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On the $\mathscr{C}^\infty$ regularity of CR mappings of positive codimension ☆

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ABSTRACT

The present paper tackles the $\mathscr{C}^\infty$ regularity problem for CR maps $h\colon M \to M'$ between $\mathscr{C}^\infty$-smooth CR submanifolds M, M' embedded in complex spaces of possibly different dimensions. For real hypersurfaces $M \subset \mathbb{C}^{n+1}$ and $M' \subset \mathbb{C}^{n'+1}$ with $n' > n \geq 1$ and M strongly pseudoconvex, we prove that every CR transversal map of class $\mathscr{C}^{n'-n+1}$ that is nowhere $\mathscr{C}^\infty$ on some non-empty open subset of M must send this open subset to the set of D'Angelo infinite points of M'. As a corollary, we obtain that every CR transversal map $h\colon M \to M'$ of class $\mathscr{C}^{n'-n+1}$ must be $\mathscr{C}^\infty$-smooth on a dense open subset of M when M' is of D'Angelo finite type. Another consequence establishes the following boundary regularity result for proper holomorphic maps in positive codimension: given $\Omega \subset \mathbb{C}^{n+1}$ and $\Omega' \subset \mathbb{C}^{n'+1}$ pseudoconvex domains with smooth boundaries $\partial\Omega$ and $\partial\Omega'$ both of D'Angelo finite type, $n' > n \geq 1$, any proper holomorphic map $h\colon \Omega \to \Omega'$ that extends $\mathscr{C}^{n'-n+1}$-smoothly up to $\partial\Omega$ must be $\mathscr{C}^\infty$-smooth on a dense open subset of $\partial\Omega$. More generally, for CR submanifolds M and M' of higher codimensions, our main result describes the impact of the existence of a nowhere smooth CR map $h\colon M \to M'$ on the CR geometry of M', allowing to extend the previously mentioned results in the hypersurface case to

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any codimension, as well as deriving a number of regularity results for CR maps with D'Angelo infinite type targets.

1. Introduction and results for real hypersurfaces

In this paper, we are interested in the following question: Under which conditions on $\mathscr{C}^\infty$-smooth CR manifolds $M \subset \mathbb{C}^N$ and $M' \subset \mathbb{C}^{N'}$ can we guarantee that a CR map $h: M \to M'$, which we assume to be of some finite smoothness $\mathscr{C}^k$ a priori, is actually $\mathscr{C}^\infty$-smooth on an open, dense subset of M?

This question is motivated by the problem of boundary regularity of holomorphic maps between smoothly bounded domains in $\mathbb{C}^N$: CR maps arise as their boundary values. The case most well-studied is when $N = N' = 1$ and the domains are simply connected; in that case, the boundary regularity of the Riemann map, as studied by Painlevé, Caratheodory, Kellogg, and many others serve as an answer to that problem.

In several dimensions, the Riemann map becomes unavailable as a tool, as there are many different equivalence classes of simply connected domains of holomorphy. However, Fefferman's mapping theorem [11] proved that biholomorphic mappings between smoothly bounded strictly pseudoconvex domains in $\mathbb{C}^N$, $N > 1$, necessarily extend smoothly up to the boundary. The proof of Fefferman's mapping theorem and also the proof of its generalization due to Bell and Ligocka [1], which reduced the assumptions on the domains to "condition (R)", rely on inherently global objects associated to the domain, in particular on properties of its Bergman kernel. Such methods however stop short of covering all pseudoconvex domains, as there exist smoothly bounded pseudoconvex domains which do not satisfy condition (R) by work of Christ [6]. Furthermore they also are not applicable when it comes to studying the boundary regularity of proper holomorphic mappings between smoothly bounded domains in complex spaces of different dimensions (see [13]). One natural alternative is then to derive global boundary regularity after investigating *local regularity* along smooth boundary patches.

Historically, the starting point for investigating the local question was again the case of (bijective) CR mappings between smooth, strongly pseudoconvex hypersurfaces in $\mathbb{C}^N$ studied by Nirenberg, Webster, and Yang [24]. The case of mappings of positive codimension, i.e. $N' > N$, from a strictly pseudoconvex hypersurface in $\mathbb{C}^N$ to one in $\mathbb{C}^{N'}$ is remarkably different, and harder. One of the reasons is that there actually exist continuous, and even Hölder continuous of exponent α for small α, CR embeddings of the sphere into a sphere in a higher dimensional space which fail to be smooth anywhere, by results due to Dor [10], Hakim [14] and Stensones [27]. It turns out that, in contrast with the equidimensional case, one can make up for that lack of smoothness by requiring a certain amount of a priori regularity for the map; this has, for example, been illustrated in the works of Forstnerič [12] and Huang [16,17] where it was shown that any $\mathscr{C}^k$-smooth, for a suitable integer k, CR map between spheres must be $\mathscr{C}^\infty$-smooth (and in fact even rational). Since then, the natural question of whether a similar regularity result holds for CR maps of positive codimension between general strongly pseudoconvex real hypersurfaces had been open for a while (see e.g. [15]), until the recent breakthrough by Berhanu–Xiao [4] who settled the problem in the affirmative for CR maps that are a priori $\mathscr{C}^{N'-N+1}$-smooth to start with. In a subsequent paper, Berhanu–Xiao [5] were also able to extend their approach to deal with Levi-nondegenerate target hypersurfaces as well (see also [18] for recent related results in the codimension one case).

In this paper, we carry out a study of the $\mathscr{C}^\infty$ regularity problem without assuming any geometric condition on the target manifold. Our basic approach differs significantly from all of these previous works: Our main result shows that if a CR mapping $h\colon M \to M'$ (of a certain a priori $\mathscr{C}^k$ regularity) fails to be $\mathscr{C}^\infty$-smooth on a large set in M, then M' has to carry a certain amount of complex structure (along the image of M under h). More precisely, we shall prove (see Theorems 1.1 and 2.2) that the image of any generic point in such a large set of bad points has a formal holomorphic manifold that is tangent to M' to infinite order, and hence must be a point of *infinite type* in the sense of D'Angelo [8]. To our knowledge, exhibiting such an explicit link between failure of regularity of a CR map and impact on the CR geometry of the target manifold seems to be a completely new point of view in the $\mathscr{C}^\infty$ CR regularity problem. As a consequence, our present approach not only allows us to provide sharper and more general results than earlier works, but also recovers many of the previously known results. The approach we are taking is, at least in philosophy, akin to our recent work [21] on the convergence of formal power series mappings. We will apply ideas from [21], adapted to the $\mathscr{C}^\infty$ setting, to the problem at hand. However, the implementation of these ideas require different strategies and new ingredients because of the different nature of the $\mathscr{C}^\infty$ CR regularity problem.

We will discuss results valid for hypersurfaces in the introduction and leave more general results for later. Before stating our first theorem, let us start by recalling the notion of *infinite type* of a point $q \in M'$ introduced by D'Angelo [8], which means that the order of contact of M' at q with (possibly singular) complex curves is unbounded. To be more precise, let ϱ be a defining function for M' near q. One defines the 1-*type* of M' at q as

$$\Delta(M',q) = \sup_{\substack{\gamma\colon \Delta\to\mathbb{C}^{N'}\\ \gamma(0)=q,\gamma\not\equiv q}} \frac{\nu_0(\rho\circ\gamma)}{\nu_0(\gamma)} \in \mathbb{R}\cup\{\infty\},$$

where γ runs over all (non-trivial) holomorphic curves in $\mathbb{C}^{N'}$ centered at q and ν_0 denotes the vanishing order at 0. We say that q is a *D'Angelo finite type point* of M' if $\Delta(M',q) < \infty$, and an *infinite type point* of M' if $\Delta(M',q) = \infty$. We denote the set of infinite type points in M' by $\mathscr{E}_{M'}$ and recall that $\mathscr{E}_{M'}$ is closed in M' by e.g. [8,9]. We say that M' is of D'Angelo finite type if $\mathscr{E}_{M'} = \emptyset$.

We also need to recall that a CR map $h\colon M \to M'$ between real hypersurfaces $M \subset \mathbb{C}^N$, $M' \subset \mathbb{C}^{N'}$, with respective CR bundles $T^{(0,1)}M$ and $T^{(0,1)}M'$, is said to be *CR transversal* if

$$T^{(1,0)}_{h(p)}M' + T^{(0,1)}_{h(p)}M' + dh(\mathbb{C}T_pM) = \mathbb{C}T_{h(p)}M'.$$

for every point $p \in M$.

We may now state our first main result, which, as mentioned above, highlights how the failure of being $\mathscr{C}^\infty$-smooth for a CR map impacts the geometry of the target manifold M'.

Theorem 1.1. *Let $M \subset \mathbb{C}^{n+1}$ and $M' \subset \mathbb{C}^{n'+1}$ be $\mathscr{C}^\infty$-smooth real hypersurfaces, $n' > n \geq 1$. Assume that M is strongly pseudoconvex and that $h\colon M \to M'$ is a CR transversal mapping of class $\mathscr{C}^{n'-n+1}$. If there exists a non-empty open subset Ω of M where h is nowhere $\mathscr{C}^\infty$, then $h(\Omega) \subset \mathscr{E}_{M'}$.*

As an immediate consequence of Theorem 1.1, we obtain the following regularity result:

Theorem 1.2. *Let $M \subset \mathbb{C}^{n+1}$ and $M' \subset \mathbb{C}^{n'+1}$ be $\mathscr{C}^\infty$-smooth real hypersurfaces, $n' > n \geq 1$. Assume that M is strongly pseudoconvex and that M' is of D'Angelo finite type. Then every CR transversal mapping $h\colon M \to M'$ of class $\mathscr{C}^{n'-n+1}$ is $\mathscr{C}^\infty$-smooth on a dense open subset of M.*

In the special case where M' is strongly pseudoconvex, Theorem 1.2 recovers Berhanu–Xiao's result alluded above [4] (for an embedded hypersurface M) since every CR map between strictly pseudoconvex hypersurfaces is CR transversal by the Hopf Lemma.

When both hypersurfaces are pseudoconvex, using results from the known literature, we will show that Theorem 1.2 also yields the following.

Corollary 1.3. *Let $\Omega \subset \mathbb{C}^{n+1}$ and $\Omega' \subset \mathbb{C}^{n'+1}$ be pseudoconvex domains and $h\colon \Omega \to \Omega'$ be a holomorphic map, $n' > n \geq 1$. Assume that $M \subset \partial\Omega$ and $M' \subset \partial\Omega'$ are $\mathscr{C}^\infty$-smooth real hypersurfaces of D'Angelo finite type. If h extends $\mathscr{C}^{n'-n+1}$-smoothly up to M and satisfies $h(M) \subset M'$, then h extends $\mathscr{C}^\infty$-smoothly up to a dense open subset of M.*

Finally, let us also mention the following new result which follows as an application of Corollary 1.3 to the boundary regularity of (global) proper holomorphic mappings of positive codimension.

Corollary 1.4. *Let $\Omega \subset \mathbb{C}^{n+1}$ and $\Omega' \subset \mathbb{C}^{n'+1}$ be pseudoconvex domains with smooth boundaries $\partial\Omega$ and $\partial\Omega'$ both of D'Angelo finite type, $n' > n \geq 1$. Let $h\colon \Omega \to \Omega'$ be a proper holomorphic map that extends $\mathscr{C}^{n'-n+1}$-smoothly up to a dense open subset of $\partial\Omega$. Then h extends $\mathscr{C}^\infty$-smoothly up to a dense open subset of $\partial\Omega$.*

Let us remark that the preceding results, Theorem 1.2, Theorem 1.1, Corollary 1.3, and Corollary 1.4 hold without any changes for weakly pseudoconvex sources having a dense open subset of strongly pseudoconvex points. For instance, they all can be applied in the setting where M is pseudoconvex and does not contain any analytic disc.

We finish the introduction with an outline of the organization of the paper. In §2, we state the general main result, Theorem 2.2, which applies to minimal source CR manifolds $M \subset \mathbb{C}^N$ of arbitrary codimension. It also implies a number of further new regularity results, which not only extend Theorem 1.2 to the setting where the source manifold is allowed to be of higher codimension but are also valid for target manifolds of infinite D'Angelo type.

The next sections provide the proof of Theorem 2.2 which splits naturally into an analytic part and a geometric part. The first part is developed in §3 and corresponds to the analytic piece of the proof. In it, we prove a smooth regularity result for CR maps that satisfy a smooth system of equations. The result, Theorem 3.1, generalizes a result due to the second author [19], and may be of independent interest. The second, geometric, part of the proof is carried out in §4 and §5. We first introduce in §4 some new numerical invariants associated to any continuous CR map $h\colon M \to \mathbb{C}^{N'}$, establish some of their basic properties and then associate to these invariants an open subset decomposition of (part of) the CR manifold M. In §5 we relate this decomposition to the $\mathscr{C}^\infty$-regularity of the mapping (Proposition 5.1) as well as to the CR geometry of the image set $h(M)$ (Proposition 5.2).

Finally, in §6, we show, among other things, that the decomposition obtained in §4 covers, at least in the situations discussed in §2, a dense open subset of M. The proofs of all theorems and corollaries stated in §1 and §2 are then completed in §7.

2. Statement of further results for CR manifolds of any codimension

This section is devoted to the formulation of the more general results already alluded to in the introduction. We let $M \subset \mathbb{C}^N$ be a $\mathscr{C}^\infty$-smooth CR submanifold, with $N \geq 2$, and recall that a map $h\colon M \to \mathbb{C}^{N'}$ of class $\mathscr{C}^1$ is CR if $h = (h_1, \ldots, h_{N'})$ where each h_j a CR function on M. (If h is assumed to be only continuous, then the preceding definition needs to be understood in the sense of distributions.)

Let us now consider a subset $M' \subset \mathbb{C}^{N'}$ (not necessarily CR nor a manifold). For every $q \in M'$, denote by $\mathscr{I}_{M'}(q) \subset \mathscr{C}^\infty(\mathbb{C}^{N'}, q)$ the ideal of all germs at q of $\mathscr{C}^\infty$-smooth functions $\rho\colon (\mathbb{C}^{N'}_w, q) \to \mathbb{R}$ that vanish on M' near q and denote by $\Gamma_p(M)$ the set of all germs at p of CR vector fields of M.

The definition of a *D'Angelo infinite type point* naturally extends to the more general setting of an arbitrary subset $M' \subset \mathbb{C}^{N'}$ in analogy to the hypersurface case. We define the 1-*type* of M' at q as

$$\Delta(M', q) = \sup_{\substack{\gamma\colon \Delta \to \mathbb{C}^{N'} \\ \gamma(0)=q,\, \gamma \not\equiv q}} \left(\inf_{\rho \in \mathscr{I}_{M'}(q)} \frac{\nu_0(\rho \circ \gamma)}{\nu_0(\gamma)} \right) \in \mathbb{R} \cup \{\infty\},$$

and say that q is a *D'Angelo infinite type point* of M' if $\Delta(M',q)=\infty$. We denote the set of points in M' which are of infinite type by $\mathscr{E}_{M'}$. Observe that if M', M'' are two subsets of $\mathbb{C}^{N'}$ with $M''\subset M'$ then for $q\in M''$, $\Delta(M'',q)\le\Delta(M',q)$ and therefore $\mathscr{E}_{M''}\subset\mathscr{E}_{M'}$.

We also recall that a formal holomorphic subvariety $X\subset\mathbb{C}^{N'}$ through q is given by a (radical) ideal $\mathscr{I}_q(X)\subset\mathbb{C}[\![Z'-q]\!]$. We say that a formal holomorphic subvariety $X\subset\mathbb{C}^{N'}$ through the point $q\in M'$ is tangent to infinite order to M' at q if for any formal holomorphic map $\varphi(t)\in\mathbb{C}[\![t]\!]^{N'}$ with $\varphi(0)=q$ and $\psi\circ\varphi(t)=0$ for every $\psi\in\mathscr{I}_q(X)$ we have $\nu_0\left(\varrho\left(\varphi(t),\overline{\varphi(t)}\right)\right)=\infty$, for every $\varrho\in\mathscr{I}_{M'}(q)$. Note that it follows from this definition that if there exists a nontrivial formal holomorphic subvariety through q which is tangent to M' up to infinite order then q is an infinite type point.

Let us now assume that we are given a CR map $h\colon M\to\mathbb{C}^{N'}$. For every $p\in M$, we set

$$r_0(p):=\dim_{\mathbb{C}}\operatorname{span}\left\{\rho_w(h(p),\overline{h(p)}):\rho\in\mathscr{I}_{h(M)}(h(p))\right\} \tag{2.1}$$

and more generally, if h is of class $\mathscr{C}^\ell$, for some $\ell\ge1$,

$$\begin{aligned} r_k(p):=\dim_{\mathbb{C}}\operatorname{span}\Big\{&\bar L_1\dots\bar L_j\rho_w(h(p),\overline{h(p)}):\rho\in\mathscr{I}_{h(M)}(h(p)),\\ &\bar L_1,\dots,\bar L_j\in\Gamma_p(M),0\le j\le k\Big\},\quad k\le\ell. \end{aligned} \tag{2.2}$$

In the second equation, the case $j=0$ refers to no application of a CR vector field. The complex gradients

$$\rho_w(h(p),\overline{h(p)})=\left(\frac{\partial\rho}{\partial w_1}\left(h(p),\overline{h(p)}\right),\dots,\frac{\partial\rho}{\partial w_{N'}}\left(h(p),\overline{h(p)}\right)\right),$$

and their CR derivatives

$$\bar L_1\dots\bar L_j\rho_w(h(p),\overline{h(p)})=\left(\bar L_1\dots\bar L_j\frac{\partial\rho}{\partial w_1}\left(h(p),\overline{h(p)}\right),\dots,\bar L_1\dots\bar L_j\frac{\partial\rho}{\partial w_{N'}}\left(h(p),\overline{h(p)}\right)\right)$$

are considered as vectors in $\mathbb{C}^{N'}$.

