \frac{f(\tau)}{\tau - t} \, d\tau
\] (2.84)

where \( \mathcal{H}\{\cdot\} \) denotes the Hilbert transform operator, \( \mathfrak{P} \) represents the Cauchy’s principal value and it is defined as:

\[
\mathfrak{P} = \int_{a}^{b} g(x) \, dx = \lim_{\epsilon \to 0^+} \left( \int_{a}^{c-\epsilon} g(x) \, dx + \int_{c+\epsilon}^{b} g(x) \, dx \right)
\] (2.85)

when it exists for a complex–valued function \( g(\cdot) \) defined on a real interval \([a,b]\).

The inverse Hilbert transform is defined in terms of a similar integral:

\[
\mathcal{H}^{-1}\{\Delta f(t)\} = \frac{1}{\pi} \mathfrak{P} \int_{-\infty}^{\infty} \frac{f(\tau)}{\tau - t} \, d\tau.
\] (2.86)

By definition, Hilbert transform can also be defined in terms of a linear convolution operation:

\[
\mathcal{H}\{f(t)\} = \Delta f(t) = f(t) * \frac{1}{\pi t},
\] (2.87)

whereas the inverse Hilbert transform is expressed as:

\[
\mathcal{H}^{-1}\{\Delta f(t)\} = -\Delta f(t) * \frac{1}{\pi t}.
\] (2.88)

From both (2.87) and (2.88), it follows that applying Hilbert transform consecutively leads to the inverse Hilbert transform.

2.8.1 Properties of Hilbert Transform

There are many applications and properties for the Hilbert transform. The main properties are reviewed below.

Linearity. If \( \Delta f_1(t) = \mathcal{H}\{f_1(t)\} \) and \( \Delta f_2(t) = \mathcal{H}\{f_2(t)\} \), then:

\[
\mathcal{H}\{a_1 f_1(t) + a_2 f_2(t)\} = a_1 \mathcal{H}\{f_1(t)\} + a_2 \mathcal{H}\{f_2(t)\} = \Delta a_1 f_1(t) + \Delta a_2 f_2(t),
\] (2.89)

for arbitrary scalars \( a_1 \) and \( a_2 \).
Scaling. If \( \hat{f}(t) = \mathcal{H}\{f(t)\} \), then:

\[
\mathcal{H}\{f(at)\} = \hat{f}(at),
\]

(2.90)

and:

\[
\mathcal{H}\{f(-at)\} = -\hat{f}(-at),
\]

(2.91)

for all \( a > 0 \).

Shifting. If \( \hat{f}(t) = \mathcal{H}\{f(t)\} \), then:

\[
\mathcal{H}\{f(t-t_0)\} = \hat{f}(t-t_0).
\]

(2.92)

Differentiation. If \( \hat{f}(t) = \mathcal{H}\{f(t)\} \), then:

\[
\mathcal{H}\left\{ \frac{d}{dt}f(t) \right\} = \frac{-1}{\pi t} \star \frac{d}{dt} (\hat{f}(t)).
\]

(2.93)

Convolution. If \( \hat{f}_1(t) = \mathcal{H}\{f_1(t)\} \) and \( \hat{f}_2(t) = \mathcal{H}\{f_2(t)\} \), then:

\[
\mathcal{H}\{f_1(t) \ast f_2(t)\} = -\hat{f}_1(t) \ast \hat{f}_2(t).
\]

(2.94)

Also:

\[
\mathcal{H}\left\{ f_1(t) \bigast \hat{f}_2(t) \right\} = \hat{f}_1(t) \ast f_2(t).
\]

(2.95)

Product. If \( \hat{f}_1(t) = \mathcal{H}\{f_1(t)\} \) and \( \hat{f}_2(t) = \mathcal{H}\{f_2(t)\} \), then:

\[
\mathcal{H}\{f_1(t)f_2(t)\} = f_1(t)\hat{f}_2(t)
\]

(2.96)

where \( f_1(t) \) and \( f_2(t) \) present non–overlapping power spectra such as one being low–pass and the other being high–pass.

