Additional Problem set: Chapter 1

1. Consider the 2π-periodic function defined by $f(x) = e^{(x-\pi)}$ for $0 \le x < 2\pi$.
a) Draw the graph of f and that of f'.
b) Compute the Fourier series coefficients of f. Find the whole Fourier series $F(f)$.
c) Find the value of $F(f)(x)$ at every point $0 \le x \le 2\pi$.
d) Show that

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^2} = \frac{\pi}{e^{\pi} - e^{-\pi}}$$

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2. For every number number α that is NOT an integer, we consider the function

$$f(x) = \cos(\alpha x), \ \ \pi \le x < \pi.$$

a) Find the Fourier series of f.
b) Using a), deduce the following formula

$$\sum_{n=1}^{\infty} \frac{1}{n^2 - \alpha^2} = \frac{1}{2\alpha^2} - \frac{\pi}{2\alpha \tan(\alpha\pi)}$$

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3. Let f and g be two piecewise continuous 2π-periodic functions. We define the 2π-periodic function

$$h(x) = \int_0^{2\pi} f(x-s)\, g(s)\, ds$$

Find a formula between the complex Fourier coefficients of h and those of f and g.

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4. a) Compute the Fourier series of the 2π-periodic odd function given by $f(x) = x(\pi - x)$ for $0 \le x \le \pi$.
b) Using the previous question and Parseval's identity, deduce

$$\sum_{k=1}^{\infty} \frac{1}{(2k-1)^6}$$
