Math 414, Spring 2023
Final (1h45mn)

Name: ______________________________

UIN: ______________________________

1. Calculators, phones, smart-watches any other e-device, are not allowed throughout the examination.
2. Present your solutions in the space provided. Show all your work neatly and concisely.You will be graded, not merely on the final answer, but also on the quality and correctness of the work leading up to it. You have to fully justify every answer.
3. As per the course syllabus, you understand that, after taking this optional final exam, your present course letter grade will either remain the same, or get better or get worse and that will be your final course grade.
4. THE AGGIE CODE OF HONOR "An Aggie does not lie, cheat or steal, or tolerate those who do
   Student Signature:______________________________

Question	Value	Score
1	5	
2	10	
3	10	
4	12	
Clarity/Accuracy	1	
Total	38	

1. (5 points) Give the exact statement of the Cauchy-Schwarz inequality in an inner product space $(V, < \cdot >)$ and give the full proof in case V is a real vector space.

2. (10 points)

   (a) (7 points) Reconstruct $f \in V_3$ given these coefficients in its Haar wavelet decomposition:

   $$a^1 = [3/2, -1], \quad b^1 = [-1, -3/2], \quad b^2 = [-3/2, -3/2, -1/2, -1/2]$$

   Write your final answer in the form $a^3 = [.., .., ..]$

(b) (3 points) Draw separately the graphs of each function corresponding to a^1, b^1, a^2, b^2 and a^3.

3. (10 points) Consider the sequence of functions $(f_n)_{n\geq 0}$ defined on the interval $[-1,1]$ given by

$$f_n(x) = \begin{cases} 0 & -1 \leq x < -\frac{1}{n} \\ nx+1 & -\frac{1}{n} \leq x < 0 \\ 1-nx & 0 \leq x < \frac{1}{n} \\ 0 & \frac{1}{n} \leq x \leq 1 \end{cases}$$

(a) (2 points) Sketch the graphs of f_1, f_2 and f_n for n arbitrary.

(b) (3 points) Show that the sequence (f_n) converge to 0 in the mean.

(c) (2 points) Does the sequence (f_n) converge uniformly to 0? Explain fully your reasoning. No points will be given for any unjustified answer, even if the answer is correct.

(d) (1 point) Show that the sequence (f_n) does not converge pointwise to the zero function.

(e) (2 points) Find a function g that is the pointwise limit of (f_n) and explain why the function g you found is the pointwise limit.

4. (12 points) Consider the function defined on $(-\pi, \pi]$ as follows

$$f(x) = \begin{cases} 0 & -\pi < x < 0 \\ x & 0 \leq x \leq \pi \end{cases}$$

We extend f as a 2π-periodic function on the whole real line that we still denote by f.

(a) (1 point) Sketch the graph of f on the interval $[-2\pi, 2\pi]$.

(b) (2 points) Does the Fourier series of f converge uniformly to f on $[-\pi, \pi]$? Explain your reasoning.

(c) (3 points) Does the Fourier series of f converge pointwise on $[-\pi, \pi]$? If yes, what is the pointwise limit? Explain fully your reasoning.

(d) (4 points) Compute the Fourier series of f.

(e) (2 points) Using what you found in previous questions find the value of the following series

$$\sum_{k=1}^{\infty} \frac{1+(-1)^{k+1}}{k^2}$$