When formal analytic signals are of interest, several interesting properties of the Hilbert transform can be observed. Formally, an analytic signals is a complex–valued continuous-time function with a Fourier transform that vanishes for negative frequencies \([6]\). Therefore, for analytic signals one can write:

\[
\mathcal{H}\{\phi_1(t)\phi_2(t)\} = \phi_1(t)\mathcal{H}\{\phi_2(t)\}
\]

(2.97)

which is identical to:

\[
\mathcal{H}\{\phi_1(t)\phi_2(t)\} = \mathcal{H}\{\phi_1(t)\} \phi_2(t)
\]

(2.98)

where the analytic signals can be represented with:

\[
\phi(t) = f(t) + j\mathcal{H}\{f(t)\}
\]

(2.99)

### 2.9 Discrete–Time Fourier Transform

In parallel with the discussion of continuous–time Fourier transform, similar transformation can be established for discrete–time signals. This idea leads to the discrete–time Fourier transform (DTFT).

If a signal in discrete time is represented as \( f[n] \), then the DTFT of \( f[n] \) is given by:

\[
F(e^{j\omega}) = \sum_{n=-\infty}^{\infty} f[n]e^{-j\omega n},
\]

(2.100)
where \( F(e^{j\omega}) \) is called the “discrete–time Fourier transform of \( f[n] \)” and equivalently represented with \( \mathfrak{F}_D \{ f[n] \} \). Then, \( \mathfrak{F}_D \{ f(t) \} = F(j\omega) \). The inverse DTFT is given by:

\[
\mathfrak{F}_D^{-1} \{ \mathfrak{F}_D \{ f[n] \} \} = f[n]
\]

(2.101a) and can also be represented with \( \mathfrak{F}_D^{-1} \{ f[n] \} \). Both (2.100) and (2.101) are referred to as a Fourier transform pair. A direct consequence of both (2.100) and (2.101) is:

\[
\mathfrak{F}_D^{-1} \{ \mathfrak{F}_D \{ f[n] \} \} = f[n]
\]

and can also be represented with \( \mathfrak{F}_D^{-1} \{ f[n] \} \). Both (2.100) and (2.101) are referred to as a Fourier transform pair. A direct consequence of both (2.100) and (2.101) is:

\[
\mathfrak{F}_D^{-1} \{ \mathfrak{F}_D \{ f[n] \} \} = F(e^{j\omega})
\]

(2.102b)

\section*{Properties of Discrete–time Fourier Transform}

\section*{Linearity.}

If \( F_1(e^{j\omega}) = \mathfrak{F}_D \{ f_1[n] \} \) and \( F_2(e^{j\omega}) = \mathfrak{F}_D \{ f_2[n] \} \), then

\[
\mathfrak{F}_D \{ a_1 f_1[n] + a_2 f_2[n] \} = a_1 \mathfrak{F}_D \{ f_1[n] \} + a_2 \mathfrak{F}_D \{ f_2[n] \}
\]

(2.103)

for arbitrary scalars \( a_1 \) and \( a_2 \).

As a consequence of both (2.100) and (2.101), linearity applies also to \( \mathfrak{F}_D^{-1} \{ \cdot \} \).

\[
\mathfrak{F}_D^{-1} \{ a_1 f_1[n] + a_2 f_2[n] \} = a_1 \mathfrak{F}_D^{-1} \{ F_1(e^{j\omega}) \} + a_2 \mathfrak{F}_D^{-1} \{ F_2(e^{j\omega}) \} = a_1 f_1[n] + a_2 f_2[n]
\]

(2.104)

\section*{Symmetry.}

If \( F(e^{j\omega}) = \mathfrak{F}_D \{ f[n] \} \) for \( f[n] \in \mathbb{R} \), then:

\[
F(e^{j\omega}) = F^* (e^{-j\omega})
\]

(2.105)

It is important to state that even though \( f[n] \in \mathbb{R} \), the DTFT can easily be extended to complex–valued sequences due to the linearity property: \( f_z[n] = f_z[n] + j f_y[n] \), where \( f_z[n] = \Re(f_z[n]) \) and \( f_y[n] = \Im(f_z[n]) \), and \( \Re(\cdot) \) and \( \Im(\cdot) \) denote the real and imaginary parts, respectively.