We note that for $0\le k\le\ell$, $p\mapsto r_k(p)\in\{1,\dots,N'\}$ is an integer-valued, lower semicontinuous function on M. We define

$$r_k:=\max\left\{e\in\mathbb{Z}_+:r_k(p)\ge e\text{ for }p\text{ on some dense subset of }M\right\},\ k\le\ell.$$

Let us recall that M is said to be minimal at the point $p\in M$ if there does not exist any CR submanifold $\Sigma\subset M$ through p, with $\dim\Sigma<\dim M$, of the same CR dimension as M (see [28,2]). We say that M is minimal if it is minimal at each of its points.

Before we state our general main result, let us introduce one more notion.

Definition 2.1. Let $M \subset \mathbb{C}^N$ be a $\mathscr{C}^\infty$-smooth CR submanifold, and $h\colon M \to \mathbb{C}^{N'}$ be a $\mathscr{C}^k$-smooth CR map. A $\mathscr{C}^k$-smooth CR family of formal (complex) submanifolds of (complex) dimension r through $h(M)$ is given by a collection $(\Gamma_\xi)_{\xi \in M}$ of formal (complex) submanifolds of $\mathbb{C}^{N'}$ of dimension r in such that, for every $\xi \in M$, Γ_ξ passes through $h(\xi)$ and such that Γ_ξ is parameterized by a formal holomorphic map of the form

$$(\mathbb{C}^r, 0) \ni t \mapsto \varphi_\xi(t) = h(\xi) + \sum_{\substack{\alpha \in \mathbb{N}^r \\ |\alpha| \geq 1}} \varphi_\alpha(\xi) t^\alpha,$$

where for every $\alpha \in \mathbb{N}^r$ the function φ_α is a $\mathscr{C}^k$-smooth CR function on M.

Theorem 2.2. *Let $M \subset \mathbb{C}^N$ be a $\mathscr{C}^\infty$-smooth CR minimal submanifold, $k, \ell \in \mathbb{N}$ with $1 \leq k \leq \ell \leq N'$ be given integers and $h\colon M \to \mathbb{C}^{N'}$ be a CR mapping of class $\mathscr{C}^{N'-\ell+k}$. Assume that $r_k \geq \ell$ and that there exists a non-empty open subset M_1 of M where h is nowhere $\mathscr{C}^\infty$.*

Then there exists a dense open subset $M_2 \subset M_1$ such that for every $p \in M_2$, there exists a neighborhood $V \subset M_2$ of p, an integer $r \geq 1$, and a $\mathscr{C}^1$-smooth CR family of formal (complex) submanifolds $(\Gamma_\xi)_{\xi \in V}$ of dimension r through $h(V)$ for which Γ_ξ is tangent to infinite order to $h(M)$ at $h(\xi)$, for every $\xi \in V$.

In particular, there exists a dense open subset M_2 of M_1 with $h(M_2) \subset \mathscr{E}_{h(M)}$.

Theorem 2.2 provides a detailed picture of how "irregularity" of a given CR map affects the CR geometry of the target set $h(M)$. Images of "irregular" points under the given map must not only be of infinite type, but the image of large open subsets carries even more structure than that: One obtains a family of formal holomorphic submanifolds tangent to $h(M)$ to infinite order that depends in a CR manner on the "irregular" points. This property will be crucial in the application of Theorem 2.2 given in Corollary 2.6 below, providing a regularity result valid for targets which are foliated by complex submanifolds.

The integers r_k in the statement of the theorem appear very naturally in various geometric settings. We will discuss in §6 a number of sufficient conditions providing lower bounds on them, in particular, on r_0 and r_1, yielding a number of new corollaries (not covered by the results in the introduction). In the first one, for $M' \subset \mathbb{C}^{N'}$, we denote by $\kappa_{M'}$ the maximum dimension of real submanifolds of class $\mathscr{C}^1$ contained in $\mathscr{E}_{M'}$.

Corollary 2.3. *Let $M \subset \mathbb{C}^N$ and $M' \subset \mathbb{C}^{N'}$ be $\mathscr{C}^\infty$-smooth CR submanifolds with $n' = \dim_{CR} M'$ and assume that M is minimal. Then every CR mapping $h\colon M \to M'$ of class $\mathscr{C}^{n'}$ and of rank $> \kappa_{M'}$ is $\mathscr{C}^\infty$-smooth on a dense open subset of M. In particular, if M' is of D'Angelo finite type, then every CR mapping $h\colon M \to M'$ of class $\mathscr{C}^{n'}$ is $\mathscr{C}^\infty$-smooth on a dense open subset of M.*

If we know more about the target, we can improve the a priori smoothness assumptions significantly. Our next corollary shows that if the target is Levi-nondegenerate, then the a priori regularity can be dropped by $(n-1)$ where n is the CR dimension of the source submanifold:

Corollary 2.4. *Let $M \subset \mathbb{C}^N$ and $M' \subset \mathbb{C}^{N'}$ be $\mathscr{C}^\infty$-smooth CR submanifolds with $n = \dim_{CR} M$, $n' = \dim_{CR} M'$. Assume that M is minimal and that M' is Levi-nondegenerate and of D'Angelo finite type. Then every CR immersion $h\colon M \to M'$ of class $\mathscr{C}^{n'-n+1}$ is $\mathscr{C}^\infty$-smooth on a dense open subset of M.*

If we want to allow complex manifolds in the target, then we can use geometric information given by Theorem 2.2 on how those complex manifolds are situated in the target (and how large they can be) in conjunction with the formal submanifolds Γ_ξ provided by Theorem 2.2 in order to rule out maps which are nowhere smooth on an open subset of M. We can for instance recover the following result by Berhanu–Xiao [5] (referring to their paper for the standard notion of signature):

Corollary 2.5. *Let $M \subset \mathbb{C}^{n+1}$ and $M' \subset \mathbb{C}^{n'+1}$ be (connected) $\mathscr{C}^\infty$-smooth real hypersurfaces with M strongly pseudoconvex and M' Levi-nondegenerate of signature ℓ', $n' > n \geq 1$. If $n' - \ell' \leq n$, then every CR transversal map $h\colon M \to M'$, of class $\mathscr{C}^{n'-n+1}$, is $\mathscr{C}^\infty$-smooth on some dense open subset of M.*

Our following result uses not only the formal submanifolds Γ_ξ constructed in Theorem 2.2, but also the CR dependence of Γ_ξ on ξ. This is in contrast to Corollary 2.3 and 2.4, where we just use the fact that the Γ_ξ exist. We recall that the tube over the light cone (in $\mathbb{C}^{N'}_w$), defined by the equation

$$\sum_{j=1}^{N'-1} (\mathrm{Re}\ w_j)^2 = (\mathrm{Re}\ w_{N'})^2, \tag{2.3}$$

is one of the basic examples of a uniformly 2-nondegenerate hypersurface. The precise statement given by Theorem 2.2 allows us, in a way similar to the case of convergence of formal maps in [21], to treat the case of maps taking values in the tube over the light cone.

Corollary 2.6. *Let $M \subset \mathbb{C}^N$ be a $\mathscr{C}^\infty$-smooth minimal CR submanifold and $M' \subset \mathbb{C}^{N'}$ be the tube over the light cone. Then every CR map $h\colon M \to M'$, of class $\mathscr{C}^{N'-1}$ and of rank ≥ 3, is $\mathscr{C}^\infty$-smooth on a dense open subset of M.*

Let us remark that both in Corollary 2.6 and also in the preceding Corollary 2.3 the rank of the map is measured in terms of its rank as a real $\mathscr{C}^1$ map (from the real manifold M to the real manifold M'). Since h, in both cases, is a CR map, its linear part at each point $p \in M$ also gives rise to a complex linear map $L(p)$. In the setting of Corollary 2.6, the requirement that the real rank of h is at least 3 corresponds to requiring that the complex rank of $L(p)$ is at least 2 for every p.

The last corollaries we are going to mention will provide a regularity result for finitely nondegenerate source manifolds and in particular, for Levi-nondegenerate sources. Before we formulate this result, we need to introduce the property which will allow us to use the finite nondegeneracy of M. While in many respects similar to the notion of CR transversality, the crucial definition needed here is in some sense dual to transversality. Recall that if $M' \subset \mathbb{C}^{N'}$ is a smooth CR submanifold, then its *complex tangent spaces* $T^c_q M'$, $q \in M'$, form a subbundle $T^c M'$ of the tangent bundle TM'. The *characteristic bundle* of M' is the annihilator of this bundle, i.e. $T^0_q M' := (T^c_q M')^\perp \subset T^*_q M'$. One can check that if h is CR, then $h^* T^0 M' \subset T^0 M$. We use the following definition:

Definition 2.7. We say that a CR map $h\colon M \to M'$ between CR submanifolds $M \subset \mathbb{C}^N$ and $M' \subset \mathbb{C}^{N'}$, of CR codimension d and d' respectively, is *strictly noncharacteristic* (at the point $p \in M$) if

$$h^*(T^0_{h(p)} M') = T^0_p M.$$

Remark 2.8. We recall that a map h is CR transversal at $p \in M$ if

$$T^{(1,0)}_{h(p)} M' + T^{(0,1)}_{h(p)} M' + h'(p)(\mathbb{C}T_p M) = \mathbb{C}T_{h(p)} M'.$$

Clearly, CR transversality implies that $d' \le d$. On the other hand, if h is strictly noncharacteristic, then $d \le d'$. If $d = d'$ one may check that a map is CR transversal if and only if it is strictly noncharacteristic. This conclusion holds in particular when M and M' are hypersurfaces.

Let us recall that a CR submanifold $M \subset \mathbb{C}^N_z$ is *σ-finitely nondegenerate* for some $\sigma \in \mathbb{Z}_+$ (see [2]) if and only if for every $p \in M$, and for any (real) defining function $\varrho = (\varrho^1, \ldots, \varrho^d)$ for M near p, we have

$$\operatorname{span}\left\{ \left(\bar{L}_1 \ldots \bar{L}_k \varrho^r_z\right)(p, \bar{p}) : \bar{L}_j \in \Gamma_p(M), 0 \le j \le k \le \sigma,\ 1 \le r \le d \right\} = \mathbb{C}^N.$$

Corollary 2.9. *Let $M \subset \mathbb{C}^N$ and $M' \subset \mathbb{C}^{N'}$ be $\mathscr{C}^\infty$-smooth CR submanifolds. Assume that M is minimal and σ-finitely nondegenerate for some $\sigma \in \mathbb{Z}_+$ and that M' is of D'Angelo finite type. Then every strictly noncharacteristic CR map $h\colon M \to M'$ of class $\mathscr{C}^{N'-N+\sigma}$ is $\mathscr{C}^\infty$-smooth on some dense open subset of M.*

A particular case of the preceding corollary is the case of a Levi-nondegenerate manifold M (meaning $\sigma = 1$). Even in this case, the regularity result given by Corollary 2.9 is new, and provides, using Remark 2.8, a generalization of Theorem 1.2 to higher codimensions:

Corollary 2.10. *Let $M \subset \mathbb{C}^N$ and $M' \subset \mathbb{C}^{N'}$ be $\mathscr{C}^\infty$-smooth CR submanifolds. Assume that M is minimal and Levi-nondegenerate and that M' is of D'Angelo finite type. Then* *every strictly noncharacteristic CR map* $h\colon M \to M'$ *of class* $\mathscr{C}^{N'-N+1}$ *is* $\mathscr{C}^\infty$*-smooth on some dense open subset of* M*.*

3. A smooth regularity result

In this section, we state and prove our main technical tool to be used later in the paper. It provides a criterion that exhibits sufficient conditions ensuring that a CR map, of class $\mathscr{C}^1$, is in fact $\mathscr{C}^\infty$-smooth. We note that a (weaker) similar result was obtained by the first author in [19], based in part on the work of Roberts [26]. However, for the purpose of this paper, we really need the stronger form stated below.

Theorem 3.1. *Let* $M \subset \mathbb{C}^N$ *be a* $\mathscr{C}^\infty$*-smooth generic submanifold,* $p_0 \in M$*, and let* $h\colon (M,p_0) \to \mathbb{C}^\ell$ *be a germ of a* $\mathscr{C}^1$ *CR mapping at* p_0*,* $g\colon (M,p_0) \to \mathbb{C}^k$ *be a germ of a continuous CR mapping at* p_0*. Let* $U \times V \times O$ *be an open neighborhood of* $(p_0, h(p_0), \overline{g(p_0)}) \in \mathbb{C}^N_z \times \mathbb{C}^\ell_w \times \mathbb{C}^k_\Lambda$*, and* $R\colon U \times V \times O \to \mathbb{C}^\ell$ *be a* $\mathscr{C}^\infty$*-smooth mapping, holomorphic in* $\Lambda \in O$*. Assume that:*

- (i) $R(z,\bar z, h(z,\bar z), \overline{h(z,\bar z)}, \overline{g(z,\bar z)}) = 0$ *for* $z \in M$ *near* p_0*.*
- (ii) $\operatorname{Rk} R_w(p_0,\bar p_0, h(p_0,\bar p_0), \overline{h(p_0,\bar p_0)}, \overline{g(p_0,\bar p_0)}) = \ell$*.*
- (iii) *All components of* h *and* g *extend holomorphically to a common wedge with edge* M *at* p_0*.*

Then h *is* $\mathscr{C}^\infty$*-smooth in a neighborhood of* p_0*.*

Even though the theorem is similar to the almost holomorphic implicit function theorem in [19], we cannot directly apply that theorem. We also include a number of details which are missing from the proof of the theorem in [19]. We split the proof into several steps.

3.1. Smooth wedge coordinates

Let $M \subset \mathbb{C}^N$ be a $\mathscr{C}^\infty$-smooth generic submanifold of codimension d, $p_0 \in M$, and let ρ be a $\mathbb{R}^d$-valued defining function of M near p_0. Recall that a wedge of edge M at p_0 is an open subset of $\mathbb{C}^N$ of the form $\mathscr{W} = \{z \in U : \rho(z,\bar z) \in \Gamma\}$ for some open neighborhood U of p_0 in $\mathbb{C}^N$ and some open convex cone Γ with vertex the origin in $\mathbb{R}^d$, see e.g. [2]. In what follows, we write $B^r_\epsilon(x)$ for the ball of radius $\epsilon > 0$, centered at the point $x \in \mathbb{R}^r$.

We start with the following known fact.

Proposition 3.2. *Let* $M \subset \mathbb{C}^N$ *be a generic* $\mathscr{C}^\infty$*-smooth submanifold of CR dimension* n *and codimension* d*. Let* $p_0 \in M$*,* $\mathscr{W}$ *be a wedge with edge* M *at* p_0*. Then there exist a wedge* $\mathscr{W}' \subset\subset \mathscr{W}$ *with compact closure,* $\epsilon_1, \epsilon_2, r > 0$ *and smooth coordinates* $(\eta, s, t) =$

$\Phi^{-1}(\eta,\bar\eta,\zeta,\bar\zeta) \in \mathbb{C}^n \times \mathbb{R}^d \times \mathbb{R}^d$ *for* $\mathbb{C}^N$ *near* p_0*, where* $\Phi\colon B^{2n}_{\epsilon_1}(0) \times B^{2d}_{\varepsilon_2}(0) \to \mathbb{C}^N$ *is a smooth diffeomorphism, with the following properties, where we write* $\sigma = s + it$*:*

i) $\Phi(0,0,0) = p_0$, $\Phi(\eta,s,0) \subset M$;
ii) $\Phi(B^{2n}_{\epsilon_1} \times B^d_{\epsilon_2} \times (0,r)^d) \subset \mathscr{W}'$
iii) *For every* $\alpha,\beta \in \mathbb{N}^n$ *and every* $\gamma,\delta \in \mathbb{N}^d$ *and every* $a \in \mathbb{N}$ *there exist constants* $C_{\alpha,\beta,\gamma,\delta}$ *and* $C_{\alpha,\beta,\gamma,\delta,a}$ *such that for every continuous CR function* φ *on* M *extending to a holomorphic function* $\tilde\varphi$ *on* $\mathscr{W}$*, we have that the function* $f = \tilde\varphi \circ \Phi$ *satisfies the following:*

$$\left|\frac{\partial^{|\alpha|+|\beta|+|\gamma|+|\delta|} f}{\partial\eta^\alpha\bar\eta^\beta s^\gamma t^\delta}(\eta,\bar\eta,s,t)\right| \le C_{\alpha,\beta,\gamma,\delta} \ \sup_{\mathscr{W}'}|\tilde\varphi| \ \ \|t\|^{-(|\alpha|+|\beta|+|\gamma|+|\delta|)}, \quad (\eta,s,t) \in B^{2n}_{\epsilon_1} \times B^d_{\epsilon_2} \times (0,r)^d, \tag{3.1}$$

and

$$\left|\frac{\partial^{|\alpha|+|\beta|+|\gamma|+|\delta|}}{\partial\eta^\alpha\bar\eta^\beta s^\gamma t^\delta}\frac{\partial f}{\partial\bar\sigma_j}(\eta,\bar\eta,s,t)\right| \le C_{\alpha,\beta,\gamma,\delta,a} \ \sup_{\mathscr{W}'}|\tilde\varphi| \ \ \|t\|^a, \quad (\eta,s,t) \in B^{2n}_{\epsilon_1} \times B^d_{\epsilon_2} \times (0,r)^d,\ 1 \le j \le d. \tag{3.2}$$

Proof. We assume that $p_0 = 0$. We consider a smooth defining function of M near 0 of the form $\mathsf{Im}\,\zeta = \varphi(\eta,\bar\eta,\mathsf{Re}\,\zeta)$, where $\mathbb{C}^N_z = \mathbb{C}^n_\eta \times \mathbb{C}^d_\zeta$, and furthermore $\nabla\varphi(0) = 0$ (so that $T^c_0 M = \{\zeta = 0\}$). Thus, for some neighborhoods U_1, U_2 of 0 in $\mathbb{C}^n$ and $\mathbb{R}^d$ respectively, the map

$$\Psi\colon \mathbb{C}^n_\eta \times \mathbb{R}^d_s \ni (\eta,s) \mapsto (\eta, s + i\varphi(\eta,\bar\eta,s))$$

parametrizes M near 0 for $(\eta,s) \in U_1 \times U_2$. We choose an almost holomorphic extension of Ψ to $U_1 \times U_2 \times \mathbb{R}^d$, again denoted by Ψ, in the s-variable (see for this e.g. [23]). After possibly shrinking U_1 and U_2 a bit, we can assume that for $a \in \mathbb{N}$ there exist constants $C_a > 0$ such that this new $\Psi\colon U_1 \times U_2 \times \mathbb{R}^d$ is a smooth map which satisfies:

$$\begin{gathered}\Psi(\eta,\bar\eta,s,0) = (\eta, s + i\varphi(\eta,\bar\eta,s)) \in M;\\ \left|\frac{\partial\Psi}{\partial\bar\sigma_j}(\eta,\bar\eta,s,t)\right| \le C_a \|t\|^a, \quad j = 1,\dots,d,\ \eta \in U_1, s \in U_2.\end{gathered} \tag{3.3}$$

Note that since $\nabla\varphi(0) = 0$, we have that $\Psi'(0) = \mathsf{id}$; hence, again after possibly shrinking U_1 and U_2 a bit, we can assume that $\Psi\colon U_1 \times U_2 \times \tilde U_2 \to \Psi(U_1 \times U_2 \times \tilde U_2)$ is a diffeomorphism from $U_1 \times U_2 \times \tilde U_2$ onto a neighborhood of 0.

