MATH 414, Spring 2023
Midterm 1

Name: ____________________

UIN: ____________________

1. Calculators are not allowed throughout the examination.
2. Present your solutions in the space provided. Show all your work neatly and concisely.You will be graded, not merely on the final answer, but also on the quality and correctness of the work leading up to it. You have to fully justify every answer.
3. THE AGGIE CODE OF HONOR "An Aggie does not lie, cheat or steal, or tolerate those who do
Student Signature:____________________

Question	Value	Score
1	4	
2	7	
3	9	
4	5	
Total	25	

1. (4 points) Find the Fourier series of the 2π-periodic function given by

$$f(x) = \begin{cases} -1, & -\pi < x \le 0 \\ 1, & 0 < x \le \pi \end{cases}$$

2. (7 points) Consider the inner product space $V = L^2([0,1])$. Let

$$h_1(x) = 1, \qquad h_2(x) = \sqrt{3}(2x-1)$$

(a) (3 points) Prove that $\{h_1, h_2\}$ is an orthonormal basis of the vector subspace $W = \operatorname{Span}\{h_1, h_2\}$.

(b) (4 points) Let $f(x) = e^x$ for $0 \leq x \leq 1$. Compute the orthogonal projection of f onto the subspace W.

3. (9 points) Consider the sequence of functions (f_n) defined on the interval $[-1, 1]$ given by

$$f_n(x) = 1 - \frac{x^2}{n^3}$$

(a) (2 points) Sketch the graph of f_1, f_2, f_3 and f_n for arbitrary n.

(b) (2 points) Find the function g that is the pointwise limit of the sequence $(f_n)_n$. Explain how you find the function g.

(c) (3 points) Show that the sequence (f_n) converges in the mean to the function g you have identified in the previous question.

(d) (2 points) Does the sequence of functions (f_n) converge uniformly to g? No credit will be given for any unjustified answer.

4. (5 points) We denote by j the standard complex number satisfying $j^2 = 1$. Prove that if $u = (u_1, u_2)$ and $v = (v_1, v_2)$ are vectors in $\mathbb{C}^2$ then

$$< u, v >= (\bar{v}_1 \ \bar{v}_2) \begin{pmatrix} 4 & -j \\ j & 4 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

defines an inner product on $\mathbb{C}^2$.