\section*{Scaling.}

In DTFT scaling requires more attention than its continuous–time domain counterpart, since the discrete–time values are defined for indices satisfying \( n \in \mathbb{Z}^+ \). In the case \( n \in \mathbb{Q} \setminus \mathbb{Z} \), \( f[n] \) should be zero. Therefore, let \( f[n] \) be defined as:

\[
f_{(a)}[n] = \begin{cases} f[n/a], & \text{if } n \text{ is a multiple integer of } a, \\ 0, & \text{otherwise}. \end{cases}
\]

(2.106)

If \( F(e^{j\omega}) = \mathfrak{F}_D \{ f[n] \} \), then

\[
\mathfrak{F}_D \{ f_{(a)}[n] \} = F(e^{ja\omega}).
\]

(2.107)

\section*{Shifting.}

If \( F(e^{j\omega}) = \mathfrak{F}_D \{ f[n] \} \), then

\[
\mathfrak{F}_D \{ f[n - n_0] \} = e^{-j\omega n_0} F(e^{j\omega}).
\]

(2.108)

Similarly,

\[
\mathfrak{F}_D \{ e^{j\omega_0 n} f[n] \} = F(e^{j(\omega-\omega_0)}).
\]

(2.109)

\section*{Differencing.}

If \( F(e^{j\omega}) = \mathfrak{F}_D \{ f[n] \} \), then

\[
\mathfrak{F}_D \{ \mathfrak{D}^{(1)} \{ f[n] \} \} = (1 - e^{-j\omega}) F(e^{j\omega}),
\]

(2.110)

where \( \mathfrak{D}^{(k)} \{ \cdot \} \) is the difference operator of \( k \)-th order.

An interesting property occurs when differentiation (not differencing) is performed in the frequency domain. In this case:

\[
\mathfrak{F}_D \{ nf[n] \} = j \frac{d}{d\omega} (F(e^{j\omega})).
\]

(2.111)
Accumulation. If \( F(e^{j\omega}) = \mathcal{F}_D\{f[n]\} \), then
\[
\mathcal{F}_D\left\{ \sum_{m=-\infty}^{n} f[m] \right\} = \frac{1}{e^{-j\omega}} F(e^{j\omega}) + \pi F(e^{0}) \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k).
\] (2.112)

Properties of Discrete–Time Fourier Transform in Linear Systems

In the preceding subsection, very important fundamental properties of DTFT have been reviewed. However, the power of DTFT is better understood when linear systems are analyzed. As will be shown subsequently, the fundamental properties mentioned above will be of help in the analysis of discrete–time LTI systems.

Parseval’s Equality. If \( F_1(e^{j\omega}) = \mathcal{F}_D\{f_1[n]\} \) and \( F_2(e^{j\omega}) = \mathcal{F}_D\{f_2[n]\} \), then
\[
\sum_{n=-\infty}^{\infty} f_1[n]f_2^*[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} F_1(e^{j\omega}) F_2^*(e^{j\omega}) \, d\omega
\] (2.113)

As a special case of Parseval’s equality is obtained for \( f_1[n] = f_2[n] \), which is known to be “Bessel’s equality,” or conservation of energy property.

Convolution. It is known that the output of a LTI system that is fed by an input signal \( x[n] \) is characterized by the convolution of the “impulse response” of the LTI system, say \( h[n] \), with the input signal \( x[n] \). Formally, this input–output relationship can be expressed as follows:
\[
y[n] = h[n] \ast x[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m],
\] (2.114)

where \( y[n] \) is the output of the LTI system at the time index \( n \). Now, if the frequency domain representation of this output is considered, then the following very important result is obtained:
\[
\mathcal{F}_D\{y[n]\} = Y(e^{j\omega}) = H(e^{j\omega}) X(e^{j\omega}),
\] (2.115)

where \( H(e^{j\omega}) \) is called the “frequency response” of the LTI system of interest. As shown in (2.115), DTFT allows one to investigate the output of LTI systems in the frequency domain as well. Note that the convolution property implies a very important simplicity in the analysis of frequency domain response of LTI systems. As an example, because of the convolution property of DTFT, ordering is not important in cascaded systems from the perspective of overall system response.