Now consider a wedge $\mathscr{W}$ with edge M near 0. This means that in a small neighborhood of 0, we can assume that we can write $\mathscr{W}$ (in our chosen coordinates) as $\mathbb{C}^n_\eta \times \mathbb{R}^d_s \times \Gamma$, for some open, convex cone $\Gamma \subset \mathbb{R}^d$. Let us also choose an arbitrary $\xi \in T_0\mathbb{C}^N$ with

$\xi \in \mathscr{W}$. It follows that $\Psi^{-1}(\mathscr{W})$ has the property that we can find a (closed) convex cone $\Gamma' \subset \mathbb{R}^d \setminus \{0\}$, with $\Gamma' \cup \{0\} = \mathrm{CH}\{u_1, \dots, u_d\}$ for some vectors $u_1, \dots, u_d$ in $\mathbb{R}^d$, linearly equivalent to $\mathbb{R}^d_+$, such that for some small balls $B^{2n}_{\varepsilon_1}(0) \subset \mathbb{C}^n$, $B^d_{\tilde{\varepsilon}_2}(0) \subset \mathbb{R}^d$, and some $\tilde{r} > 0$ we have $B^{2n}_{\varepsilon_1}(0) \times B^d_{\tilde{\varepsilon}_2}(0) \times \Gamma'_{\tilde{r}} \subset \Psi^{-1}(\mathscr{W})$, where $\Gamma'_{\tilde{r}} = \{t \in \Gamma' \colon \|t\| < \tilde{r}\}$. Now consider the complex linear transformation $U \colon \mathbb{C}^d \to \mathbb{C}^d$ defined by $U(\sigma_1, \dots, \sigma_d) = \sum_{j=1}^d \sigma_j u_j$. By choice of Γ', we have $U(i\mathbb{R}^d_+) = \{0\} \times \Gamma'$. By choosing an appropriate ε_2 and r we can assume that $U(B^d_{\varepsilon_2}(0) + i(0,r)^d) \subset B^d_{\tilde{\varepsilon}_2}(0) \times \Gamma'_{\tilde{r}}$.

We define the map $\Phi \colon B^{2n}_{\varepsilon_1}(0) \times B^d_{\varepsilon_2}(0) \times (-r,r)^d \to \mathbb{C}^N$,

$$\Phi(\eta, \bar{\eta}, s, t) = \Psi(\eta, \bar{\eta}, Us, Ut).$$

Note that since $\Gamma' \subset \Gamma$ was a closed cone, and r can be chosen as small as needed, we can find a wedge $\mathscr{W}' \subset\subset \mathscr{W}$ and a constant $C > 0$ such that

$$\frac{1}{C} \|t\| \le d(\Psi(\eta, s, t), \partial \mathscr{W}') \le C \|t\| . \tag{3.4}$$

Also note that since $U(s + it) = Us + iUt$ is complex linear, the estimates (3.3) hold also for the corresponding derivatives of Φ (where we might to use different constants C_a, $a \in \mathbb{N}$, of course):

$$\begin{aligned} &\Phi(\eta, \bar{\eta}, s, 0) = (\eta, Us + i\varphi(\eta, \bar{\eta}, Us)) \in M; \\ &\left| \frac{\partial \Phi}{\partial \bar{\sigma}_j}(\eta, \bar{\eta}, s, t) \right| \le C_a \|t\|^a, \quad j = 1, \dots, d, \ \eta \in B^{2n}_{\varepsilon_1}(0), s \in B^d_{\varepsilon_2}(0), \ \forall a \in \mathbb{N}. \end{aligned} \tag{3.5}$$

If a continuous CR function $\varphi \colon M \to \mathbb{C}$ extends holomorphically to $\mathscr{W}$ near 0, we know by a result of Rosay [25] that the extension, which we are still going to denote by $\tilde{\varphi}$, is actually continuous up to the edge M on any finer wedge than the given $\mathscr{W}$. Therefore, we can apply Cauchy's inequalities to the domain $\mathscr{W}'$: since $\tilde{\varphi}$ is continuous up to the edge, and holomorphic in $\mathscr{W}$, we have that

$$\left| \frac{\partial^{|\alpha|+|\beta|} \tilde{\varphi}}{\partial \eta^\alpha \zeta^\beta}(\eta, \zeta) \right| \le \frac{\alpha! \beta! \ \sup_{\mathscr{W}'} |\tilde{\varphi}|}{(Kd((\eta, \zeta), \partial \mathscr{W}'))^{|\alpha|+|\beta|}},$$

with a constant K just depending on the metric used.

Combining this inequality with (3.4), applying the chain rule, and using the fact that Ψ is smooth, we can therefore find, for any $\alpha, \beta \in \mathbb{N}^n$, and every $\gamma, \delta \in \mathbb{N}^d$, a constant $C_{\alpha,\beta,\gamma,\delta}$ (independent of $\tilde{\varphi}$) such that (3.1) holds.

Furthermore, if we appeal to (3.5), a similar argument shows that (3.2) holds. □

3.2. Edge of the wedge theory

In this subsection, we discuss the necessary smooth edge of the wedge theory. We consider $H = B^{2n}_{\varepsilon_1}(0) \times B^d_{\epsilon_2}(0) \times \{0\}^d \subset \mathbb{C}^n_\eta \times \mathbb{R}^d_s \times \mathbb{R}^d_t$, and $H_+ = B^{2n}_{\varepsilon_1}(0) \times B^d_{\epsilon_2}(0) \times (0,r)^d$,

$H_- = B^{2n}_{\varepsilon_1}(0) \times B^d_{\epsilon_2}(0) \times (-r,0)^d$. We use, as before, the complex variables $\sigma = s + it \in \mathbb{C}^d$. We define $\mathfrak{A}_\infty(H_+)$ to be the set of all functions $f \in \mathscr{C}^\infty(H_+)$ which have the following property: For every $\alpha, \beta \in \mathbb{N}^n$, every $\gamma, \delta \in \mathbb{N}^d$, and every $a \in \mathbb{N}$ there exist constants $C_{\alpha,\beta,\gamma,\delta}$, $C_{\alpha,\beta,\gamma,\delta,a}$, and $b \in \mathbb{N}$ such that

$$\begin{aligned} \left| \frac{\partial^{|\alpha|+|\beta|+|\gamma|+|\delta|} f}{\partial \eta^\alpha \bar\eta^\beta s^\gamma t^\delta}(\eta, \bar\eta, s, t) \right| &\le C_{\alpha,\beta,\gamma,\delta} \left\| t \right\|^{-b}, \quad (\eta, s, t) H_+, \\ \left| \frac{\partial^{|\alpha|+|\beta|+|\gamma|+|\delta|}}{\partial \eta^\alpha \bar\eta^\beta s^\gamma t^\delta} \frac{\partial f}{\partial \bar\sigma_j}(\eta, \bar\eta, s, t) \right| &\le C_{\alpha,\beta,\gamma,\delta,a} \left\| t \right\|^{a}, \quad (\eta, s, t) \in H_+. \end{aligned} \tag{3.6}$$

The analogous definition is given for $\mathfrak{A}_\infty(H_-)$. It is well known, see e.g. [2,3], that every function f in $\mathfrak{A}(H_\pm)$ has a boundary value distribution defined for $\chi \in \mathscr{D}(H)$ by

$$\langle \mathrm{bv}\, f, \chi \rangle = \lim_{\substack{t \to 0 \\ t \in \mathbb{R}^d_\pm}} \int_{\mathbb{C}^n \times \mathbb{R}^d} f(z, s, t) \chi(z, s) \, dm.$$

The edge of the wedge theorem that we are going to use is the following.

Theorem 3.3. *Assume that $U \in \mathscr{D}'(H)$ is both a boundary value from above and from below, i.e. there exist $f_+ \in \mathfrak{A}_\infty(H_+)$ and $f_- \in \mathfrak{A}_\infty(H_-)$ such that* $\mathrm{bv}\, f_+ = \mathrm{bv}\, f_- = U$. *Then $U \in \mathscr{C}^\infty(H)$.*

A proof of Theorem 3.3 can be found in e.g. [19].

There are a number of interesting properties for the sets $\mathfrak{A}_\infty(H_\pm)$. The most important of them is probably the inclusion $\mathscr{C}^\infty(H) \subset \mathfrak{A}_\infty(H_\pm)$ which follows from the existence of an almost analytic extension of a smooth function in the s variables : If $U \in \mathscr{C}^\infty(H)$, then there exists a function $\tilde U \in \mathscr{C}^\infty(\mathbb{C}^n \times \mathbb{R}^d \times \mathbb{R}^d)$ with $\tilde U|_H = U$ and such that $\frac{\partial \tilde U}{\partial \bar\sigma_j}$ vanishes to infinite order on H for $j = 1, \dots, d$ (see [23]).

Also, if X is a partial differential operator in the (η, s)-variables with smooth coefficients, and $\tilde X$ denotes the extension given by almost analytic extension of the coefficients of X, then $\tilde X f \in \mathfrak{A}_\infty(H_\pm)$ for $f \in \mathfrak{A}_\infty(H_\pm)$ and $X\, \mathrm{bv}\, f = \mathrm{bv}\, \tilde X f$.

3.3. A priori regularity for $\bar\partial$-bounded extensions

Our goal in this section is to recall a Hölder regularity result for extensions of Hölder continuous functions which are $\bar\partial$-bounded and whose (first order) derivatives are of a certain growth (later to be applied to extensions of continuous CR functions).

We first introduce some notation: a continuous function $f \colon \Omega \to \mathbb{C}$ is Hölder continuous on a set $\Omega \subset \mathbb{R}^p$ with Hölder exponent $\alpha \in (0, 1]$ if there exists a constant C such that

$$|f(x) - f(y)| \le C \left\| x - y \right\|^\alpha.$$

The space of all Hölder continuous functions on Ω with Hölder exponent α is denoted by $\mathscr{C}^{0,\alpha}(\Omega)$. If Ω is compact, it becomes a Banach space if endowed with the norm

$$\|f\|_{0,\alpha} = \|f\|_\infty + \|f\|_\alpha ,$$

where

$$\|f\|_\infty = \max_{x\in\Omega} |f(x)|, \quad \|f\|_\alpha = \max_{x\neq y\in\Omega} \frac{|f(x)-f(y)|}{\|x-y\|^\alpha}.$$

Let $H \subset \mathbb{C}^n_\eta \times \mathbb{R}^d_s$ be open, and write (for some $r > 0$)

$$H_+ = H \times (0,r)^d, \quad H_- = H \times (-r,0)^d, \; H_+, H_- \subset H \times \mathbb{R}^d_t.$$

The following result follows from inspecting the proof of Coupet's paper [7] including his proposition 1, and is stated in the context we need to refer to:

Proposition 3.4. *Let $\tilde H \subset\subset H$, $0<\alpha<1$. Set $\beta = \frac{\alpha}{1+\alpha}$ and write $\sigma = s+it \in \mathbb{C}^d$. There exists a constant $K = K(\alpha, \tilde H)$ such that if $h \in \mathscr{C}^1(H_+)$ is continuous up to $H \times \{0\}$ with*

$$\forall(\eta,s,t) \in H_+, \; \left|\frac{\partial h}{\partial \bar\sigma_j}(\eta,\bar\eta,s,t)\right| \le C, \quad h|_{t=0} \in \mathscr{C}^{0,\alpha}(H),$$

$$\forall(\eta,s,t) \in H_+, \; |h_{s_j}(\eta,\bar\eta,s,t)| \le \frac{C}{\|t\|}, \quad |h_{\eta_k}(\eta,\bar\eta,s,t)| \le \frac{C}{\|t\|}, \quad |h_{\bar\eta_k}(\eta,\bar\eta,s,t)| \le \frac{C}{\|t\|},$$

for some constant $C>0$ and $j=1,\dots,d$, $k=1,\dots,n$, then

$$h \in \mathscr{C}^{0,\beta}(\tilde H_+), \qquad \text{with } \|h\|_\beta \le K(C + \|h|_{t=0}\|_{0,\alpha}),$$

where $\tilde H_+ = \tilde H \times (0,\tilde r)^d$ for arbitrary $\tilde r < r$.

3.4. Proof of Theorem 3.1

Proof. For the proof of the theorem, we need to extend R almost analytically in (most) of its variables. This will allow us to consider h and $\bar h$ (mostly) as independent variables. We will from now on choose coordinates for M as in Proposition 3.2, adapted to the wedge $\mathscr{W}$ to which we assume that h and g extend. In these coordinates, h and g extending continuously to functions $h_+(\eta,\bar\eta,s,t), g_+(\eta,\bar\eta,s,t) \in \mathfrak{A}_\infty(H_+)$ where $H_+ = B^{2n}_{\epsilon_1} \times B^d_{\epsilon_2} \times (0,r)^d$ and $H = B^{2n}_{\epsilon_1} \times B^d_{\epsilon_2}$. The plan is to use the smooth identity

$$R(q,\bar q, h(q,\bar q), \overline{h(q,\bar q)}, \overline{g(q,\bar q)}) = 0$$

for q in some neighborhood of p_0 in M, to express h in a second way through an "almost reflection identity", which will show that it also extends continuously to a function

$h_- \in \mathfrak{A}_\infty(H_-)$. An application of Theorem 3.3 then implies the smoothness of h near p_0.

We write $w = x + iy$ with $x, y \in \mathbb{R}^{N'}$, and for simplicity assume that $h(p_0) = 0$ and $g(p_0) = 0$. We write R as a map in the following way: $R(\eta, \bar\eta, s, x, y, \Lambda)$ is defined on a set of the form $B^{2n}_{\varepsilon_1}(0) \times B^d_{\varepsilon_2}(0) \times U_2 \times U_2 \times O$, where $U_2 \subset \mathbb{R}^{N'}$ and $O \subset \mathbb{C}^k$ are neighborhoods of the origin. We can extend R almost analytically in s, x, and y, to a smooth map defined on $B^{2n}_{\varepsilon_1}(0) \times B^d_{\varepsilon_2}(0) \times \mathbb{R}^d_t \times U_2 \times \mathbb{R}^{N'}_{x'} \times U_2 \times \mathbb{R}^{N'}_{y'} \times O$. We write complex coordinates $\sigma = s + it$, $\chi = x + ix'$, and $\upsilon = y + iy'$. The extended map will be denoted by $\tilde R(\eta, \bar\eta, s, t, x, x', y, y', \Lambda) = \tilde R(\eta, \bar\eta, s, t, \chi, \bar\chi, \upsilon, \bar\upsilon, \Lambda)$. It relates to R by

$$\tilde R(\eta, \bar\eta, s, 0, x, 0, y, 0, \Lambda) = R(\eta, \bar\eta, s, x, y, \Lambda)$$

and satisfies that

$$\frac{\partial}{\partial\bar\sigma_j}\tilde R,\ j = 1, \dots, d, \quad \frac{\partial}{\partial\bar\chi_\ell}\tilde R,\ \text{and}\ \frac{\partial}{\partial\bar\upsilon_\ell}\tilde R,\ \ell = 1, \dots, N',$$

all vanish to infinite order along $t = 0$, $x' = y' = 0$ (actually, locally uniformly in Λ).

