Math 414, Fall 2016
Final

Name: ______________________________

UIN: ______________________________

1. Calculators are not allowed throughout the examination.
2. Present your solutions in the space provided. Show all your work neatly and concisely.You will be graded, not merely on the final answer, but also on the quality and correctness of the work leading up to it. You have to fully justify every answer.
3. We recall the following identities regarding the Fourier transform $\mathcal{F}$:
   $\mathcal{F}[f(t-a)](\lambda) = e^{-i\lambda a}\mathcal{F}[f](\lambda)$
   $\mathcal{F}[f(bt)](\lambda) = \frac{1}{b}\mathcal{F}[f]\left(\frac{\lambda}{b}\right)$
4. THE AGGIE CODE OF HONOR "An Aggie does not lie, cheat or steal, or tolerate those who do
   Student Signature:______________________________

Question	Value	Score
1	8	
2	13	
3	4	
4	8	
5	22	
6	6	
7	9	
Total	70	

1. (8 points) Let (f_n) be a sequence of functions on an interval $[a, b]$.

   (a) (2 points) Recall what it means for the sequence of functions (f_n) to converge pointwise to a function f on $[a, b]$.

   (b) (2 points) Recall what it means for the sequence of functions (f_n) to converge uniformly to a function f on $[a, b]$.

(c) (2 points) Recall what it means for the sequence of functions (f_n) to converge in the mean to a function f on $[a, b]$?

(d) (2 points) Which one of the above three notions of convergence implies the two other ones?

2. (13 points) Let $f(x)$ be the 2π-periodic extension to all of the real line of the function $f(x) = x$ for $-\pi \leq x < \pi$.

   (a) (5 points) Compute the Fourier series of $f(x)$.

(b) (4 points) What is the limit of the partial sums $S_N(x)$ of the Fourier series of f for $-\pi \leq x \leq \pi$ as $N \to +\infty$? Justify fully your answer.

(c) (2 points) Do the partial sums $S_N(x)$ of the Fourier series converge uniformly on $[-\pi, \pi]$? Justify fully your answer.

(d) (2 points) Do the partial sums $S_N(x)$ of the Fourier series converge in the mean on $[-\pi, \pi]$? Justify fully your answer.

3. (4 points) Compute $f * g$ where $f(x) = e^x$ and $g(x) = \begin{cases} e^{-2x} & 0 \leq x < 2 \\ 0 & otherwise \end{cases}$.

4. (8 points) Prove the following version of the sampling theorem : if $f(x) \in L^2(R)$ is such that $\hat{f}(\lambda) = 0$ for $|\lambda| > \pi$, then $f(x)$ can be written (in $L^2(R)$) as follows

$$f(x) = \sum_{k=-\infty}^{\infty} \sqrt{2\pi} b_k \, sinc(x-k),$$

where

$$b_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{f}(\lambda) e^{ik\lambda} d\lambda.$$

If you prefer, you can instead prove the following equivalent statement

$$f(x) = \sum_{k=-\infty}^{\infty} \sqrt{2\pi} c_k \, sinc(x+k)$$

where

$$c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{f}(\lambda) e^{-ik\lambda} d\lambda.$$

(Hint : Start writing the Fourier expansion of $\hat{f}(\lambda)$ on $[-\pi, \pi]$ and use that if $\Phi(x) = sinc(x)$ then $\hat{\Phi}(\lambda) = \begin{cases} \frac{1}{\sqrt{2\pi}}, & -\pi \le \lambda \le \pi \\ 0, & otherwise \end{cases}$.)

5. (22 points) This problem aims at reproving that the Shannon MRA indeed satisfies some the properties that a MRA has to fulfill.

   (a) (4 points) Recall how is exactly defined the sequence of subspaces $(V_j)_{j \in Z}$ in the Shannon case. Recall also what is the scaling function Φ in this case.

   (b) (6 points) State and prove the density property (again only in the Shannon case).

(c) (5 points) State and prove the scaling property (again only in the Shannon case).

(d) (7 points) Prove that Φ is in V_0 and that $\{\Phi(x-k) : k \in Z\}$ is an orthonormal basis for V_0 [again only in the Shannon case. You may use, without proving it again, the sampling theorem stated in the previous problem.]

6. (6 points) Find the Fourier transform of the function

$$f(x) = \begin{cases} x + \pi & -\pi \le x \le 0 \\ \pi - x & 0 < x \le \pi \\ 0 & otherwise \end{cases}$$

7. (9 points) In this problem, we are in the Haar setting. Consider the signal $f \in V_2$ given by $[2, 6, -1, 5]$. Find its decomposition in $V_0 \oplus W_0 \oplus W_1$ i.e find $f_0 \in V_0$, $w_0 \in W_0$ and $w_1 \in W_1$ such that $f = f_0 + w_0 + w_1$.