Multiplication. Through the use of duality, if convolution is applied in frequency domain, then the corresponding time domain sequence is expressed in terms of multiplication of original sequences. Formally, this is stated by:
\[
y[n] = x_1[n]x_2[n] = \mathcal{F}_D^{-1}\left\{ X_1(e^{j\omega}) \ast X_1(e^{j\omega}) \right\}.
\] (2.116)

2.10 The Z–Transform

With the same reasoning used in Section 2.6, one might wonder what happens if the transform kernel is generalized for the discrete–time case in such a way that it employs a complex exponential of a general complex form. In the light of the generalized kernel, Z–transform is defined as:
\[
F(z) = \sum_{n=-\infty}^{\infty} f[n]z^{-n},
\] (2.117)

where \( z = re^{j\omega} \). At this point, it is important to make the following observations.
In analogy with (2.117) which represents the bilateral Laplace transform, then “unilateral” or “one–sided” Z–transform is defined as follows:

\[ F(z) = \sum_{n=0}^{\infty} f[n]z^{-n}, \quad (2.118) \]

where the convergence of the sum is assumed.

The inverse Z–transform can be found by exploiting the relationship between the DTFT and the Z–transform, and assumes the following expression:

\[ f[n] = \frac{1}{2\pi j} \oint F(z)z^{-n}dz. \quad (2.119) \]

The integral (2.119) requires the use of contour integration and is calculated using Cauchy’s residue formula.

### 2.10.1 Properties of Z–Transform

Since the Z–transform can be considered as an extended version of DTFT, similar properties can be observed for Z–transform.

**Linearity.** If \( F_1(z) = \mathcal{Z}\{f_1[n]\} \) and \( F_2(z) = \mathcal{Z}\{f_2[n]\} \) with \( \text{RoCs} \) \( R_1 \) and \( R_2 \), respectively, then:

\[ 3\{a_1 f_1[n] + a_2 f_2[n]\} = a_1 3\{f_1[n]\} + a_2 3\{f_2[n]\} \]

\[ = a_1 F_1(z) + a_2 F_2(z), \quad (2.120) \]

with a resulting \( \text{RoC} \) containing \( R_1 \cap R_2 \), for arbitrary scalars \( a_1 \) and \( a_2 \). Here, it is worth mentioning that if \( R_1 \cap R_2 = \emptyset \), then the linear combination of interest does not admit a Z–transform.

**Symmetry.** If \( F(z) = 3\{f[n]\} \) has the \( \text{RoC} \) \( R \), then:

\[ 3\{f^*[n]\} = F^*(z^*), \quad (2.121) \]

and admits the same \( \text{RoC} \) \( R \).

**Scaling.** If \( F(z) = 3\{f[n]\} \) has the \( \text{RoC} \) \( R \), then:

\[ 3\{z^n f[n]\} = F\left(\frac{z}{z_0}\right), \quad (2.122) \]

admits the \( \text{RoC} \) \( R_1 \) satisfying \( R_1 = |z_0| \cdot R \). Here, note that the scaling is performed in the Z–domain.

**Shifting.** If \( F(z) = 3\{f[n]\} \) has the \( \text{RoC} \) \( R \), then:

\[ 3\{f[n - n_0]\} = z^{-n_0} F(z), \quad (2.123) \]

with the same \( \text{RoC} \) \( R \) except for the possible appending or deletion of the origin or infinity. Similar to the time domain shifting, if the \( \text{RoC} \) for \( F(z) \) includes the unit circle, a shift in the complex frequency plane admits in the time–domain the modulated sequence \( e^{j\omega_0 n f[n]} \):

\[ 3\{e^{j\omega_0 n f[n]\} = F(\omega - \omega_0) \quad (2.124) \]

with the same \( \text{RoC} \) which is referred to as the “frequency translation.”

**Differentiation.** If \( F(z) = 3\{f[n]\} \) has the \( \text{RoC} \) \( R \), then:

\[ 3\{nf[n]\} = -z \frac{d}{dz} (F(z)), \quad (2.125) \]

with the same \( \text{RoC} \) \( R \). Similar to the differentiation in time domain, differentiation in the complex frequency plane is given by:

\[ 3\{-tf(t)\} = \frac{d}{ds} F(s) \quad (2.126) \]

with the same \( \text{RoC} \) \( R \).
2.11. CONCLUSION AND FURTHER READING

Convolutions. If \( F_1(z) = 3 \{ f_1[n] \} \) and \( F_2(z) = 3 \{ f_2[n] \} \) with RoCs \( R_1 \) and \( R_2 \), respectively, then:

\[
3 \{ f_1[n] * f_2[n] \} = F_1(z)F_2(z),
\]

(2.127)

with a RoC containing \( R_1 \cap R_2 \). Here, it is worth mentioning that when \( R_1 \cap R_2 = \emptyset \), then the linear convolution does not admit a Z–transform.

