3.2 Subspace

Given a vector space V , it is possible to form another vector space by taking a subset S of V and using the same operations (addition and multiplication) of V . For a set S to be a vector space, it must satisfy addition and multiplication closure conditions on top of the other 8 conditions on addition and multiplication which are naturally satisfied since S is a subset of V , so we just need to check addition and multiplication closure conditions. If the subset S is a vector space, we call vector space S is a subspace of V .

Definition: If S is a non-empty subset of a vector space V , and S satisfies the following conditions:

(i). $\alpha \bullet x \in S,$ whenever $x \in S$ for any scalar α

(ii). $x + y \in S,$ whenever $x \in S, y \in S$

Then we call S is a s**ubspace** of V .

Remark: Every vector space V , has at least two subspaces. Namely V itself and $S = 0$ (the zero space). These two subspaces are called trivial subspaces.

Example 1: Determine if the given set is a subspace of the given vector space.

(1). Let $S = \{(x_1, -x_1)\}$. Is S a subspace of vector space R^2?

Let $x = (x_1, -x_1) \in S, y = (y_1, -y_1) \in S, \alpha$ is a scalar.

$$\alpha x = (\alpha x_1, -\alpha x_1) \in S$$
$$x + y = (x_1 + y_1, -x_1 - y_1) = (x_1 + y_1, -(x_1 + y_1)) \in S$$

S is a subspace.

(2). Let $S = \{(x_1, x_2) \mid x_1 \neq 0, x_2 \neq 0\}$. Is S a subspace of vector space R^2?

$$x = (1,2) \in S, y = (-1,1) \in S$$
$$x + y = (0,3) \notin S$$

S is not a subspace.

(3). Let $S = \{(x_1, x_2) \mid x_1 \geq 0\}$. Is S a subspace of vector space R^2?

$$x = (1,2) \in S,$$
$$-2x = (-2,-4) \notin S$$

S is not a subspace.

(3). Let $S=\{(x_1,0)\}$. Is S a subspace of vector space R^2?

Let $x=(x_1,0)\in S, y=(y_1,0)\in S, \alpha$ is a scalar.

$$x+y=(x_1+y_1,0)\in S$$
$$\alpha x=(\alpha x_1,0)\in S$$

S is a subspace.

(3). Let $S=\{(x_1,1)\}$. Is S a subspace of vector space R^2?

$x=(2,1)\in S$, but $2x=(4,2)\notin S$

S is not a subspace.

Exercise: Determine if the given set is a subspace of the given vector space.

(1). Let S be the set of 2×2 diagonal matrices. Is S a subspace of vector space M_{22}? (yes)

(2). Let S be the set of matrices of the form $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$. Is S a subspace of vector space M_{22}? (yes)

(2). Let S be the set of matrices of the form $\begin{pmatrix} a & b \\ 1 & c \end{pmatrix}$. Is S a subspace of vector space M_{22}? (No)

Exercise: Determine if the given set is a subspace of the given vector space.

(1). Let $C[a,b]$ be the set of all continuous functions on the interval $[a,b]$. Is this a subspace of $F[a,b]$, the set of all real valued functions on the interval $[a,b]$? (yes)

(2). Let P_n be the set of all polynomials of degree n or less. Is this a subspace of $F[a,b]$? (yes)

(3). Let S be the set of all polynomials of degree exactly 2. Is this a subspace of $F[a,b]$? (no)

$$\begin{aligned} &f(x)=x^2\in S, g(x)=-x^2+x\in S \\ &(f+g)(x)=x\notin S \end{aligned}$$, so it is not a subspace

(4). Let S be the set of all functions such $f(2)=1$. Is this a subspace of $F[a,b]$? (no)

$f(x) = x-1 \in S, g(x) \equiv 2 \in S$
$(f+g)(x) = x-1+2 = x+1$
$(f+g)(2) = 2+1 \neq 2 \notin S$

so it is not a subspace

(5). Let S be the set of all functions such $f(2) = 0$. Is this a subspace of $F[a,b]$? (yes)

Exercise: Let A be a particular vector in $R^{2\times 2}$. Determine whether the following are subspaces of $R^{2\times 2}$.