We introduce new complex coordinates $(Z, \zeta) \in \mathbb{C}^{N'} \times \mathbb{C}^{N'}$ by

$$\chi = \frac{Z + \zeta}{2}, \quad \upsilon = \frac{Z - \zeta}{2i}.$$

Let us set

$$\hat R(\eta, \bar\eta, s, t, Z, \bar Z, \zeta, \bar\zeta, \Lambda) = \tilde R\left(\eta, \bar\eta, s, t, \frac{Z + \zeta}{2}, \overline{\frac{Z + \zeta}{2}}, \frac{Z - \zeta}{2i}, \overline{\frac{Z - \zeta}{2i}}, \Lambda\right).$$

Note that $\hat R(\eta, \bar\eta, s, 0, h(\eta, \bar\eta, s), \overline{h(\eta, \bar\eta, s)}, \overline{h(\eta, \bar\eta, s)}, h(\eta, \bar\eta, s), \overline{g(\eta, \bar\eta, s)}) = 0$ for $(\eta, s) \in H$ and that since (Z, ζ) are *complex* coordinates, the derivatives

$$\frac{\partial}{\partial\bar\sigma_j}\hat R,\ j = 1, \dots, d, \quad \frac{\partial}{\partial\bar Z_\ell}\hat R,\ \text{and}\ \frac{\partial}{\partial\bar\zeta_\ell}\hat R,\ \ell = 1, \dots, N',$$

vanish to infinite order along $t = 0$, $\zeta = \bar Z$; to be more exact, we can assume (after possibly shrinking the neighborhoods a bit near the origin) that for any $a \in \mathbb{N}$ there exists a constant $C = C_a$, depending also on the chosen neighborhood, such that

$$\sum_{j=1}^{d} \left\| \frac{\partial \hat R}{\partial \bar\sigma_j} \right\| + \sum_{\ell=1}^{N'} \left\| \frac{\partial \hat R}{\partial \bar Z_\ell} \right\| + \sum_{\ell=1}^{N'} \left\| \frac{\partial \hat R}{\partial \bar\zeta_\ell} \right\| \le C_a \left(\|t\| + \left\|\bar Z - \zeta\right\|\right)^a \tag{3.7}$$

Let us now compute the (real) Jacobian of $\hat R$ with respect to Z (at 0), that is, the Jacobian with respect to all of the underlying real variables of Z. For this, we note that for each ℓ, $\ell = 1, \dots, N'$, we have

$$\hat{R}_{Z_\ell}(0) = \frac{1}{2}\underbrace{\tilde{R}_{\chi_\ell}(0)}_{=\tilde{R}_{x_\ell}(0)} + \frac{1}{2i}\underbrace{\tilde{R}_{\upsilon_\ell}(0)}_{=\tilde{R}_{y_\ell}(0)} = \frac{1}{2}\left(R_{x_\ell}(0) - iR_{y_\ell}(0)\right) = R_{w_\ell}(0),$$

and that

$$\overline{\hat{R}}_{Z_\ell}(0) = \frac{1}{2}\overline{\tilde{R}}_{\chi_\ell}(0) - \frac{1}{2i}\overline{\tilde{R}}_{\upsilon_\ell}(0) = \overline{\tilde{R}_{\bar{\chi}_\ell}(0)} + \overline{\tilde{R}_{\bar{\upsilon}_\ell}(0)} = 0.$$

Hence the Jacobian matrix of $\hat{R}$ with respect to all of the underlying real variables constituting the complex variables Z, evaluated at the origin, has the determinant

$$\left|\frac{\partial \hat{R}}{\partial(Z,\bar{Z})}(0)\right| = \begin{vmatrix} \frac{\partial \hat{R}}{\partial Z}(0) & \frac{\partial \hat{R}}{\partial \bar{Z}}(0) \\ \frac{\partial \overline{\hat{R}}}{\partial Z}(0) & \frac{\partial \overline{\hat{R}}}{\partial \bar{Z}}(0) \end{vmatrix} = \begin{vmatrix} \frac{\partial R}{\partial w}(0) & 0 \\ 0 & \overline{\frac{\partial R}{\partial w}(0)} \end{vmatrix} = \left|\det(R_w(0))\right|^2 \neq 0.$$

We can thus apply the (smooth) implicit function theorem and from it see that there exists a unique smooth function Φ, defined in a neighborhood $\hat{U}_1 \times \hat{U}_2 \times \hat{U}_3 \times \hat{U}_4 \subset \mathbb{C}^n \times \mathbb{C}^d \times \mathbb{C}^{N'} \times \mathbb{C}^k$ of 0 , taking values in some open neighborhood $\hat{V}$ of $0 \in \mathbb{C}^{N'}$, such that

$$\hat{R}(\eta,\bar{\eta},s,t,Z,\bar{Z},\zeta,\bar{\zeta},\Lambda) = 0 \Leftrightarrow Z = \Phi(\eta,\bar{\eta},s,t,\zeta,\bar{\zeta},\Lambda)$$

for $(\eta,\sigma,Z,\zeta,\Lambda) \in \hat{U}_1 \times \hat{U}_2 \times \hat{V} \times \hat{U}_3 \times \hat{U}_4$.

Differentiating with respect to $\bar{\sigma}$ and $\bar{\zeta}$, using the usual matrix notation, we see that for $Z = \Phi(\eta,\bar{\eta},s,t,\zeta,\bar{\zeta},\Lambda)$

$$\Phi_{\bar{\sigma}} = -R_Z^{-1}\left(\hat{R}_{\bar{\sigma}} + \hat{R}_{\bar{Z}}\overline{\Phi}_{\bar{\sigma}}\right)$$
$$\Phi_{\bar{\zeta}} = -R_Z^{-1}\left(\hat{R}_{\bar{\zeta}} + \hat{R}_{\bar{Z}}\overline{\Phi}_{\bar{\zeta}}\right).$$

Using these equalities, (3.7), and the fact that $\det R_Z$ does not vanish at any point, we see that for every $\alpha,\beta \in \mathbb{N}^n$, $\gamma,\delta \in \mathbb{N}^d$, $\varepsilon,\nu \in \mathbb{N}^{N'}$, every $\mu \in \mathbb{N}^k$ and every $a \in \mathbb{N}$ there exists a constant $C = C_{\alpha,\beta,\gamma,\delta,\varepsilon,\nu,\mu,a} > 0$ such that for $j = 1,\dots,d$ and $\ell = 1,\dots,N'$, and $(\eta,\sigma,\zeta,\Lambda) \in \hat{U}_1 \times \hat{U}_2 \times \hat{U}_3 \times \hat{U}_4$

$$\begin{aligned} \left\|\frac{\partial^{|\alpha|+|\beta|+|\gamma|+|\delta|+|\varepsilon|+|\nu|}}{\partial\eta^\alpha\bar{\eta}^\beta s^\gamma t^\delta \zeta^\varepsilon \bar{\zeta}^\nu \Lambda^\mu}\frac{\partial \Phi}{\partial \bar{\sigma}_j}(\eta,\bar{\eta},s,t,\zeta,\bar{\zeta},\Lambda)\right\| &\le C\left(\|t\| + \left\|\Phi(\eta,\bar{\eta},s,t,\zeta,\bar{\zeta},\Lambda) - \bar{\zeta}\right\|\right)^a \\ \left\|\frac{\partial^{|\alpha|+|\beta|+|\gamma|+|\delta|+|\varepsilon|+|\nu|}}{\partial\eta^\alpha\bar{\eta}^\beta s^\gamma t^\delta \zeta^\varepsilon \bar{\zeta}^\nu \Lambda^\mu}\frac{\partial \Phi}{\partial \bar{\zeta}_\ell}(\eta,\bar{\eta},s,t,\zeta,\bar{\zeta},\Lambda)\right\| &\le C\left(\|t\| + \left\|\Phi(\eta,\bar{\eta},s,t,\zeta,\bar{\zeta},\Lambda) - \bar{\zeta}\right\|\right)^a, \end{aligned} \tag{3.8}$$

where $\hat{U}_1, \hat{U}_2, \hat{U}_3, \hat{U}_4$ may have possibly been shrunk.

We recall that, shrinking ϵ_1, ϵ_2 if necessary, for $(\eta, s) \in H$,

$$\hat{R}\left(\eta, \bar{\eta}, s, 0, h(\eta,\bar{\eta},s), \overline{h(\eta,\bar{\eta},s)}, \overline{h(\eta,\bar{\eta},s)}, h(\eta,\bar{\eta},s), \overline{g(\eta,\bar{\eta},s)}\right) = 0,$$

from which we conclude that

$$h(\eta,\bar{\eta},s) = \Phi(\eta,\bar{\eta},s,\bar{s},\overline{h(\eta,\bar{\eta},s)}, h(\eta,s), \overline{g(\eta,\bar{\eta},s)}), \quad (\eta,s) \in H.$$

We recall that we write $h_+(\eta,\bar{\eta},s,t)$, and $g_+(\eta,\bar{\eta},s,t)$ for the almost analytic extensions of h and g to H_+, which exist by assumption.

We now set

$$\begin{aligned} &h_-(\eta,\bar{\eta},s,t) := \Phi(\eta,\bar{\eta},s,t,\overline{h_+(\eta,\bar{\eta},s,-t)}, h_+(\eta,\bar{\eta},s,-t), \overline{g_+(\eta,\bar{\eta},s,-t)}), \\ &\quad (\eta,s) \in H, t \in (-r,0)^d, \end{aligned}$$

and claim that h_- lies in $\mathfrak{A}_\infty(\tilde{H}_-)$ for some (possibly smaller) neighborhood $\tilde{H} \subset H$ of 0 in $\mathbb{C}^n \times \mathbb{R}^d$ and some $0 < \tilde{r} < r$.

One can check that the slow growth condition for $h_-(\eta,\bar{\eta},s,t)$ is satisfied on H_-, because Φ is smooth, and h_+, $\bar{h}_+$, and $\bar{g}_+$ are all of slow growth on H_+ by assumption. We therefore only have to check that for any $\alpha, \beta \in \mathbb{N}^n$, any $\gamma, \delta \in \mathbb{N}^d$, and any $a \in \mathbb{N}$, there exists a constant $C_{\alpha,\beta,\gamma,\delta,a}$ such that

$$\left| \frac{\partial^{|\alpha|+|\beta|+|\gamma|+|\delta|}}{\partial\eta^\alpha\bar{\eta}^\beta s^\gamma t^\delta} \frac{\partial h_-}{\partial\bar{\sigma}_j}(\eta,\bar{\eta},s,t) \right| \le C_{\alpha,\beta,\gamma,\delta,a} \|t\|^a, \quad (z,s,t) \in \tilde{H}_-, \quad j = 1,\dots,d. \tag{3.9}$$

So we first compute the derivative with respect to $\bar{\sigma}_j = s_j - it_j$. Recall that

$$\frac{\partial \overline{h_+(\eta,\bar{\eta},s,-t)}}{\partial\bar{\sigma}_j} = \overline{\frac{\partial h_+}{\partial\bar{\sigma}_j}(\eta,\bar{\eta},s,-t)},$$

and compute (we drop the arguments):

$$\frac{\partial h_-}{\partial\bar{\sigma}_j} = \Phi_{\bar{\sigma}_j} + \Phi_\zeta \overline{\frac{\partial h_+}{\partial\bar{\sigma}_j}} + \Phi_{\bar{\zeta}} \frac{\partial h_+}{\partial\sigma_j} + \Phi_\Lambda \overline{\frac{\partial g_+}{\partial\bar{\sigma}_j}} \tag{3.10}$$

Using similar arguments as in showing that h_- is of slow growth, one sees that the second and the fourth summand satisfy the estimate (3.9). Indeed, if $\alpha, \beta \in \mathbb{N}^n$, $\beta, \gamma \in \mathbb{N}^d$, and a are given, then we can write

$$\frac{\partial^{|\alpha|+|\beta|+|\gamma|+|\delta|}}{\partial\eta^\alpha\bar{\eta}^\beta s^\gamma t^\delta}\left(\Phi_\zeta \overline{\frac{\partial h_+}{\partial\bar{\sigma}_j}}\right), \quad \frac{\partial^{|\alpha|+|\beta|+|\gamma|+|\delta|}}{\partial\eta^\alpha\bar{\eta}^\beta s^\gamma t^\delta}\left(\Phi_\Lambda \overline{\frac{\partial g_+}{\partial\bar{\sigma}_j}}\right)$$

as (finite) sum of terms, each of which is a product of three types of factors: First, some derivative of Φ, evaluated at $(\eta,\bar{\eta},s,t,\overline{h_+(\eta,\bar{\eta},s,-t)}, h_+(\eta,\bar{\eta},s,-t), \overline{g_+(\eta,\bar{\eta},s,-t)})$, which stays uniformly bounded over H_-; second, some derivatives of h_+, $\bar h_+$, and $\bar g$, each of which are of slow growth; and third, some derivative of either $\overline{\frac{\partial h_+}{\partial \bar\sigma_j}}$ or $\overline{\frac{\partial g_+}{\partial \bar\sigma_j}}$. Since by (3.2) each of these vanishes to infinite order at $t = 0$, so does this finite sum.

In order to deal with the first and the third summand, we first need some preparation: Since $h(\eta, \bar\eta, s)$ is $\mathscr{C}^1$, by the result of Rosay [25] already mentioned above, h_+ is actually $\mathscr{C}^1$ up to the edge; therefore, (3.10) and (3.2) imply that there exists a constant $C > 0$ with $\left|\frac{\partial h_-}{\partial \bar\sigma_j}(\eta, \bar\eta, s, t)\right| \le C$ for $(\eta, \bar\eta, s, t) \in H_-$, $j = 1, \dots, d$, and that $h_- \in \mathscr{C}(H_- \cup H)$. Also choose C so large that we have that

$$\left|\frac{\partial h_-}{\partial s_j}(\eta, \bar\eta, s, t)\right| < \frac{C}{\|t\|}, \quad \left|\frac{\partial h_-}{\partial \eta_\ell}(\eta, \bar\eta, s, t)\right| < \frac{C}{\|t\|}, \quad \left|\frac{\partial h_-}{\partial \bar\eta_\ell}(\eta, \bar\eta, s, t)\right| < \frac{C}{\|t\|},$$
$$j = 1, \dots, d, \ell = 1, \dots, n.$$

Recalling that for $(\eta, s) \in H$

$$h(\eta, \bar\eta, s) = \Phi\left(\eta, \bar\eta, s, 0, \overline{h_+(\eta, \bar\eta, s, 0)}, h_+(\eta, \bar\eta, s, 0), \overline{g(\eta, \bar\eta, s, 0)}\right) = h_-(\eta, \bar\eta, s, 0),$$

we thus see that $h_-(\eta, \bar\eta, s, t)$ satisfies the assumptions of Proposition 3.4 for any $\alpha < 1$. Therefore, h_-, when restricted to any set of the form $\tilde H_- = \tilde H \times (\tilde r, 0)^d$ as in that corollary, is $\mathscr{C}^{0,\beta_0}(\tilde H)$ for every $\beta_0 < \frac{1}{2}$. Fix any such β_0 for the remainder of the proof.

For $(\eta, s, t) \in \tilde H_-$, we can therefore estimate

$$\|h_+(\eta, \bar\eta, s, -t) - h_-(\eta, \bar\eta, s, t)\| \le C \|t\|^{\beta_0}.$$

We now return to the terms of interest. We claim that both

$$\Phi_{\bar\sigma}\left(\eta, \bar\eta, s, t, \overline{h_+(\eta, \bar\eta, s, -t)}, h_+(\eta, \bar\eta, s, -t), \overline{g_+(\eta, \bar\eta, s, -t)}\right),$$
$$\Phi_{\bar\zeta}\left(\eta, \bar\eta, s, t, \overline{h_+(\eta, \bar\eta, s, -t)}, h_+(\eta, \bar\eta, s, -t), \overline{g_+(\eta, \bar\eta, s, -t)}\right)$$

are flat along $t = 0$ on $\tilde H_-$ that is, we will check that for $j = 1, \dots, d$, $\ell = 1, \dots, N'$ and given $\alpha, \beta \in \mathbb{N}^n$, $\gamma, \delta \in \mathbb{N}^d$, and $a \in \mathbb{N}$ there exists a constant $C_{\alpha,\beta,\gamma,\delta,a} > 0$ such that for $(\eta, s) \in \tilde H_-$

$$\begin{aligned} &\left\|\frac{\partial^{|\alpha|+|\beta|+|\gamma|+|\delta|}}{\partial \eta^\alpha \bar\eta^\beta s^\gamma t^\delta} \Phi_{\bar\sigma_j}\left(\eta, \bar\eta, s, t, \overline{h_+(\eta, \bar\eta, s, -t)}, h_+(\eta, \bar\eta, s, -t), \overline{g_+(\eta, \bar\eta, s, -t)}\right)\right\| \\ &\quad \le C_{\alpha,\beta,\gamma,\delta,a} \|t\|^a \\ &\left\|\frac{\partial^{|\alpha|+|\beta|+|\gamma|+|\delta|}}{\partial \eta^\alpha \bar\eta^\beta s^\gamma t^\delta} \Phi_{\bar\zeta_\ell}\left(\eta, \bar\eta, s, t, \overline{h_+(\eta, \bar\eta, s, -t)}, h_+(\eta, \bar\eta, s, -t), \overline{g_+(\eta, \bar\eta, s, -t)}\right)\right\| \\ &\quad \le C_{\alpha,\beta,\gamma,\delta,a} \|t\|^a \end{aligned} \tag{3.11}$$

Write $A = |\alpha| + |\beta| + |\gamma| + |\delta|$. First choose a constant $\tilde{C} > 0$ and $b \in \mathbb{N}$ such that for ever $\tilde{\alpha}, \tilde{\beta} \in \mathbb{N}^n$ and every $\tilde{\gamma}, \tilde{\delta} \in \mathbb{N}^d$ with $|\tilde{\alpha}| + |\tilde{\beta}| + |\tilde{\gamma}| + |\tilde{\delta}| \leq A$ we have

$$\begin{aligned} \left\| \frac{\partial^{|\tilde{\alpha}|+|\tilde{\beta}|+|\tilde{\gamma}|+|\tilde{\delta}|}}{\partial \eta^{\tilde{\alpha}} \bar{\eta}^{\tilde{\beta}} s^{\tilde{\gamma}} t^{\tilde{\delta}}} h_+(\eta, \bar{\eta}, s, -t) \right\| &\leq \frac{\tilde{C}}{\|t\|^b}, \\ \left\| \frac{\partial^{|\tilde{\alpha}|+|\tilde{\beta}|+|\tilde{\gamma}|+|\tilde{\delta}|}}{\partial \eta^{\tilde{\alpha}} \bar{\eta}^{\tilde{\beta}} s^{\tilde{\gamma}} t^{\tilde{\delta}}} g_+(\eta, \bar{\eta}, s, -t) \right\| &\leq \frac{\tilde{C}}{\|t\|^b}, \quad (\eta, s, t) \in H_-. \end{aligned} \tag{3.12}$$

