Math 151: 6.3 Definite Integrals from Riemann Sums

We just saw that the exact area bounded by a continuous function $f$ and the $x$-axis on the interval $x \in [a, b]$ was given as $A_{\text{exact}} = \lim_{n \to \infty} A_{\text{RAM},n}$, where $n$ is the number of rectangles in the Rectangular Approximation Method.

If we don’t restrict our functions to having positive heights AND widths, then we may replace the special case idea of area with “directed or net area” or accumulation.

The Rule of Four representations of the DEFINITE INTEGRAL, $\int_a^b f(x) \, dx$ are:

1. Verbal: The accumulation of $f$ over the interval $[a, b]$.
2. Numerical: The finite net area (positive, negative or zero), possessing the units of $f$ multiplied by the units of $x$.
3. Graphical: The directed area bounded between the graphs of the function $y = f(x)$ and the $x$-axis $y = 0$ from $x = a$ to $x = b$.
4. Analytical or Limit Definition: $\int_a^b f(x) \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*) \Delta x$, where $\Delta x = \frac{b - a}{n}$ and $x_i^*$ is the $x$-coordinate on the $i^{th}$ rectangle for the given RAM, if the limit exists.

Calculus Notation for Derivatives and Definite Integrals:

Leibniz’s notation for derivatives $\frac{dy}{dx}$ resembles the limit of the quotient: $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$.

In the limit, delta for difference has been replaced with $d$ for derivative. Leibniz notation allows us to see the derivative as an infinitely small DIFFERENCE QUOTIENT (a combination of limits, SUBTRACTION & DIVISION). Furthermore, it makes the Chain Rule meaningful: $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$ (where $y$ is the outer function of $u$, and $u$ is the inner function of $x$).

Likewise, the limit of the sum of products $\lim_{\Delta x \to 0} \sum f(x) \cdot \Delta x$ resembles $\int f(x) \, dx$.

Again, due to the LIMIT, $\Delta x$ has been replaced with $dx$. Now sigma for sum is replaced by the elongated S for integral, the infinite SUM of PRODUCTS, or SUMMED PRODUCTS (where $A_{\text{rectangle}} = l \cdot w = f(x) \cdot dx$; $f(x)$ is height and $\Delta x$ is width of the rectangle).
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If derivatives are the limit of infinitely small ______________ ______________ and integrals are the limit of infinitely small ______________ ______________, what relationship do you suspect exists between derivatives and integrals?

Definite Integral as Net Area

The net or directed (positive, negative or zero) area between the curve $f(x)$ and the $x$-axis and bounded between the vertical lines $x=a$ and $x=b$ is given by the DEFINITE INTEGRAL

$$A = \int_{a}^{b} f(x) \, dx.$$ 

Theorem: The Existence of Definite Integrals

All continuous functions are integrable. In other words, if a function is continuous on an interval $x \in [a, b]$, then its definite integral on $x \in [a, b]$ exists.

Exercises: For #1-7, sketch the function & shade the region represented by each definite integral. State if the definite integral ("net area") is positive, negative or 0.

1. $\int_{0}^{\pi} \sin x \, dx$
2. $\int_{0}^{2\pi} \sin x \, dx$
3. $\int_{0}^{\frac{2\pi}{2}} \sin x \, dx$
4. $\int_{\frac{2\pi}{2}}^{2\pi} \sin x \, dx$

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5. \[ \int_{a}^{b} 3 \, dx \]

6. \[ \int_{a}^{b} 3 \, dx, \text{ where } a < b \]

7. \[ \int_{b}^{a} 3 \, dx, \text{ where } a < b \]

8. Sketch the semi-circle \( y = \sqrt{4 - x^2} \). Express the area between the semi-circle \( x \in [-2, 2] \) and the \( x \)-axis as a definite integral and evaluate it BY HAND.

9. Evaluate \[ \int_{-2}^{2} \sqrt{4 - x^2} \, dx \] on the TI. Is this answer different from your answer in the previous question? Explain.

10. ALL continuous functions and some discontinuous functions are integrable ("able to have integrals" or finite accumulations over a finite domain).

   a) Show that \[ \int_{-3}^{3} \frac{x}{|x|} \, dx \] exists. Include a sketch in your solution.

   What kind of discontinuous functions (defined on closed domains) are
   b) integrable? c) not integrable?
Properties of Definite Integrals:

Let \( f \) and \( g \) be functions such that the following integrals exist. Furthermore, let \( x \in [a, b] \) where \( a \leq b \). Then the following definitions and properties of definite integrals hold.

1. Analytic/Limit Definition of a Definite Integral (by Riemann Approximation Method):
   \[
   \int_{a}^{b} f(x) \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*) \cdot \Delta x_i
   \]

2. Verbal Definition of a Definite Integral: \( \int_{a}^{b} f(x) \, dx \) represents the accumulation of a function \( f \) over the interval \([a, b]\).

3. Graphical Definition:
   - \( \int_{a}^{b} f(x) \, dx = \int_{a}^{b} [f(x) - 0] \, dx \) represents the net or directed area between the curves \( y = f(x) \) and \( y = 0 \) and the vertical lines \( x = a \) and \( x = b \).
   - \( \int_{a}^{b} |f(x)| \, dx \) represents the (nonnegative) area of the region bounded between \( y = f(x), y = 0, x = a \) and \( x = b \).
   - **Special Case:** If \( f(x) \geq 0 \) for \( x \in [a, b] \), then \( f(x) = |f(x)| \), and so \( \int_{a}^{b} f(x) \, dx = \int_{a}^{b} |f(x)| \, dx \geq 0 \) is the (nonnegative) area of the bounded region.
   - **Caution:** In general, \( \left| \int_{a}^{b} f(x) \, dx \right| \neq \int_{a}^{b} |f(x)| \, dx \). Rather, \( \left| \int_{a}^{b} f(x) \, dx \right| \leq \int_{a}^{b} |f(x)| \, dx \).