(a). $S = \{B \in R^{2\times 2} \mid AB = BA\}$

Let $B \in S, C \in S$, α is a scalar.

$\because B \in S, C \in S$
$\therefore BA = AB, CA = AC$
$\therefore (B+C)A = BA + CA = AB + AC = A(B+C)$
$\therefore B + C \in S$
$\because (\alpha B)A = \alpha BA = \alpha AB = A(\alpha B)$
$\alpha B \in S$

So it is a subspace

(b). $S = \{B \in R^{2\times 2} \mid AB \neq BA\}$

Let $B \in S$.

$0A = 0 = A0 \Rightarrow 0 \notin S \Rightarrow 0B = 0 \notin S$

So it is not a subspace

(c). $S = \{B \in R^{2\times 2} \mid AB = 0\}$

(yes).

Null Space of a matrix.

Definition: Let A be an $m \times n$ matrix. The **null space** $N(A)$ of A is the set

$$N(A) = \{x \in R^n \mid Ax = 0\}$$

Property: $N(A)$ is a subspace.

Let $x \in A, y \in A, \alpha$ be a scalar.

$x \in A, y \in A$
$Ax = Ay = 0$
$A(x+y) = Ax + Ay = 0 + 0 = 0$
$x + y \in A$
$A(\alpha x) = \alpha Ax = \alpha 0 = 0$
$\alpha x \in A$

So $N(A)$ is a subspace.

Example 2: Determine $N(A)$ if $A = \begin{pmatrix} 2 & 1 & 0 & 2 \\ 1 & 1 & 1 & 0 \end{pmatrix}$

$$A = \begin{pmatrix} 2 & 1 & 0 & 2 \\ 1 & 1 & 1 & 0 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 2 \end{pmatrix} \xrightarrow{-2R_1 + R_2 \to R_2} \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & -1 & -2 & 2 \end{pmatrix}$$
$$\xrightarrow{-R_2 \to R_2} \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & -2 \end{pmatrix} \xrightarrow{-R_2 + R_1 \to R_1} \begin{pmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 2 & -2 \end{pmatrix}$$

So, we have

$$\begin{cases} x_1 = x_3 - 2x_4 \\ x_2 = -2x_3 + 2x_4 \end{cases}$$

Let $x_3 = \alpha, x_4 = \beta$, we have

$$\begin{cases} x_1 = \alpha - 2\beta \\ x_2 = -2\alpha + 2\beta \\ x_3 = \alpha \\ x_4 = \beta \end{cases}$$
$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} \alpha - 2\beta \\ -2\alpha + 2\beta \\ \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix} \alpha + \begin{pmatrix} -2 \\ 2 \\ 0 \\ 1 \end{pmatrix} \beta$$

$N(A)$ contains all vectors of the form:

$$\begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix} \alpha + \begin{pmatrix} -2 \\ 2 \\ 0 \\ 1 \end{pmatrix} \beta$$

where α, β are any scalars.

$N(A)$ is a linear combination of vector $\begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 2 \\ 0 \\ 1 \end{pmatrix}$ or the span of $\begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 2 \\ 0 \\ 1 \end{pmatrix}$. We give these definitions below.

Definition: Let $v_1, v_2, \cdots, v_n$ be vectors in a vector space V, $\alpha_1, \alpha_2, \cdots, \alpha_n$ be scalars,

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$$

is called the **linear combinations** of vectors $v_1, v_2, \cdots, v_n$, while

$$W = \{\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n \mid \alpha_1, \alpha_2, \cdots, \alpha_n \text{ are scalars}\}$$

be called the **span** of $v_1, v_2, \cdots, v_n$, denoted by $span(v_1, v_2, \cdots, v_n)$, or $W = span(v_1, v_2, \cdots, v_n)$.