By (3.8) we can choose a $K > 0$ such that for $j = 1, \dots, d$ and $\ell = 1, \dots, N'$,

$$\begin{aligned} &\left\| \frac{\partial^{|\tilde{\alpha}|+|\tilde{\beta}|+|\tilde{\gamma}|+|\tilde{\delta}|+|\tilde{\varepsilon}|+|\tilde{\nu}|}}{\partial \eta^{\tilde{\alpha}} \bar{\eta}^{\tilde{\beta}} s^{\tilde{\gamma}} t^{\tilde{\delta}} \zeta^{\tilde{\varepsilon}} \bar{\zeta}^{\tilde{\nu}} \Lambda^{\tilde{\mu}}} \Phi_{\bar{\sigma}_j}(\eta, \bar{\eta}, \sigma, \bar{\sigma}, \zeta, \bar{\zeta}, \Lambda) \right\| \\ &\qquad \leq K \left(\|t\| + \left\| \Phi(\eta, \bar{\eta}, \sigma, \bar{\sigma}, \zeta, \bar{\zeta}, \Lambda) - \bar{\zeta} \right\| \right)^{\frac{a+Ab}{\beta_0}}, \\ &\left\| \frac{\partial^{|\tilde{\alpha}|+|\tilde{\beta}|+|\tilde{\gamma}|+|\tilde{\delta}|+|\tilde{\varepsilon}|+|\tilde{\nu}|}}{\partial \eta^{\tilde{\alpha}} \bar{\eta}^{\tilde{\beta}} s^{\tilde{\gamma}} t^{\tilde{\delta}} \zeta^{\tilde{\varepsilon}} \bar{\zeta}^{\tilde{\nu}} \Lambda^{\tilde{\mu}}} \Phi_{\bar{\zeta}_\ell}(\eta, \bar{\eta}, \sigma, \bar{\sigma}, \zeta, \bar{\zeta}, \Lambda) \right\| \\ &\qquad \leq K \left(\|t\| + \left\| \Phi(\eta, \bar{\eta}, \sigma, \bar{\sigma}, \zeta, \bar{\zeta}, \Lambda) - \bar{\zeta} \right\| \right)^{\frac{a+Ab}{\beta_0}}, \end{aligned}$$

holds on $\hat{U}_1 \times \hat{U}_2 \times \hat{U}_3 \times \hat{U}_4$ for all $\tilde{\alpha}, \tilde{\beta} \in \mathbb{N}^n$, $\tilde{\gamma}, \tilde{\delta} \in \mathbb{N}^d$, $\tilde{\varepsilon}, \tilde{\nu} \in \mathbb{N}^{N'}$, and $\tilde{\mu} \in \mathbb{N}^k$ such that

$$|\tilde{\alpha}| + |\tilde{\beta}| + |\tilde{\gamma}| + |\tilde{\delta}| + |\tilde{\varepsilon}| + |\tilde{\nu}| \leq A.$$

We thus see that for $(\eta, s, t) \in \tilde{H}_-$ and $\ell = 1, \dots, N'$

$$\begin{aligned} &\left\| \frac{\partial^{|\tilde{\alpha}|+|\tilde{\beta}|+|\tilde{\gamma}|+|\tilde{\delta}|+|\tilde{\varepsilon}|+|\tilde{\nu}|}}{\partial \eta^{\tilde{\alpha}} \bar{\eta}^{\tilde{\beta}} s^{\tilde{\gamma}} t^{\tilde{\delta}} \zeta^{\tilde{\varepsilon}} \bar{\zeta}^{\tilde{\nu}} \Lambda^{\tilde{\mu}}} \Phi_{\bar{\zeta}_\ell} \left(\eta, \bar{\eta}, s, t, \overline{h_+(\eta, \bar{\eta}, s, -t)}, h_+(\eta, \bar{\eta}, s, -t), \overline{g_+(\eta, \bar{\eta}, s, -t)} \right) \right\| \\ &\leq K \left(\|t\| + \left\| \Phi(\eta, \bar{\eta}, \sigma, \bar{\sigma}, \overline{h_+(\eta, \bar{\eta}, s, -t)}, h_+(\eta, \bar{\eta}, s, -t), \overline{g_+(\eta, \bar{\eta}, s, -t)}) \right. \right. \\ &\qquad \left. \left. - h_+(\eta, \bar{\eta}, s, -t) \right\| \right)^{\frac{a+Ab}{\beta_0}} \\ &= K \left(\|t\| + \|h_-(\eta, \bar{\eta}, s, t) - h_+(\eta, \bar{\eta}, s, -t)\| \right)^{\frac{a+Ab}{\beta_0}} \\ &\leq K \left(\|t\| + C \|t\|^{\beta_0} \right)^{\frac{a+Ab}{\beta_0}} \\ &\leq \tilde{K} \|t\|^{a+Ab}, \end{aligned} \tag{3.13}$$

and with the same argument, for $j = 1, \dots, d$ and $(\eta, s, t) \in \tilde{H}_-$

$$\begin{aligned} &\left\| \frac{\partial^{|\tilde{\alpha}|+|\tilde{\beta}|+|\tilde{\gamma}|+|\tilde{\delta}|+|\tilde{\varepsilon}|+|\tilde{\nu}|}}{\partial \eta^{\tilde{\alpha}} \bar{\eta}^{\tilde{\beta}} s^{\tilde{\gamma}} t^{\tilde{\delta}} \zeta^{\tilde{\varepsilon}} \bar{\zeta}^{\tilde{\nu}} \Lambda^{\tilde{\mu}}} \Phi_{\bar{\sigma}_j} \left(\eta, \bar{\eta}, s, t, \overline{h_+(\eta, \bar{\eta}, s, -t)}, h_+(\eta, \bar{\eta}, s, -t), \overline{g_+(\eta, \bar{\eta}, s, -t)} \right) \right\| \\ &\qquad \leq \tilde{K} \|t\|^{a+Ab}, \end{aligned}$$

for the same range of $\tilde\alpha, \tilde\beta \in \mathbb{N}^n$, $\tilde\gamma, \tilde\delta \in \mathbb{N}^d$, $\tilde\varepsilon, \tilde\nu \in \mathbb{N}^{N'}$, and $\tilde\mu \in \mathbb{N}^k$ as above.

If we now consider the term

$$\frac{\partial^{|\alpha|+|\beta|+|\gamma|+|\delta|}}{\partial\eta^\alpha\bar\eta^\beta s^\gamma t^\delta}\Phi_{\bar\zeta_\ell}\left(\eta,\bar\eta,s,t,\overline{h_+(\eta,\bar\eta,s,-t)},h_+(\eta,\bar\eta,s,-t),\overline{g_+(\eta,\bar\eta,s,-t)}\right),$$

then by the chain rule, we can write it as a sum of $M \in \mathbb{N}$ terms (where M is a combinatorial constant involving the multiindices $\alpha,\beta,\gamma,\delta$) each of which is a product of a derivative of the form

$$\frac{\partial^{|\tilde\alpha|+|\tilde\beta|+|\tilde\gamma|+|\tilde\delta|+|\tilde\varepsilon|+|\tilde\nu|}}{\partial\eta^{\tilde\alpha}\bar\eta^{\tilde\beta} s^{\tilde\gamma} t^{\tilde\delta}\zeta^{\tilde\varepsilon}\bar\zeta^{\tilde\nu}\Lambda^{\tilde\mu}}\Phi_{\bar\zeta}\left(\eta,\bar\eta,s,t,\overline{h_+(\eta,\bar\eta,s,-t)},h_+(\eta,\bar\eta,s,-t),\overline{g_+(\eta,\bar\eta,s,-t)}\right)$$

with at most A factors of derivatives of the form

$$\frac{\partial^{|\tilde\alpha|+|\tilde\beta|+|\tilde\gamma|+|\tilde\delta|}}{\partial\eta^{\tilde\alpha}\bar\eta^{\tilde\beta} s^{\tilde\gamma} t^{\tilde\delta}}h_+(\eta,\bar\eta,s,-t), \quad \frac{\partial^{|\tilde\alpha|+|\tilde\beta|+|\tilde\gamma|+|\tilde\delta|}}{\partial\eta^{\tilde\alpha}\bar\eta^{\tilde\beta} s^{\tilde\gamma} t^{\tilde\delta}}\overline{h_+(\eta,\bar\eta,s,-t)},$$
$$\frac{\partial^{|\tilde\alpha|+|\tilde\beta|+|\tilde\gamma|+|\tilde\delta|}}{\partial\eta^{\tilde\alpha}\bar\eta^{\tilde\beta} s^{\tilde\gamma} t^{\tilde\delta}}g_+(\eta,\bar\eta,s,-t).$$

Using this observation together with (3.12) and (3.13) we see that for $\ell = 1,\dots,N'$,

$$\begin{aligned}&\left\|\frac{\partial^{|\alpha|+|\beta|+|\gamma|+|\delta|}}{\partial\eta^\alpha\bar\eta^\beta s^\gamma t^\delta}\Phi_{\bar\zeta_\ell}\left(\eta,\bar\eta,s,t,\overline{h_+(\eta,\bar\eta,s,-t)},h_+(\eta,\bar\eta,s,-t),\overline{g_+(\eta,\bar\eta,s,-t)}\right)\right\|\\ &\le M\tilde K\left\|t\right\|^{a+Ab}\left(\frac{\tilde C}{\left\|t\right\|^b}\right)^A\\ &\le M\tilde K\tilde C^A\left\|t\right\|^a,\end{aligned} \tag{3.14}$$

and thus, (3.11) holds for $\Phi_{\bar\zeta}$. As the same argument applies to $\Phi_{\bar\sigma_j}$, $j = 1,\dots,d$, we get that h_- lies in $\mathfrak{A}_\infty(\tilde H_-)$ as claimed. The final conclusion follows by applying Theorem 3.3. □

4. Numerical invariants for a CR map and associated open subsets decomposition

4.1. Admissible rings of functions, numerical invariants and some basic properties

Here we introduce a new sequence of invariants attached to a CR map that relates to its smoothness properties. If X is a real manifold, $x_0 \in X$ and $\ell \in \mathbb{Z}_+ \cup \{\infty\}$, we denote by $\mathscr{C}^\ell(X, x_0)$ the ring of germs of $\mathscr{C}^\ell$-smooth functions at x_0 and by $\mathscr{C}^\ell(X)$ the ring of $\mathscr{C}^\ell$-functions over X.

In this section we assume that $M \subset \mathbb{C}^N$ is a $\mathscr{C}^\infty$-smooth generic submanifold of CR dimension n, and $h\colon M \to \mathbb{C}_w^{N'}$ is a continuous CR map. We denote by $\mathscr{C}^k_{CR}(M,p)$,

$\ell \in \mathbb{Z}_+ \cup \{\infty\}$, the ring of germs of $\mathscr{C}^k$-smooth CR functions at a point $p \in M$. For a given $\psi \in \mathscr{C}^1(M,p)$, we denote by $\bar{L}\psi = (\bar{L}_1\psi, \dots, \bar{L}_n\psi)$ where $\bar{L}_1, \dots, \bar{L}_n$ is a given choice of a basis of $\mathscr{C}^\infty$-smooth CR vector fields near p. The reader should note that, wherever we use this notation in what follows, the conditions involved will not depend on the choice of the basis of CR vector fields.

It will be convenient to introduce the following:

Definition 4.1. Let M and h be as above, $\mu \in \mathbb{Z}_+$, $p \in M$, and j be an integer satisfying $0 \le j \le \mu$.

a) We denote by $\mathscr{A}_p^{j,\mu}$ the set of all pairs (g, R) with $g = (g_1, \dots, g_k) \in (\mathscr{C}_{CR}^{\mu-j}(M,p))^k$ for some integer k and $R(z, \bar{z}, w, \bar{w}, \Lambda) \in \mathscr{C}^\infty(M \times \mathbb{C}^{N'} \times \mathbb{C}^k, (p, h(p), \overline{g(p)}))$, which have the property that R is holomorphic in Λ and which satisfy

$$R\left(\xi, \bar{\xi}, h(\xi,\bar{\xi}), \overline{h(\xi,\bar{\xi})}, \overline{g(\xi,\bar{\xi})}\right) = 0$$

for $\xi \in M$ near p.

b) If h is assumed to be $\mathscr{C}^{\mu-j}$-smooth, we denote by $\mathscr{F}_p^{j,\mu}$ the subring of $\mathscr{C}^{\mu-j}(M,p)$ consisting of those functions ψ that may written in the form

$$\psi(\xi,\bar{\xi}) = R\left(\xi, \bar{\xi}, h(\xi,\bar{\xi}), \overline{h(\xi,\bar{\xi})}, \overline{g(\xi,\bar{\xi})}\right)$$

for $\xi \in M$ near p where $g = (g_1, \dots, g_k) \in (\mathscr{C}_{CR}^{\mu-j}(M,p))^k$ for some integer k, and $R(z, \bar{z}, w, \bar{w}, \Lambda) \in \mathscr{C}^\infty(M \times \mathbb{C}^{N'} \times \mathbb{C}^k, (p, h(p), \overline{g(p)}))$ is holomorphic in Λ.

c) For $(g, R) \in \mathscr{A}_p^{j,\mu}$, we define

$$\begin{aligned} R_w :&= R_w\left(\xi, \bar{\xi}, h(\xi,\bar{\xi}), \overline{h(\xi,\bar{\xi})}, \overline{g(\xi,\bar{\xi})}\right) \\ &= \left(R_{w_1}\left(\xi, \bar{\xi}, h(\xi,\bar{\xi}), \overline{h(\xi,\bar{\xi})}, \overline{g(\xi,\bar{\xi})}\right), \dots, R_{w_{N'}}\left(\xi, \bar{\xi}, h(\xi,\bar{\xi}), \overline{h(\xi,\bar{\xi})}, \overline{g(\xi,\bar{\xi})}\right)\right), \end{aligned}$$

for $\xi \in M$ near p.

Remark 4.2. Note that if $\psi \in \mathscr{F}_p^{j,\mu}$ then there is a neighborhood of p in M such that for any z in that neighborhood, (the germ at z of) $\psi \in \mathscr{F}_z^{j,\mu}$.

We note that for any $p \in M$, the space

$$\mathscr{D}_j^\mu(M,p) = \left\{R_w\left(p, \bar{p}, h(p,\bar{p}), \overline{h(p,\bar{p})}, \overline{g(p,\bar{p})}\right) : (g,R) \in \mathscr{A}_p^{j,\mu}\right\} \subset \mathbb{C}^{N'}$$

is a vector space. We define, for $p \in M$ and any integer $0 \le j \le \mu$:

$$\mathscr{S}_j^\mu(M,p) := \dim_{\mathbb{C}} \mathscr{D}_j^\mu(M,p) \tag{4.1}$$

For every $p \in M$ and each $\mu \in \mathbb{Z}_+$, we have

$$\mathscr{D}_0^\mu(M,p) \subset \mathscr{D}_1^\mu(M,p) \subset \cdots \subset \mathscr{D}_\mu^\mu(M,p),$$

and hence

$$\mathscr{S}_0^\mu(M,p) \leq \mathscr{S}_1^\mu(M,p) \leq \ldots \leq \mathscr{S}_\mu^\mu(M,p).$$

Remark 4.3. We note that even though $\mathscr{S}_j^\mu(M,p)$ was defined using specific coordinates in $\mathbb{C}^{N'}$, it is not hard to see that $\mathscr{S}_j^\mu(M,p)$ is actually independent of the specific choice of (local) holomorphic coordinates in $\mathbb{C}^{N'}$ near $h(p)$. The same is true for the numbers $r_j(p)$ defined by (2.2).

We do need to be careful as the sequence $\mathscr{S}_j^\mu(M,p)$ might be strictly increasing up to a certain j, then stabilize, and then can start to strictly increase again. Stabilization, however, is crucial for what follows.

For $p \in M$, we set

$$\begin{aligned}\mathbb{V}_p^{j,\mu} &= \left(\mathscr{D}_j^\mu(M,p)\right)^\perp \\ &= \left\{V \in \mathbb{C}^{N'} : V \cdot R_w(p,\bar p, h(p,\bar p), \overline{h(p,\bar p)}, \overline{g(p,\bar p)}) = 0,\ \forall (g,R) \in \mathscr{A}_p^{j,\mu}\right\}.\end{aligned} \tag{4.2}$$

Since $\mathscr{D}_j^\mu(M,p)$ is increasing in j, we have that

$$\mathbb{V}_p^{\mu,\mu} \subset \mathbb{V}_p^{\mu-1,\mu} \subset \cdots \subset \mathbb{V}_p^{0,\mu} \text{ and } \dim \mathbb{V}_p^{j,\mu} = N' - \mathscr{S}_j^\mu(M,p).$$

In the following remark, we define the "holomorphic" derivatives of elements of $\mathscr{F}_p^{j,\mu}$.

Remark 4.4. Let $\mu \in \mathbb{Z}_+$, $p \in M$, and j be an integer satisfying $0 \leq j \leq \mu$.

(i) For $\psi \in \mathscr{F}_p^{j,\mu}$ and $V \in \mathbb{V}_p^{j,\mu}$, one can define $V \cdot \psi_w$ (at p) in a unique way. Indeed, if $\psi \in \mathscr{F}_p^{j,\mu}$ can be written in two different ways, using (g^1,R^1) and (g^2,R^2), so that

$$\psi(\xi,\bar\xi) = R^1\left(\xi,\bar\xi,h(\xi,\bar\xi),\overline{h(\xi,\bar\xi)},\overline{g^1(\xi,\bar\xi)}\right) = R^2\left(\xi,\bar\xi,h(\xi,\bar\xi),\overline{h(\xi,\bar\xi)},\overline{g^2(\xi,\bar\xi)}\right)$$

for $\xi \in M$ near p, where each $g^i \in (\mathscr{C}_{CR}^{\mu-j}(M,p))^{k_i}$ for some integer k_i, and $R^i \in \mathscr{C}^\infty(M \times \mathbb{C}^{N'} \times \mathbb{C}_{\Lambda_i}^{k_i}, (p,h(p),\overline{g^i(p)}))$ is holomorphic in its last argument, $i=1,2$, then we have for $g=(g^1,g^2)$ and R defined by

$$R\left(\xi,\bar\xi,w,\bar w,\Lambda_1,\Lambda_2\right) = R^1\left(\xi,\bar\xi,w,\bar w,\Lambda_1\right) - R^2\left(\xi,\bar\xi,w,\bar w,\Lambda_2\right)$$

that $(R,g) \in \mathscr{A}_p^{j,\mu}$. Then for every $V \in \mathbb{V}_p^{j,\mu}$, we have $V \cdot R_w = 0$ and so

$$V \cdot R^1_w = V \cdot R^2_w \quad (\text{at } p). \tag{4.3}$$

It follows that for every $V \in \mathbb{V}^{j,\mu}_p$, the natural definition

$$V \cdot \psi_w := V \cdot R^1_w \quad (\text{at } p) \tag{4.4}$$

is well defined, since (4.3) shows that the right hand side of (4.4) is independent of a particular choice of representative (g^i, R^i) for ψ.