4. Reversing Bounds of Integration:
   \( \int_{a}^{b} f(x) \, dx = -\int_{b}^{a} f(x) \, dx \)

5. Zero Width:
   \( \int_{a}^{a} f(x) \, dx = 0 \)

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6. Definite integral of the constant function \( f(x) = 1 \):
\[
\int_a^b dx = b - a
\]

7. Linearity of Integrals:
   a) Homogeneity:
   \[
   \int_a^b k \, dx = k \int_a^b dx = k(b - a)
   \]
   b) Distributivity over add/substr:
   \[
   \int_a^b f(x) \, dx \pm \int_a^b g(x) \, dx = \int_a^b [f(x) \pm g(x)] \, dx
   \]

8. Subdivision of Interval \([a, b]\):
\[
\int_a^b f(x) \, dx = \int_a^c f(x) \, dx + \int_c^b f(x) \, dx
\]
NOTE: Do not confuse this with subdivision of a summation:
\[
\sum_{i=1}^n a_i = \sum_{i=1}^m a_i + \sum_{i=m+1}^n a_i
\]

9. If \( f(x) \geq g(x) \) for \( a \leq x \leq b \), then
\[
\int_a^b f(x) \, dx \geq \int_a^b g(x) \, dx
\]

10. Under- and Over-Estimates:
If \( f_{\text{min}} \leq f(x) \leq f_{\text{max}} \) for \( a \leq x \leq b \), then
\[
\int_a^b f_{\text{min}} \cdot (b-a) \leq \int_a^b f(x) \, dx \leq \int_a^b f_{\text{max}} \cdot (b-a)
\]

11. Average Function Value: The average value of an integrable function \( f \) on the interval \([a, b]\) is the numerical value that the \( f \) would be if \( f \) were constant. Let’s refer to this constant as \( f_{\text{ave}} \). Now we have two functions: \( y = f(x) \) (a nonconstant function, in general) and \( y = f_{\text{ave}} \) (a constant function). If \( f_{\text{ave}} \) is the average value of the function \( f \), then there accumulations over the interval \([a, b]\) are equivalent. In other words, their definite integrals are equal:
\[
A_{\text{red}} = A_{\text{blue}}
\]
\[
\int_a^b f_{\text{ave}} \, dx = \int_a^b f(x) \, dx
\]
\[
\therefore f_{\text{ave}} \cdot (b-a) = \int_a^b f(x) \, dx
\]
\[
\therefore f_{\text{ave}} = \frac{1}{b-a} \int_a^b f(x) \, dx
\]
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11. Given that \( \int_0^1 x^3 \, dx = \frac{1}{4} \), evaluate the following BY HAND.

\[
\begin{align*}
&\text{a) } \int_0^1 (x^3 + 1) \, dx \\
&\text{b) } \int_0^1 (-x^3) \, dx \\
&\text{c) } \int_0^1 x^3 \, dx \\
&\text{d) } \int_{-1}^0 x^3 \, dx \\
&\text{e) } \int_1^1 (x-1)^3 \, dx
\end{align*}
\]

12. If \( \int_2^5 f(x) \, dx = m \), \( \int_2^4 f(x) \, dx = n \) and \( \int_2^5 g(x) \, dx = k \), evaluate:

\[
\begin{align*}
&\text{a) } \int_5^5 f(x) \, dx \\
&\text{b) } \int_4^4 f(x) \, dx \\
&\text{c) } \int_4^5 f(x) \, dx \\
&\text{d) } \int_2^5 [f(x) + 3g(x)] \, dx
\end{align*}
\]

13. Determine the boundary values for the integral \( \int_2^5 \sin x \, dx \) (constants \( m \) and \( M \) such that \( m \leq \int_2^5 \sin x \, dx \leq M \)) without actually evaluating the integral.

14. True or false and explain: \( \int_2^5 x \, dx \leq \int_2^5 x^2 \, dx \) on \( x \in [2, 5] \).

Mean Value Theorem for Definite Integrals: If \( f \) is continuous on \([a, b] \), then there exists at least one \( x \)-value of \( c \in [a, b] \) such that \( f(c) = f_\text{ave} = \frac{1}{b-a} \int_a^b f(x) \, dx \).

Note that \( f(c) = f_\text{ave} \) is the mean or average (function) value. Do not confuse the average function value (defined by integrals) with the average rate of change of a function (defined using the secant slope and, by the Mean Value Theorem - for derivatives - is equal to the derivative at \( c \), as long as the function satisfies the theorem’s criteria).

15. For \( f(x) = x^2 - 6x + 8 \) defined on the interval \([0, \sqrt{3}] \), evaluate

\[
\begin{align*}
&\text{a) } \text{the average rate of change of } f(x) = x^2 - 6x + 8 \text{ over the interval } [0, \sqrt{3}] . \\
&\text{b) } \text{the average value of } f(x) = x^2 - 6x + 8 \text{ over the interval } [0, \sqrt{3}] . \\
&\text{c) } \text{Does the Mean Value Theorem for Integrals apply? If yes, at what point(s) in the interval does the function assume its average value? If no, why not?}
\end{align*}
\]

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