Example: Let $e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, what are $span(e_1), span(e_1, e_2), span(e_1, e_2, e_3)$ in R^3?

$span(e_1)$ is a line (x-axis)
$span(e_1, e_2)$ is a plane (xy-plane)
$span(e_1, e_2, e_3) = R^3$

Definition: The set $\{v_1, v_2, \cdots, v_n\}$ is called a **spanning set** for vector space V if every vector in V can be written as a linear combination of $v_1, v_2, \cdots, v_n$.

Example: Let $E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, E_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, E_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, E_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, is $\{E_{11}, E_{12}, E_{21}, E_{22}\}$ a spanning set for vector space: all 2×2 matrices?

For any matrix, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = aE_{11} + bE_{12} + cE_{21} + dE_{22}$, so it is a spannng set.

Exercise: Let $E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, E_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, E_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, E_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, v = \begin{pmatrix} 3 & -1 \\ 2 & 1 \end{pmatrix}$, is $\{E_{11}, E_{12}, E_{21}, E_{22}\}$ a spanning set for vector space: all 2×2 matrices?.

For any matrix, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = aE_{11} + bE_{12} + cE_{21} + dE_{22} + 0v$, so it is a spannng set.

Exercise: Let $v_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, v_3 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$, $\{v_1, v_2, v_3\}$ is a spanning set for vector space R^3?

For any vector $u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$ whether we can find scalars $\alpha_1, \alpha_2, \alpha_3$ such that $\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = u$ or

$$\alpha_1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \alpha_3 \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \Leftrightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

(What we need to do is to determine if this system has at least one solution for every possible vector u. If the coefficient matrix determinant is non-zero, then the system has a unique solution, so we just need to compute the determinant of coefficient matrix.)

$$\begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix} + 0 + \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = -2 - 1 = -3 \neq 0$$

Thus, there are unique scalars $\alpha_1, \alpha_2, \alpha_3$ such that $\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = u$. Thus, $\{v_1, v_2, v_3\}$ is a spanning set for vector space R^3.

Exercise: Let $v_1 = x + 1, v_2 = x^2 + 2x, v_3 = x^2 - 1$ $\{v_1, v_2, v_3\}$ is a spanning set for vector space $P_3 = \{ax^2 + bx + c \mid a, b, c \text{ are scalars}\}$, the polynomial with degree 2 or less?

For any vector $P(x) = ax^2 + bx + c \in P_3$ whether we can find scalars $\alpha_1, \alpha_2, \alpha_3$ such that $\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = P(x)$ or $\alpha_1(x+1) + \alpha_2(x^2 + 2x) + \alpha_3(x^2 - 1) = ax^2 + bx + c$.

$$\alpha_1(x+1) + \alpha_2(x^2 + 2x) + \alpha_3(x^2 - 1) = ax^2 + bx + c \Rightarrow (\alpha_2 + \alpha_3)x^2 + (\alpha_1 + 2\alpha_2)x + \alpha_1 - \alpha_3 = ax^2 + bx + c \Rightarrow$$

$$\begin{cases} \alpha_2 + \alpha_3 = a \\ \alpha_1 + 2\alpha_2 = b \\ \alpha_1 - \alpha_3 = c \end{cases} \Rightarrow \begin{pmatrix} 0 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \qquad (3.2-1)$$

$$\begin{vmatrix} 0 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & -1 \end{vmatrix} = -\begin{vmatrix} 1 & 0 \\ 1 & -1 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix} = 1 - 2 = -1 \neq 0$$

Thus, there are unique solution to equation (3.2-1). Thus, there are unique scalars $\alpha_1, \alpha_2, \alpha_3$ such that $\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = P(x)$. Thus, $\{v_1, v_2, v_3\}$ is a spanning set for vector space P_3.

HW: 1,10(d), 14(c).