(ii) For any polynomial $P(t,\bar t) = \sum_{\alpha,\beta} P^{\alpha,\beta} t^\alpha \bar t^\beta \in \mathscr{F}^{j,\mu}_p[t,\bar t]$, $t \in \mathbb{C}^r$, and any $V \in \mathbb{V}^{j,\mu}_p$, we define

$$V \cdot P_w(t,\bar t) := \sum_{\alpha,\beta} (V \cdot P^{\alpha,\beta}_w) t^\alpha \bar t^\beta,$$

which is well defined by (i).

Lemma 4.5. *Let $M \subset \mathbb{C}^N$ be a $\mathscr{C}^\infty$-smooth generic minimal submanifold, of CR dimension n, and $p \in M$. Let μ, j be integers satisfying $0 \le j < \mu$ and let $h\colon M \to \mathbb{C}^{N'}$ be a CR map of class $\mathscr{C}^{\mu-j}$. Let $\bar K$ be a $\mathscr{C}^\infty$-smooth CR vector field on M defined near p.*

(i) *Let $\psi \in \mathscr{F}^{j,\mu}_p$ and assume that both ψ and $\bar K$ are defined on a neighborhood U_p of p. Then $\bar K\psi \in \mathscr{F}^{j+1,\mu}_p$, and for every $z \in U_p$, (the germ at z) of $\bar K \psi$ belongs to $\mathscr{F}^{j+1,\mu}_z$. Furthermore, if $V\colon U_p \to \mathbb{C}^{N'}$ is a CR map of class $\mathscr{C}^1$ and satisfies $V(z) \in \mathbb{V}^{j+1,\mu}_z$ for $z \in U_p$, then $V \cdot (\bar K\psi)_w$ is defined all over U_p and one has*

$$V \cdot (\bar K \psi)_w = \bar K (V \cdot \psi_w), \quad \text{on } U_p.$$

(ii) *Let $(g, R) \in \mathscr{A}^{j,\mu}_p$. Then there exists $(\widehat g^{\bar K}, \widehat R^{\bar K}) \in \mathscr{A}^{j+1,\mu}_p$ such that $\bar K R_w = \widehat R^{\bar K}_w$.*

In applications of Lemma 4.5, the place of $\bar K$ will be taken up by entries of a local basis $\bar L_1, \dots, \bar L_n$ of CR vector fields on M near p. In order to simplify notation, we will in that case write $\widehat R^{\bar L_j} =: \widehat R^j$.

Proof. Let $\psi \in \mathscr{F}^{j,\mu}_p$. By definition there exist $g \in (\mathscr{C}^{\mu-j}_{CR}(M,p))^k$ for some integer k and $R \in \mathscr{C}^\infty(M \times \mathbb{C}^{N'} \times \mathbb{C}^k, (p, h(p), \overline{g(p)}))$, holomorphic in its last argument (denoted by Λ in what follows) such that

$$\psi(z,\bar z) = R\left(z, \bar z, h(z,\bar z), \overline{h(z,\bar z)}, \overline{g(z,\bar z)}\right), \quad z \in M \text{ near } p.$$

Hence for $z \in M$ near p,

$$\begin{aligned}(\bar K \psi)(z,\bar z) &= R_{\bar z}(z,\bar z, h(z,\bar z), \overline{h(z,\bar z)}, \overline{g(z,\bar z)}) \cdot \bar K(\bar z) \\ &\quad + R_{\bar w}(z,\bar z, h(z,\bar z), \overline{h(z,\bar z)}, \overline{g(z,\bar z)}) \cdot (\bar K \bar h)(z,\bar z) \\ &\quad + R_{\Lambda}(z,\bar z, h(z,\bar z), \overline{h(z,\bar z)}, \overline{g(z,\bar z)}) \cdot (\bar K \bar g)(z,\bar z).\end{aligned}$$

Since g and h are CR maps of class $\mathscr{C}^{\mu-j}$, their components all extend holomorphically to a (common) wedge of edge M at p by Tumanov's theorem [28] and the extensions are of class $\mathscr{C}^{\mu-j}$ up to the edge (on any strictly finer wedge), see e.g. [25,2]. Keeping the same notation for the maps g, h and for their extension on some appropriate finer wedge, we then may write

$$\bar{K}\bar{g} = \sum_{j=1}^{N} a_j(z,\bar{z}) \overline{\frac{\partial g}{\partial z_j}}(z,\bar{z}), \quad \bar{K}\bar{h} = \sum_{j=1}^{N} a_j(z,\bar{z}) \overline{\frac{\partial h}{\partial z_j}}(z,\bar{z})$$

for $z \in M$ near p, where the a_j are $\mathscr{C}^\infty$ functions defined on U_p. Using the notation $\partial g = \left(\frac{\partial g}{\partial z_1}, \ldots, \frac{\partial g}{\partial z_N}\right)$ and similarly for h, we can therefore write

$$(\bar{K}\psi)(z,\bar{z}) = \widetilde{R}\left(z, \bar{z}, h(z,\bar{z}), \overline{h(z,\bar{z})}, \overline{g(z,\bar{z})}, \overline{(\partial h)(z,\bar{z})}, \overline{(\partial g)(z,\bar{z})}\right),$$

with $\widetilde{R} \in \mathscr{C}^\infty\left(M \times \mathbb{C}^{N'} \times \mathbb{C}^{k+NN'+kN}, (p, h(p), \overline{g(p)}, \overline{\partial h(p)}, \overline{\partial g(p)})\right)$, and holomorphic in its last three arguments. Hence $\bar{L}\psi \in \mathscr{F}_p^{j+1,\mu}$, and as observed in Remark 4.2, for $z \in U_p$, the germ at z of $\bar{L}\psi$ belongs to $\mathscr{F}_z^{j+1,\mu}$.

Next, suppose that we are given a neighborhood U_p of p in M, as in Lemma 4.5, and $U_p \ni z \mapsto V(z) \in \mathbb{V}_z^{j+1,\mu}$ of class $\mathscr{C}^1$ and CR. Then we have on U_p

$$V \cdot (\bar{K}\psi)_w = V \cdot \bar{K}(\psi_w) = \bar{K}(V \cdot \psi_w),$$

since V is CR. This completes the proof of part (i) of the lemma. Part (ii) can be proven as well by using the same type of arguments as in (i). The proof of the lemma is therefore complete. □

4.2. Open subset decomposition associated to the numerical invariants

For $k, \ell \in \mathbb{N}$, $\ell \le N'$ and $\nu \in \mathbb{N}$ with $k \le \nu \le N' - \ell + k - 1$, we define

$$\begin{aligned}\Omega_{k,\nu}^{\ell} = \big\{ z \in M : \mathscr{S}_j^{N'-\ell+k}(M,\xi) = \mathscr{S}_j^{N'-\ell+k}(M,z) \text{ for } \xi \text{ near } z,\ k \le j \le \nu+1, \text{ and} \\ \ell \le \mathscr{S}_k^{N'-\ell+k}(M,z) < \ldots < \mathscr{S}_\nu^{N'-\ell+k}(M,z) = \mathscr{S}_{\nu+1}^{N'-\ell+k}(M,z) \big\},\end{aligned} \tag{4.5}$$

$$\begin{aligned}\Omega_{k,N'-\ell+k}^{\ell} = \big\{ z \in M : \mathscr{S}_j^{N'-\ell+k}(M,\xi) = \mathscr{S}_j^{N'-\ell+k}(M,z) \text{ for } \xi \\ \text{near } z,\ k \le j \le N'-\ell+k, \text{ and} \\ \ell \le \mathscr{S}_k^{N'-\ell+k}(M,z) < \ldots < \mathscr{S}_{N'-\ell+k}^{N'-\ell+k}(M,z) = N' \big\}.\end{aligned} \tag{4.6}$$

We also define, for $k, \ell \in \mathbb{N}$, $k \le \nu \le N' - \ell + k$, and $\ell \le m \le N'$,

$$\widehat{\Omega}_{k,\nu}^{\ell,m} := \left\{ z \in \Omega_{k,\nu}^{\ell} : \mathscr{S}_\nu^{N'-\ell+k}(M,z) = m \right\}. \tag{4.7}$$

Note that by construction, for each $k, \ell \in \mathbb{N}$ with $\ell \leq N'$ and every ν with $k \leq \nu \leq N' - \ell + k$ we have

$$\bigcup_{m=\ell}^{N'} \widehat{\Omega}_{k,\nu}^{\ell,m} = \Omega_{k,\nu}^{\ell} \tag{4.8}$$

and that each $\widehat{\Omega}_{k,\nu}^{\ell,m}$ is open in $\Omega_{k,\nu}^{\ell}$ and also open in M. Let us finally note that the definition (4.6) implies that

$$\widehat{\Omega}_{k,N'-\ell+k}^{\ell,m} = \emptyset, \text{ for } m < N'. \tag{4.9}$$

5. Relating the smoothness of a CR map to the open subset decomposition

For M and h as in §4 we denote by M_h^{∞} the open subset of M consisting of those points $p \in M$ such that h is $\mathscr{C}^{\infty}$-smooth in a neighborhood of p. The relevance of the introduction of the open subsets $\widehat{\Omega}_{k,\nu}^{\ell,m}$ in §4.2 to the study of the smoothness properties of the map h and the CR geometry of $h(M)$ is explained by our next two results.

Proposition 5.1. *Let $M \subset \mathbb{C}^N$ be a $\mathscr{C}^{\infty}$-smooth generic minimal submanifold, and $h\colon M \to \mathbb{C}^{N'}$ be a CR map of class $\mathscr{C}^1$. Let $\ell, k \in \mathbb{N}$ with $\ell \leq N'$, $k \leq \nu \leq N' - \ell + k$, and let the sets $\Omega_{k,\nu}^{\ell}$ be defined as above. Then $\bigcup_{\nu=k}^{N'-\ell+k} \widehat{\Omega}_{k,\nu}^{\ell,N'} \subset M_h^{\infty}$.*

Proof. Let $z \in \bigcup_{\nu=k}^{N'-\ell+k} \widehat{\Omega}_{k,\nu}^{\ell,N'}$. Hence there is $k \leq \nu \leq N' - \ell + k$ such that $\mathscr{S}_{\nu}^{N'-\ell+k}(M,z) = N'$. Hence we can find $(g, R^1), \dots, (g, R^{N'}) \in \mathscr{A}_z^{\nu, N'-\ell+k}$ such that for $\xi \in M$ near z

$$R^j(\xi, \bar{\xi}, h(\xi,\bar{\xi}), \overline{h(\xi,\bar{\xi})}, \overline{g(\xi,\bar{\xi})}) = 0, \; j = 1, \dots, N',$$

and

$$\mathrm{Rk}\{R_w^j(z, \bar{z}, h(z,\bar{z}), \overline{h(z,\bar{z})}, \overline{g(z,\bar{z})}), 1 \leq j \leq N'\} = N'.$$

Since M is minimal, all components of h and g extend holomorphically to a common wedge of edge M at z by Tumanov's theorem [28]. Observing that h is of class $\mathscr{C}^1$ and g of class $\mathscr{C}^{N'-\ell+k-\nu}$ and hence at least continuous, we may apply Theorem 3.1 to conclude that h is $\mathscr{C}^{\infty}$-smooth in a neighborhood of z. The proof of Proposition 5.1 is complete. □

Proposition 5.2. *Let $M \subset \mathbb{C}^N$ be a $\mathscr{C}^{\infty}$-smooth generic minimal submanifold and $h\colon M \to \mathbb{C}^{N'}$ be a CR map of class $\mathscr{C}^1$. Let $k, \ell, m, \nu \in \mathbb{N}$ with $k \leq \nu \leq N' - \ell + k - 1$ and $\ell \leq m < N'$. If h is of class $\mathscr{C}^{N'-\ell+k-\nu}$ on $\widehat{\Omega}_{k,\nu}^{\ell,m}$, then for every $p \in \widehat{\Omega}_{k,\nu}^{\ell,m}$, there exists a neighborhood U_p of p in $\widehat{\Omega}_{k,\nu}^{\ell,m}$, and for every $\xi \in U_p$, an $(N'-m)$-dimensional* *formal holomorphic submanifold* Γ_ξ *through* $h(\xi)$ *that is tangent to* $h(M)$ *to infinite order at* $h(\xi)$. *Furthermore, the family of formal holomorphic submanifolds* $(\Gamma_\xi)_{\xi\in U_p}$ *can be parametrized in such a way that the dependence on* $\xi \in U_p$ *is CR and of class* $\mathscr{C}^{N'-\ell+k-\nu}$.

The proof of Proposition 5.2 is more involved than that of the previous proposition and is mainly inspired by some arguments originating from our previous work on convergence of formal maps [21].

Throughout the rest of §5, we fix $k, \ell, m, \nu \in \mathbb{N}$ with $k \le \nu \le N' - \ell + k - 1$ and $\ell \le m < N'$.

For $z \in \widehat{\Omega}^{\ell,m}_{k,\nu}$, we have by definition $\dim \mathbb{V}_z^{\nu,N'-\ell+k} = N' - m$. However, locally around any point $p \in \widehat{\Omega}^{\ell,m}_{k,\nu}$ we can actually give a basis of vectors spanning $\mathbb{V}_z^{\nu,N'-\ell+k}$ for z close to p which depend on z in a CR manner. The next proposition gives an exact statement.

Proposition 5.3. *Under the assumptions of Proposition 5.2, for every* $p \in \widehat{\Omega}^{\ell,m}_{k,\nu}$, *there exists a neighborhood* $W_p \subset \widehat{\Omega}^{\ell,m}_{k,\nu}$ *of* p *and CR maps* $V^j : W_p \to \mathbb{C}^{N'}$ *of class* $\mathscr{C}^{N'-\ell+k-\nu}$, $j = 1, \dots, N' - m$, *whose components belong to* $\mathscr{F}_p^{\nu,N'-\ell+k}$, *such that* $\{V^1(z), \dots, V^{N'-\ell}(z)\}$ *forms a basis of* $\mathbb{V}_z^{\nu,N'-\ell+k}$ *for every* $z \in W_p$.

For the proof of Proposition 5.3, we shall need the following lemma.

Lemma 5.4. *Let* $M \subset \mathbb{C}^N$ *be a* $\mathscr{C}^\infty$*-smooth generic submanifold of CR dimension* n, $p \in M$, *and* $\mathscr{R}_p$ *be a subring of* $\mathscr{C}^\tau(M,p)$, *for some* $\tau \in \mathbb{Z}_+$, *satisfying the following condition: for every* $\psi \in \mathscr{R}_p$, *if* $\psi(p) \neq 0$ *then* $1/\psi \in \mathscr{R}_p$. *Let* $N' \ge 1$, $1 \le \delta < N'$, *and* $A^1, \dots, A^\delta$ *be germs of* p *of* $\mathbb{C}^{N'}$*-valued mappings with components in* $\mathscr{R}_p$. *Assume that:*

(i) *The rank of the* $N' \times \delta$ *matrix* $A := (A^1, \dots, A^\delta)$ *is equal to* δ *at* p*;*
(ii) *For any smooth CR vector field* $\bar{L}$ *of* M *near* p, *the rank of the* $N' \times 2\delta$ *matrix* $(A, \bar{L}A)$ *is constantly equal to* δ *in a neighborhood of* p.

Then there exist $N' - \delta$ *germs at* p *of* $\mathbb{C}^{N'}$*-valued mappings, with components in* $\mathscr{R}_p \cap \mathscr{C}^\tau_{CR}(M,p)$, *denoted by* $V^1, \dots, V^{N'-\delta}$ *such that for* $1 \le j \le N' - \delta$ *and* $1 \le \gamma \le \delta$, *we have*

$$V^j \cdot A^\gamma := \sum_{i=1}^{N'} V_i^j A_i^\gamma = 0 \quad \text{in} \quad \mathscr{R}_p, \tag{5.1}$$

and such that $V_1(p,\bar{p}), \dots, V_{N'-\delta}(p,\bar{p})$ *are linearly independent.*

The proof of Lemma 5.4 can be obtained by elementary linear algebra by following e.g. the steps of [20, Lemma 4.5] and will therefore be left to the reader.

Proof of Proposition 5.3. Let $p \in \widehat{\Omega}^{\ell,m}_{k,\nu}$. We may choose $(g, R^1), \dots, (g, R^m) \in \mathscr{A}_p^{\nu,N'-\ell+k}$ such that

$$\mathrm{Rk}\{R^j_w(p,\bar p,h(p,\bar p),\overline{h(p,\bar p)},\overline{g(p,\bar p)}),1\le j\le m\}=m. \tag{5.2}$$

We shall apply Lemma 5.4 to the subring $\mathscr{R}_p:=\mathscr{F}_p^{\nu,N'-\ell+k}$ of $\mathscr{C}^{N'-\ell+k-\nu}(M,p)$ and to

$$A^j(z,\bar z):=R^j_w(z,\bar z,h(z,\bar z),\overline{h(z,\bar z)},\overline{g(z,\bar z)}),\quad 1\le j\le m.$$

One can check that for every $\psi\in\mathscr{F}_p^{\nu,N'-\ell+k}$ with $\psi(p)\neq 0$, $1/\psi\in\mathscr{F}_p^{\nu,N'-\ell+k}$. Furthermore, (5.2) shows that condition (i) in Lemma 5.4 is already satisfied.