Product. If \( F_1(z) = 3 \{ f_1[n] \} \) and \( F_2(z) = 3 \{ f_2[n] \} \) with RoCs \( R_1 \) and \( R_2 \), respectively, then:

\[
3 \{ f_1[n]f_2[n] \} = \frac{1}{2\pi j} \int F_1(z)F_2(\frac{z}{x}) \frac{dx}{x}
\]

(2.128)

with a resulting RoC containing \( R_1 \cap R_2 \). If \( R_1 \cap R_2 = \emptyset \), then the product of signals does not admit a Z–transform.

2.11 Conclusion and Further Reading

Since transformations enable to look at things in different ways, many problems especially in the field of signal processing are somehow tackled with transformations. Before applying any transformation to a specific problem, one should always keep in mind that the suitability of the transformation considered to the problem in hand is of utmost importance. Especially in engineering applications, depending on the system design and/or constraints, some of the transformations (or just one single transformation) become more appropriate under specific conditions. Therefore, prominent characteristics of the transformations should be well investigated so that the problem in hand can be tackled in a very effective way.

Signal processing has a vast literature in terms of both applications and techniques. Transformations have a special place in these vast literature as well. In order to cover a broad perspective, the readers might refer to [7]. There are even studies dedicated to specific transformations, which provide a very comprehensive perspective for each of them [8,11]. For a detailed and one of the state-of-the-art work which covers many signal processing transformations and explores the relationships between them, the readers might refer to [4].

Acknowledgment: This work was supported by QNRF–NPRP.

2.12 Exercises

Exercise 2.12.1 (Even–Odd Signals). Assume that \( f(t) \) is an odd signal and \( g(t) \) is an even signal. Show that \( h(t) = f(t)g(t) \) is an odd signal.

Exercise 2.12.2 (Periodicity). Assume that \( f(t) = g(kt) \) where \( k \in \mathbb{R}/\{0\} \). If \( f(t) \) is periodic with \( T \), is \( g(\cdot) \) periodic?

Exercise 2.12.3 (Basic Transformation). Assume that an \( 
\end{align}

Exercise 2.12.4 (Linear Transformation and Fourier Series). Assume that \( f(t) \) is a continuous–time periodic signal with a fundamental period \( T \). Find the Fourier series coefficients for \( g(t) = f(2t – 1) \).

Exercise 2.12.5 (Differentiation and Fourier Series). Assume that \( f(t) \) is a continuous–time periodic signal with a fundamental period \( T \). Find the Fourier series coefficients for \( g(t) = \frac{d^2}{dt^2} (f(t)) \).

Exercise 2.12.6 (Non–linear Transformation and Fourier Transform). Assume that \( f(t) = e^{-3|t-1|} \). Find the Fourier transform of \( f(t) \).

Exercise 2.12.7 (Inverse Fourier Transform). Calculate the inverse Fourier transform of \( F(\omega) = \frac{1}{\omega} \).
Exercise 2.12.8 (Wiener-Khintchine Theorem). Show that $\exists \{\rho_{f,g}(\tau)\} = F(\omega)G^*(\omega)$, where $\rho_{f,g}$ denotes the cross-correlation of $f(t)$ and $g(t)$ in the delay domain $\tau$.

Exercise 2.12.9 (Sampling). Assume that for a function $f(t)$, $\exists \{f(t)\}$ exists and is denoted by $F(j\omega)$. Prove that

$$\Delta t \sum_{n=-\infty}^{\infty} f(t - n\Delta t) = \sum_{n=-\infty}^{\infty} F(nj\Delta\omega)e^{jn\Delta\omega t}$$

Exercise 2.12.10 (Fourier Cosine Transform). Derive the second-order derivative of the cosine transform of a function $f(t)$, that is $\frac{d^2}{d\omega^2} (F_c\{f(t)\})$.

Exercise 2.12.11 (Differentiation in Laplace Transform). Prove (2.70).

Exercise 2.12.12 (Hartley Transform). Show that the set of $\{\cos(n\nu_0 t)\sqrt{2\pi}\}_{n \in \mathbb{Z}}$ forms an orthonormal basis on $(-\pi, \pi]$.

Exercise 2.12.13 (Autoconvolution Hilbert Transform). Prove that $\rho_{ff}(t) = -\rho_{f\dot{f}}$. 
Bibliography


