Additional Problem set: Chapter 2

1. Find the Fourier transform of the function $f(x) = e^{-|x|}$.

2. Using what you found in 1), compute $\int_{-\infty}^{+\infty} \frac{\cos x}{1+x^2}\, dx$.

3. Using what you found in 1), find the Fourier transform of the following functions:
a) $g(t) = te^{-|t|}$
b) $h(t) = e^{-2|t-3|}$

4. Consider the following functions:

$$f(x) = \begin{cases} \sqrt{\frac{\pi}{2}} & -1 \le x \le 1 \\ 0 & \text{otherwise} \end{cases} \qquad g(x) = \begin{cases} \pi + \frac{\pi}{2}x, & -2 \le x \le 0 \\ \pi - \frac{\pi}{2}x & 0 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$

a) Sketch the graphs of these functions and find their Fourier transforms.
b) What is the link between f and g?

5. Let $\omega > 0$. Using Plancherel's theorem and the given table of Fourier transforms, compute the following integrals:

$$i) \quad \int_{-\infty}^{+\infty} \frac{1}{(x^2+\omega^2)^2}\, dx \qquad ii) \quad \int_{-\infty}^{+\infty} \frac{\sin^2(\omega x)}{x^2}\, dx$$

6. a) Let $a \neq 0$ and $b \in \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ be a piecewise continuous integrable function. Show that the function $g(x) = f(ax+b)$ is also integrable and that

$$\hat{g}(\lambda) = \frac{1}{|a|} \hat{f}\left(\frac{\lambda}{a}\right) e^{i\lambda \frac{b}{a}}$$

b) Using the table of Fourier transforms and part a), find the Fourier transform of $g(x) = e^{-4x^2+8x+8}$.

	$f(y)$	$\mathfrak{F}(f)(\alpha) = \widehat{f}(\alpha)$
1	$f(y) = \begin{cases} 1 & \text{si } \|y\| < \|b\| \\ 0 & \text{sinon} \end{cases}$	$\sqrt{\frac{2}{\pi}} \frac{\sin(\|b\|\,\alpha)}{\alpha}$
2	$f(y) = \begin{cases} 1 & \text{si } b < y < c \\ 0 & \text{sinon} \end{cases}$	$\frac{e^{-i\alpha b} - e^{-i\alpha c}}{i\alpha\sqrt{2\pi}}$
3	$f(y) = \begin{cases} e^{-\omega y} & \text{si } y > 0 \\ 0 & \text{sinon} \end{cases} \quad (\omega > 0)$	$\frac{1}{\sqrt{2\pi}} \frac{1}{(\omega + i\alpha)}$
4	$f(y) = \begin{cases} e^{-\omega y} & \text{si } b < y < c \\ 0 & \text{sinon} \end{cases}$	$\frac{1}{\sqrt{2\pi}} \frac{e^{-(\omega+i\alpha)b} - e^{-(\omega+i\alpha)c}}{(\omega + i\alpha)}$
5	$f(y) = \begin{cases} e^{-i\omega y} & \text{si } b < y < c \\ 0 & \text{sinon} \end{cases}$	$\frac{1}{i\sqrt{2\pi}} \frac{e^{-i(\omega+\alpha)b} - e^{-i(\omega+\alpha)c}}{\omega + \alpha}$
6	$\frac{1}{y^2 + \omega^2} \quad (\omega \neq 0)$	$\sqrt{\frac{\pi}{2}} \frac{e^{-\|\omega\alpha\|}}{\|\omega\|}$
7	$\frac{e^{-\|\omega y\|}}{\|\omega\|} \quad (\omega \neq 0)$	$\sqrt{\frac{2}{\pi}} \frac{1}{\omega^2 + \alpha^2}$
8	$e^{-\omega^2 y^2} \quad (\omega \neq 0)$	$\frac{1}{\sqrt{2}\,\|\omega\|} e^{-\alpha^2/4\omega^2}$
9	$ye^{-\omega^2 y^2} \quad (\omega \neq 0)$	$\frac{-i\alpha}{2\sqrt{2}\,\|\omega\|^3} e^{-\frac{\alpha^2}{4\omega^2}}$
10	$\frac{4y^2}{(\omega^2 + y^2)^2} \quad (\omega \neq 0)$	$\sqrt{2\pi}\left(\frac{1}{\|\omega\|} - \|\alpha\|\right) e^{-\|\omega\alpha\|}$