In order to apply Lemma 5.4, we now check that condition (ii) is also satisfied. For this, choose a basis $\bar L_r$, $1\le r\le n$, of $\mathscr{C}^\infty$-smooth CR vector fields for M near p. Then by Lemma 4.5 (ii), for every $1\le j\le m$, $1\le r\le n$, there exists $(g^{j,r},\widehat R^{j,r})\in\mathscr{A}_p^{\nu+1,N'-\ell+k}$ such that $\bar L_r A^j=\widehat R^{j,r}_w$. Hence for all j,r as above, we have a collection (g,R^j) and $(g^{j,r},\widehat R^{j,r})$ all belonging to $\mathscr{A}_p^{\nu+1,N'-\ell+k}$. Since $p\in\widehat\Omega^{\ell,m}_{k,\nu}$, the rank of the family of vectors in $\mathbb{C}^{N'}$ given by $R^j_w,\widehat R^{j,r}_w$, for j,r as above is constant and equal to m in a neighborhood of p. Since this latter rank coincides with that of the family of vectors $A^j,\bar L_rA^j$, $1\le j\le m$, $1\le r\le n$, the claim is proved. To conclude, we now just have to apply Lemma 5.4, recall that $\dim\mathbb{V}_z^{\nu,N'-\ell+k}=N'-m$ for all $z\in M$ near p and note that for z in some sufficiently small neighborhood of p in M, we have

$$\mathbb{V}_z^{\nu,N'-\ell+k}=\left\{V\in\mathbb{C}^{N'}:V\cdot R^j_w(z,\bar z,h(z,\bar z),\overline{h(z,\bar z)},\overline{g(z,\bar z)})=0,\ j=1,\dots,\ell\right\}.$$

The proof of Proposition 5.3 is complete now. □

In order to prove Proposition 5.2, we shall now follow and adapt the approach developed in [21]. We first make the following simple useful observation which follows from our previous construction.

Lemma 5.5. *Under the assumptions of Proposition 5.2, for every $p\in\widehat\Omega^{\ell,m}_{k,\nu}$, let $(V^1,\dots,V^{N'-m})$ and W_p be the basis and the neighborhood constructed in the proof of Proposition 5.3. Then for every $z\in W_p$, we have $\mathbb{V}_z^{\nu,N'-\ell+k}=\mathbb{V}_z^{\nu+1,N'-\ell+k}$. Furthermore, for every $\xi\in W_p$, for $j=1,\dots,N'-m$, and for every $(g,R)\in\mathscr{A}_\xi^{\nu+1,N'-\ell+k}$ defined on a neighborhood $U_\xi\subset W_p$ of ξ, we have*

$$V^j(z)\cdot R_w(z,\bar z,h(z,\bar z),\overline{h(z,\bar z)},\overline{g(z,\bar z)})=0,\quad z\in U_\xi.$$

We can now state and prove the last step towards the completion of the proof of Proposition 5.2. This next result can be thought of as a ($\mathscr{C}^\infty$–)smooth version of [21, Theorem 4.1].

Proposition 5.6. *Let $M\subset\mathbb{C}^N$ be a $\mathscr{C}^\infty$-smooth generic minimal submanifold and $h\colon M\to\mathbb{C}^{N'}$ be a CR map of class $\mathscr{C}^1$. Let $k,\ell,m,\nu\in\mathbb{N}$ with $k\le\nu\le N'-\ell+k-1$ and $\ell\le m<N'$. Assume that h is of class $\mathscr{C}^{N'-\ell+k-\nu}$ on $\widehat\Omega^{\ell,m}_{k,\nu}$ and for every $p\in\widehat\Omega^{\ell,m}_{k,\nu}$,*

let $\mathscr{V} = (V^1, \dots, V^{N'-m})$ *and* W_p *be given by Proposition 5.3. For* $t = (t_1, \dots, t_{N'-m}) \in \mathbb{C}^{N'-m}$*, we set* $t \cdot \mathscr{V} := \sum_{i=1}^{N'-m} t_i V^i$*. For every* $d \in \mathbb{Z}_+$*, define a family of homogeneous polynomial maps of degree* d *in* $(\mathscr{F}_p^{\nu,N'-\ell+k}[t])^{N'}$ *inductively by setting*

$$D^1(t) := t \cdot \mathscr{V}, \qquad D^{d+1}(t) := \frac{1}{d+1}(t \cdot \mathscr{V}) \cdot D^d_w(t), \quad d \geq 1. \tag{5.3}$$

In addition set $D(t) := \sum_{d=1}^{\infty} D^d(t) \in (\mathscr{F}_p^{\nu,N'-\ell+k}[\![t]\!])^{N'}$ *and write* $D(t) = \sum_{\alpha \in \mathbb{N}^{N'-m}} d_\alpha t^\alpha$*. Then, shrinking* W_p *if necessary, the following holds:*

(a) *for each* $\alpha \in \mathbb{N}^{N'-m}$*,* d_α *is well defined on* W_p *and is of class* $\mathscr{C}^{N'-\ell+k-\nu}$ *and CR on* W_p*.*
(b) *for every* $\xi \in W_p$*,* $t \mapsto D(\xi; t) := h(\xi) + \sum_{\alpha \in \mathbb{N}^{N'-m}} d_\alpha(\xi) t^\alpha$ *defines an* $(N'-m)$*-dimensional formal holomorphic submanifold through* $h(\xi)$*, denoted by* Γ_ξ*.*
(c) *for every* $\xi \in W_p$*,* Γ_ξ *is tangent to* $h(M)$ *to infinite order at* $h(\xi)$*.*

Proof of Proposition 5.6. a) The fact that all the d_α's, for $\alpha \in \mathbb{N}^{N'-m}$ are well defined and of class $\mathscr{C}^{N'-\ell+k-\nu}$ on W_p follows from the fact that the V^i's belong to $\mathscr{F}_p^{\nu,N'-\ell+k}$, are well defined on W_p, and from the construction given in (5.3). It remains to check that the d_α's are CR over W_p. Choose a basis of $\mathscr{C}^\infty$-smooth CR vector fields $\bar{L}_s$, $s = 1, \dots, n$, for M defined all over W_p. We show by induction on d that $\bar{L}_s(D^d(t)) = 0$, $s = 1, \dots, n$, where we consider $D^d(t)$ as a polynomial map with coefficients in $\mathscr{C}^1(W_p)$.

For $d = 1$, in view of (5.3) and Proposition 5.3, $D^1(t)$ is polynomial map with coefficients that are CR over W_p. Assume now that $D^d(t)$ has all its coefficients CR over W_p. This means that for $s = 1, \dots, n$, $\bar{L}_s(D^d(t)) = 0$ over W_p. By Lemma 4.5 (i), $\bar{L}_s(D^d(t))$ is a homogeneous polynomial in t with coefficients in $\mathscr{F}_p^{\nu+1,N'-\ell+k}$ and defined all over W_p. Furthermore, since $\mathbb{V}_z^{\nu,N'-\ell+k} = \mathbb{V}_z^{\nu+1,N'-\ell+k}$ for $z \in W_p$ (see Lemma 5.5), we have, for every $t \in \mathbb{C}^{N'-m}$, a CR map of class $\mathscr{C}^1$ given by $W_p \ni z \mapsto t \cdot \mathscr{V}(z) \in \mathbb{V}_z^{\nu+1,N'-\ell+k}$. Hence, using again Lemma 4.5 (i), we get

$$\bar{L}_s\left((d+1)D^{d+1}(t)\right) = \bar{L}_s((t \cdot \mathscr{V}) \cdot D^d_w(t)) = (t \cdot \mathscr{V}) \cdot (\bar{L}_s D^d(t))_w \quad \text{on } W_p.$$

Since $\bar{L}_s(D^d(t)) = 0$ over W_p, we have $(t \cdot \mathscr{V}) \cdot (\bar{L}_s D^d(t))_w = 0$ and hence $\bar{L}_s(D^{d+1}(t)) = 0$ for $s = 1, \dots, n$ which completes the proof of (a).

Regarding part (b), we use the fact that the vectors $V^1(\xi), \dots, V^{N'-m}(\xi)$ are of rank $N' - m$ at every $\xi \in W_p$, shrinking W_p if necessary, by Proposition 5.3. Hence

$$\frac{\partial D}{\partial t}(\xi, 0) = \begin{pmatrix} V_1^1(\xi) & \dots & V_1^{N'-m}(\xi) \\ \vdots & & \vdots \\ V_{N'}^1(\xi) & \dots & V_{N'}^{N'-m}(\xi) \end{pmatrix}$$

is of maximal rank $N' - m$ for $\xi \in W_p$.

We prove part (c) by showing that for every $\xi \in W_p$ and for every germ $\rho\colon (\mathbb{C}^{N'}, h(\xi)) \to \mathbb{R}$ of a $\mathscr{C}^\infty$-smooth function that vanishes on $h(M)$ near $h(\xi)$ the identity

$$\rho\left(h(\xi) + D(\xi;t), \overline{h(\xi) + D(\xi;t)}\right) \sim 0$$

holds in the ring of formal power series $\mathbb{R}[\![t,\bar{t}]\!]$. In the previous statement, we have identified ρ with its formal power series expansion at $h(\xi)$. From now on we fix $\xi \in W_p$ and ρ as above. We also assume that ρ is defined on some neighborhood X_ξ of $h(\xi)$ in $\mathbb{C}^{N'}$ and that V_ξ is an open neighborhood of ξ (in M) such that $h(V_\xi) \subset X_\xi$ and $V_\xi \subset W_p$.

We need the following lemma, analogous to [21, Lemma 4.2], whose proof will therefore be omitted.

Lemma 5.7. *Let $\xi \in W_p$, ρ, V_ξ and D be as above. For $z \in V_\xi$, write the formal power series expansion*

$$\rho\left(h(z) + D(z;t), \overline{h(z) + D(z;t)}\right) \sim \sum_{a,b\in\mathbb{Z}_+} \frac{1}{a!b!} R^{a,b}(z;t,\bar{t}) \in \mathbb{R}[\![t,\bar{t}]\!] \tag{5.4}$$

where each $R^{a,b}$ is homogeneous of degree a in t and of degree b in $\bar{t}$. Then for any $a,b \in \mathbb{Z}_+$, there exists a universal polynomial $\mathscr{U}_{a,b}$ in all its arguments such that

$$R^{a,b}(z;t,\bar{t}) = \mathscr{U}_{a,b}\left[\left(\rho_{w^\beta\bar{w}^\delta}(h(z),\overline{h(z)})\right)_{\substack{|\beta|\le a\\|\delta|\le b}}, (s!\,D^s(z;t))_{s\le a}, (r!\,\overline{D^r(z;t)})_{r\le b}\right]. \tag{5.5}$$

Furthermore, for $a,b \in \mathbb{Z}_+$, writing $\mathscr{U}_{a,b} = \mathscr{U}_{a,b}((\Lambda_{\beta,\delta})_{\substack{|\beta|\le a\\|\delta|\le b}}, S_1,\ldots,S_a,T_1,\ldots,T_b)$, $\Lambda_{\beta,\delta}\in\mathbb{C}$, $S_i,T_j \in \mathbb{C}^{N'}$, and $R^{a+1,b}$ for $R^{a+1,b}(z;t,\bar{t})$, we have

$$\begin{aligned} R^{a+1,b} = &\sum_{i=1}^{a}(i+1)!\,\frac{\partial\mathscr{U}_{a,b}}{\partial S_i}\left[\left(\rho_{w^\beta\bar{w}^\delta}(h(z),\overline{h(z)})\right)_{\substack{|\beta|\le a\\|\delta|\le b}}, (s!\,D^s(z;t))_{s\le a}, (r!\,\overline{D^r(z;t)})_{r\le b}\right] \\ &\cdot D^{i+1}(z;t) \\ &+ \sum_{\substack{|\gamma|\le a\\|\mu|\le b}} \frac{\partial\mathscr{U}_{a,b}}{\partial\Lambda_{\gamma,\mu}}\left[\left(\rho_{w^\beta\bar{w}^\delta}(h(z),\overline{h(z)})\right)_{\substack{|\beta|\le a\\|\delta|\le b}}, (s!\,D^s(z;t))_{s\le a}, (r!\,\overline{D^r(z;t)})_{r\le b}\right] \\ &\times D^1(z;t)\cdot\left(\rho_{w^\gamma\bar{w}^\mu}(h(z),\overline{h(z)})\right)_w \end{aligned} \tag{5.6}$$

In view of Lemma 5.7, we may now complete the proof of Proposition 5.6 (c) by showing that for $\xi \in W_p$, ρ as above, and for every $z \in V_\xi$, $R^{a,b}(z;t,\bar{t}) = 0$ for every $a,b \in \mathbb{Z}_+$ by induction on $e := b + a$ and hence in particular at $z = \xi$. First observe that $R^{0,0}(z;t,\bar{t}) = \rho(h(z),\overline{h(z)})$ and hence is identically zero for $z \in V_\xi$. Let $e \in \mathbb{Z}_+$ and suppose that $R^{a,b}(z;t,\bar{t}) = 0$ for $z \in V_\xi$ and $a+b \le e$. We are going to show that $R^{a+1,b}(z;t,\bar{t}) = R^{a,b+1}(z;t,\bar{t}) = 0$ for $z \in V_\xi$ and $a+b \le e$. By Lemma 5.7 we have for $a+b \le e$ and $z \in V_\xi$

$$R^{a,b}(z;t,\bar{t}) = \mathscr{U}_{a,b}\left[\left(\rho_{w^\beta \bar{w}^\nu}(h(z),\overline{h(z)})\right)_{\substack{|\beta|\le a,\\ |\nu|\le b}}, (s!\, D^s(z;t))_{\ell\le a}, (r!\, \overline{D^r(z;t)})_{r\le b}\right] = 0. \tag{5.7}$$

Since for every integer d, $D^d(z;t)$ is polynomial in t with coefficients that are at the same time CR and belong to $\mathscr{F}_\xi^{\nu,N'-\ell+k}$ (*cf.* Proposition 5.6 (a) proved above), we may see (5.7) as a polynomial identity in $(t,\bar{t})$, with coefficients in $\mathscr{F}_\xi^{\nu,N'-\ell+k}$. Hence it follows from Lemma 5.5 that

$$D^1(z;t)\cdot R^{a,b}_w(z;t,\bar{t}) = (t\cdot \mathscr{V}(z))\cdot R^{a,b}_w(z;t,\bar{t}) = 0, \quad z\in V_\xi. \tag{5.8}$$

But in view of (5.7), we have that for $z \in V_\xi$ the left-hand side $\mathscr{L}$ of (5.8) satisfies

$$\begin{aligned}
\mathscr{L} = {} & \sum_{i=1}^{a} i!\, \frac{\partial \mathscr{U}_{a,b}}{\partial S_i}\left[\left(\rho_{w^\beta \bar{w}^\delta}(h(z),\overline{h(z)})\right)_{\substack{|\beta|\le a,\\ |\delta|\le b}}, (s!\, D^s(z;t))_{s\le a}, (r!\, \overline{D^r(z;t)})_{r\le b}\right] \\
& \cdot D^1(z;t)\cdot D^i_w(z;t) \\
& + \sum_{\substack{|\gamma|\le a\\ |\mu|\le b}} \frac{\partial \mathscr{U}_{a,b}}{\partial \Lambda_{\gamma,\mu}}\left[\left(\rho_{w^\beta \bar{w}^\delta}(h(z),\overline{h(z)})\right)_{\substack{|\beta|\le a,\\ |\delta|\le b}}, (s!\, D^s(z;t))_{s\le a}, (r!\, \overline{D^r(z;t)})_{r\le b}\right] D^1(z;t) \\
& \cdot \left(\rho_{w^\gamma \bar{w}^\mu}(h(z),\overline{h(z)})\right)_w \\
= {} & \sum_{i=1}^{a} (i+1)!\, \frac{\partial \mathscr{U}_{a,b}}{\partial S_i}\left[\left(\rho_{w^\beta \bar{w}^\delta}(h(z),\overline{h(z)})\right)_{\substack{|\beta|\le a,\\ |\delta|\le b}}, (s!\, D^s(z;t))_{s\le a}, (r!\, \overline{D^r(z;t)})_{r\le b}\right] \\
& \cdot D^{i+1}(z;t) \\
& + \sum_{\substack{|\gamma|\le a\\ |\mu|\le b}} \frac{\partial \mathscr{U}_{a,b}}{\partial \Lambda_{\gamma,\mu}}\left[\left(\rho_{w^\beta \bar{w}^\delta}(h(z),\overline{h(z)})\right)_{\substack{|\beta|\le a,\\ |\delta|\le b}}, (s!\, D^s(z;t))_{s\le a}, (r!\, \overline{D^r(z;t)})_{r\le b}\right] D^1(z;t) \\
& \cdot \left(\rho_{w^\gamma \bar{w}^\mu}(h(z),\overline{h(z)})\right)_w .
\end{aligned} \tag{5.9}$$

In the last equality, we have used the definition given in (5.3). Now in view of Lemma 5.7, the last quantity we found for $\mathscr{L}$ in (5.9) happens to coincide with $R^{j+1,k}(z;t,\bar{t})$. Hence $R^{j+1,k}(z;t,\bar{t}) = 0$ for $z \in V_\xi$ and $j+k \le e$. Furthermore, since ρ is real-valued, we have $\overline{R^{k+1,j}(z;t,\bar{t})} = R^{j,k+1}(z;,t,\bar{t})$. Hence the induction step is complete, which finishes the proof of Proposition 5.6 (c). □

6. Density and elementary rank properties

Let us now again consider a $\mathscr{C}^\infty$-smooth CR submanifold $M \subset \mathbb{C}^N$ and a fixed subset $M' \subset \mathbb{C}^{N'}$. Recall that we defined for a CR map $h\colon M \to \mathbb{C}^{N'}$ of class at least $\mathscr{C}^m$ and every $0 \le k \le m$ the following quantities in (2.1)–(2.2):

$$r_k(p) := \dim_{\mathbb{C}} \operatorname{span}\Big\{ \bar L_1 \dots \bar L_j \rho_w(h(p), \overline{h(p)}) : \rho \in \mathscr{I}_{h(M)}(h(p)), \\ \bar L_1, \dots, \bar L_j \in \Gamma_p(M), 0 \le j \le k \Big\},$$
$$r_k := \max\{e \in \mathbb{Z}_+ : r_k(p) \ge e \text{ for } p \text{ on some dense subset of } M\}.$$

If $h(M) \subset M'$, we may also define:

$$r_{k,M'}(p) := \dim_{\mathbb{C}} \operatorname{span}\Big\{ \bar L_1 \dots \bar L_j \rho_w(h(p), \overline{h(p)}) : \rho \in \mathscr{I}_{M'}(h(p)), \\ \bar L_1, \dots, \bar L_j \in \Gamma_p(M), 0 \le j \le k \Big\},$$
$$r_{k,M'} := \max\{e \in \mathbb{Z}_+ : r_{k,M'}(p) \ge e \text{ for } p \text{ on some dense subset of } M\}.$$

In what follows, we will use the following obvious fact: when $h(M) \subset M'$, for every $p \in M$, $r_k(p) \ge r_{k,M'}(p)$ and hence $r_k \ge r_{k,M'}$.

The goal of this section is to discuss some elementary bounds on these integers r_k when one puts various geometric properties on the pair $(M, h(M))$. The first bound involves r_0.

Lemma 6.1. *Let $M \subset \mathbb{C}^N$ be a $\mathscr{C}^\infty$-smooth CR submanifold and $h\colon M \to \mathbb{C}^{N'}$ be a continuous CR map. If there exists a $\mathscr{C}^\infty$-smooth CR submanifold $M' \subset \mathbb{C}^{N'}$ such that $h(M) \subset M'$ then $r_0 \ge N' - n'$ where $n' = \dim_{CR} M'$. In particular, if M' is maximally real, then $r_0 = N'$.*

Proof. Pick $p \in M$. Then by [2, Theorem 1.8.1], there exist holomorphic coordinates $(\chi, (\zeta, \tau)) \in \mathbb{C}^{n'} \times \mathbb{C}^{N'-n'-d'} \times \mathbb{C}^{d'}$ near $h(p)$, vanishing at $h(p)$, such that M' is given by the zero set of $\mathscr{C}^\infty$-smooth functions of the form:

$$\zeta = \theta(\chi, \tau, \bar\chi, \bar\tau), \ \operatorname{Im} \tau = \Phi(\chi, \bar\chi, \operatorname{Re} \tau). \tag{6.1}$$

Here θ and Φ are defined and $\mathscr{C}^\infty$-smooth near the origin in $\mathbb{C}^{n'+d'}$ and $\mathbb{C}^{n'} \times \mathbb{R}^{d'}$, $\theta(0) = 0$, $\Phi(0) = d\Phi(0) = 0$ and θ being CR on the generic submanifold $\hat M' = \{(\chi, \tau) : \operatorname{Im} \tau = \Phi(\chi, \bar\chi, \operatorname{Re} \tau)\} \subset \mathbb{C}^{n'+d'}$. In the same vein as what mentioned in Remark 4.3, the integers $r_{j,M'}$, $j \in \mathbb{N}$, are independent of the choice of holomorphic coordinates in $\mathbb{C}^{N'}$. We therefore use $w = (\chi, \zeta, \tau)$ as coordinates near $h(p)$ and the smooth defining functions (6.1) $\rho = (\rho_1, \dots, \rho_{N'-n'})$ to see that ρ_w has rank at least $N' - n'$. Since $\rho_j \in \mathscr{I}_{M'}(h(p))$, $j = 1, \dots, N' - n'$, we have $r_{0,M'}(p) \ge N' - n'$ for every $p \in M$ and hence $r_0 \ge N' - n'$. $\square$

The next result provides a bound for r_1.

Lemma 6.2. *Let $M \subset \mathbb{C}^N$ be a $\mathscr{C}^\infty$-smooth CR submanifold and $h\colon M \to \mathbb{C}^{N'}$ be a CR map of class $\mathscr{C}^1$. Suppose that there exists a $\mathscr{C}^\infty$-smooth Levi-nondegenerate submanifold $M' \subset \mathbb{C}^{N'}$ such that $h(M) \subset M'$. If h is immersive (on a dense open subset of M), then $r_1 \geq N' - n' + n$ where $n = \dim_{CR} M$, $n' = \dim_{CR} M'$.*

The content of this Lemma is a well-known fact that can be found in other variants in the existing literature (see e.g. [22]). We give a self-contained proof of the statement we need here.

Proof. Pick a point $p \in M$ where h is immersive. We will use a defining function of M' as in the proof of Lemma 6.1, where we can (because of the assumption of Levi-nondegeneracy) achieve the additional property that if we write $\Phi = (\Phi^1, \dots, \Phi^{d'})$, then the matrices

$$\Phi^1_{\chi,\bar\chi}, \dots, \Phi^{d'}_{\chi,\bar\chi},$$

have no common kernel when evaluated at 0, and also assuming that $\Phi_{\chi s}$ vanishes at 0. Denote the components of h in the (χ, ζ, τ)-variables by $h = (f, F, g)$. When we compute $r_{1,M'}$, we have in particular amongst the $\rho_w(0)$ with $\rho \in \mathscr{I}_{M'}(0)$ the vectors

$$\left(\Phi^j_\chi(0), 0, 0, \dots, \left(\frac{1}{2i} - \frac{1}{2}\Phi^j_{\operatorname{Re}\tau}(0)\right), \dots, 0\right), \quad (0, 0, \dots, 1, \dots, 0, \dots, 0)\,,\ j = 1, \dots, d'.$$

Since the last $N' - n'$ slots in these give rise to linearly independent vectors in $\mathbb{C}^{N'-n'}$ as already noted in the computation for (i), we just need to consider the $\bar L \Phi^j_\chi(f, \bar f, \operatorname{Re} g)$ for all CR vector fields $\bar L$ on M. Choose a basis $\bar L_1, \dots, \bar L_n$ of the CR vector fields on M near p. Since h is immersive, $\bar L_1 \bar f, \dots, \bar L_n \bar f$, is of rank n at p. We claim that the vectors

$$\bar L_j \Phi^k_\chi(f, \bar f, \operatorname{Re} g) = \Phi^k_{\chi,\bar\chi} \bar L_j \bar f + \frac{1}{2}\Phi^k_{\chi,s}\, \bar L_j \bar g, \quad j = 1, \dots, n, \quad k = 1, \dots, d'$$

have rank at least n when evaluated at p. Since we have normalized Φ so that $\Phi_{\chi,s}(f, \bar f, \operatorname{Re} g)|_p = 0$, it is enough to check that the

$$\Phi^k_{\chi,\bar\chi}\, \bar L_j \bar f, \quad j = 1, \dots, n, \quad k = 1, \dots, d'$$

are of rank at least n at p. We decompose $\chi = (\chi^1, \chi^2) \in \mathbb{C}^n \times \mathbb{C}^{n'-n}$ and correspondingly $f = (f^1, f^2) \in \mathbb{C}^n \times \mathbb{C}^{n'-n}$, and write the matrix

$$\bar L \bar f := \begin{pmatrix} \bar L_1 \bar f^1 & \bar L_2 \bar f^1 & \dots & \bar L_n \bar f^1 \\ \bar L_1 \bar f^2 & \bar L_2 \bar f^2 & \dots & \bar L_n \bar f^2 \end{pmatrix} = \begin{pmatrix} \bar L \bar f^1 \\ \bar L \bar f^2 \end{pmatrix}.$$

After a complex linear change of coordinates in the χ, we may assume that $\bar{L}\bar{f}^2(p,\bar{p}) = 0$, and $\bar{L}\bar{f}^1 = \mathrm{I}_{\mathrm{n}\times\mathrm{n}}$ is the identity matrix. When we now consider

$$\Phi^k_{\chi,\bar{\chi}} \bar{L}\bar{f} = \left(\Phi^k_{\chi,\bar{\chi}^1} \bar{L}\bar{f}^1 + \Phi^k_{\chi,\bar{\chi}^2} \bar{L}\bar{f}^2 \right)$$

and evaluate at p, we obtain

$$\Phi^k_{\chi,\bar{\chi}}(f(p,\bar{p}), \overline{f(p,\bar{p})}, \mathrm{Re}\; g(p,\bar{p}))\bar{L}\bar{f}(p,\bar{p}) = \Phi^k_{\chi,\bar{\chi}^1}(f(p,\bar{p}), \overline{f(p,\bar{p})}, \mathrm{Re}\; g(p,\bar{p})).$$

We note that the vectors

$$U_1 = \begin{pmatrix} \Phi^1_{\chi_1,\bar{\chi}_1} \\ \vdots \\ \Phi^1_{\chi_{n'},\bar{\chi}_1} \\ \Phi^2_{\chi_1,\bar{\chi}_1} \\ \vdots \\ \Phi^2_{\chi_{n'},\bar{\chi}_1} \\ \vdots \\ \vdots \\ \Phi^{d'}_{\chi_1,\bar{\chi}_1} \\ \vdots \\ \Phi^{d'}_{\chi_{n'},\bar{\chi}_1} \end{pmatrix}, \quad U_2 = \begin{pmatrix} \Phi^1_{\chi_1,\bar{\chi}_2} \\ \vdots \\ \Phi^1_{\chi_{n'},\bar{\chi}_2} \\ \Phi^2_{\chi_1,\bar{\chi}_2} \\ \vdots \\ \Phi^2_{\chi_{n'},\bar{\chi}_2} \\ \vdots \\ \vdots \\ \Phi^{d'}_{\chi_1,\bar{\chi}_2} \\ \vdots \\ \Phi^{d'}_{\chi_{n'},\bar{\chi}_2} \end{pmatrix}, \; \ldots \; U_n = \begin{pmatrix} \Phi^1_{\chi_1,\bar{\chi}_n} \\ \vdots \\ \Phi^1_{\chi_{n'},\bar{\chi}_n} \\ \Phi^2_{\chi_1,\bar{\chi}_n} \\ \vdots \\ \Phi^2_{\chi_{n'},\bar{\chi}_n} \\ \vdots \\ \vdots \\ \Phi^{d'}_{\chi_1,\bar{\chi}_n} \\ \vdots \\ \Phi^{d'}_{\chi_{n'},\bar{\chi}_n} \end{pmatrix}, \ldots, U_{n'} = \begin{pmatrix} \Phi^1_{\chi_1,\bar{\chi}_{n'}} \\ \vdots \\ \Phi^1_{\chi_{n'},\bar{\chi}_{n'}} \\ \Phi^2_{\chi_1,\bar{\chi}_{n'}} \\ \vdots \\ \Phi^2_{\chi_{n'},\bar{\chi}_{n'}} \\ \vdots \\ \vdots \\ \Phi^{d'}_{\chi_1,\bar{\chi}_{n'}} \\ \vdots \\ \Phi^{d'}_{\chi_{n'},\bar{\chi}_{n'}} \end{pmatrix}.$$

are not only linearly independent in $\mathbb{C}^{n'd'}$: If we consider the space $D = \{w = (w^1, \ldots, w^{d'}) \in \mathbb{C}^{n'd'} \colon w^j \in \mathbb{C}^{n'}, w^1 = \cdots = w^{d'}\}$ as a subspace of $\mathbb{C}^{n'd'}$, then $\{w \in D : w \cdot U_1 = \cdots = w \cdot U_{n'} = 0\} = \{0\}$ since the matrices $\Phi^j_{\chi,\bar{\chi}}$ are hermitian and were assumed to have no common kernel (by Levi-nondegeneracy of M'). Therefore, for $\ell \le n'$, $\dim_{\mathbb{C}}\{w \in D \colon w \cdot U_1 = \ldots = w \cdot U_\ell = 0\} \le n' - \ell$, and those vectors w's which annihilate $U_1, \ldots, U_n$ belong to an at most $n' - n$-dimensional subspace of D. It follows that the rank of the $\Phi^j_{\chi,\bar{\chi}_\ell}$ for $j = 1, \ldots, d'$ and $\ell = 1, \ldots, n$ is at least n at p. This proves that $r_{1,M'}(p) \ge N' - n' + n$ and hence that $r_1 \ge N' - n' + n$ as desired. □

For the statement of the next lemma, we need to define the following quantities for $k \in \mathbb{Z}_+$:

$$\begin{aligned} r^M_k(p) &:= \dim_{\mathbb{C}} \mathrm{span} \left\{ \bar{L}_1 \ldots \bar{L}_j \rho_z(p,\bar{p}) : \rho \in \mathscr{I}_M(p),\ \bar{L}_1, \ldots, \bar{L}_j \in \Gamma_p(M), 0 \le j \le k \right\}, \\ r^M_k &:= \max \left\{ e \in \mathbb{Z}_+ : r^M_k(p) \ge e \text{ for } p \text{ on some dense subset of } M \right\}. \end{aligned} \tag{6.2}$$

Lemma 6.3. *Let $M \subset \mathbb{C}^N$ be a $\mathscr{C}^\infty$-smooth finitely nondegenerate CR submanifold of CR codimension d and $h \colon M \to \mathbb{C}^{N'}$ be a CR map of class $\mathscr{C}^1$. Suppose that there* *exists a smooth CR submanifold $M' \subset \mathbb{C}^{N'}$ with $h(M) \subset M'$, with $\dim_{CR} M' = n'$. If h is of class $\mathscr{C}^{k+1}$ for some $k \in \mathbb{Z}_+$ and strictly noncharacteristic at p, then $r_k(p) \geq N' - n' - d + r_k^M(p) \geq r_k^M(p)$. In particular, if M is at most ℓ-finitely nondegenerate for some $\ell \leq k_0$ on an open dense subset of M and if h is of class $\mathscr{C}^{k_0+1}$ and strictly noncharacteristic (on some open dense subset of M), then $r_{k_0} \geq N$.*

Proof. We may replace M without loss of generality by a generic submanifold, so that we assume that $M \subset \mathbb{C}^N$, where $N = n + d$ and $n = \dim_{CR} M$.

Pick a point $p \in M$ where h is strictly noncharacteristic. As in the proof of Lemma 6.1, we may choose coordinates (χ, ζ, τ) such that M' near $h(p)$ is locally defined by (6.1), and as in the proof of Lemma 6.2, we write $h = (f, F, g)$. Consider the generic manifold $\hat{M}' \subset \mathbb{C}^{n'}_\chi \times \mathbb{C}^{d'}_\tau$ defined by $\operatorname{Im} \tau = \Phi(\chi, \bar{\chi}, \operatorname{Re} \tau)$; it is locally CR-diffeomorphic to M'. We write $\hat{h} = (f, g)$ and obtain a smooth map $\hat{h}: M \to \hat{M}'$ defined in a neighborhood of p. Denoting, for $j \leq k+1$, $\hat{r}_{j,\hat{M}'}(p)$ the integers associated to the map $\hat{h}$, one easily checks that

$$r_{j,M'}(p) \geq N' - n' - d' + \hat{r}_{j,\hat{M}'}(p). \tag{6.3}$$

Note that since h is strictly noncharacteristic, and M' and $\hat{M}'$ are CR diffeomorphic, $\hat{h}$ is also strictly noncharacteristic.

This means that the pullbacks $\hat{h}^* \theta^\nu$ of the characteristic forms

$$\theta^\nu = \partial(\operatorname{Im} \tau_\nu - \Phi^\nu(\chi, \bar{\chi}, \operatorname{Re} \tau))|_{\hat{M}'}, \quad \nu = 1, \dots, d'$$

span $T^0 M$ (near p). After possibly reordering, we can assume that $\hat{h}^*\theta^1, \dots, \hat{h}^*\theta^d$ span.

We are next going to consider the generic submanifold $\tilde{M}' \subset \mathbb{C}^{n'} \times \mathbb{C}^{d'-d} \times \mathbb{C}^d$ defined by

$$\rho^\nu(\chi, \bar{\chi}, \tau, \bar{\tau}) = \operatorname{Im} \tau_\nu - \Phi^\nu(\chi, \bar{\chi}, \operatorname{Re} \tau) = 0, \quad \nu = 1, \dots, d.$$

Of course, $\hat{h}$ can also be considered as a map into the (larger) manifold $\tilde{M}' \subset \mathbb{C}^{n'+d'}$. Hence we see that $\hat{r}_{j,\hat{M}'}(p) \geq \hat{r}_{j,\tilde{M}'}(p) + d' - d$, for $0 \leq j \leq k+1$; taken together with (6.3), we see that

$$r_{j,M'}(p) \geq N' - n' - d + \hat{r}_{j,\tilde{M}'}(p), \quad j \leq k+1. \tag{6.4}$$

By construction, $\hat{h}$, viewed as a map from M into $\tilde{M}'$, is also strictly noncharacteristic.

We are now going to check that $\hat{r}_{k,\tilde{M}'}(p) \geq r_k^M(p)$ thereby finishing the proof of the Lemma. We first extend each of the components $\hat{h}_j$ of $\hat{h}$ (which are CR functions of class $\mathscr{C}^{k+1}$ by assumption) to $\mathscr{C}^{k+1}$-functions on $\mathbb{C}^N$ such that each of the derivatives $\frac{\partial \hat{h}_j}{\partial \bar{z}_\ell}$, for $1 \leq \ell \leq N$, vanish to order k on M near p. The equations

$$\tilde{\rho}^1(z, \bar{z}) = \rho^1\left(\hat{h}(z), \overline{\hat{h}(z)}\right) = 0, \dots, \tilde{\rho}^d(z, \bar{z}) = \rho^d\left(\hat{h}(z), \overline{\hat{h}(z)}\right) = 0$$

are then defining equations for M of class $\mathscr{C}^{k+1}$ (near p) since $\hat{h}$ is strictly noncharacteristic at p. We have that

$$\tilde{\rho}^j_z(z,\bar{z}) = \rho^j_w\left(\hat{h}(z),\overline{\hat{h}(z)}\right)\frac{\partial \hat{h}}{\partial z}(z,\bar{z}) + O(k+1),$$

where the $O(k+1)$-term vanishes to order (at least) k on M. Therefore, an application of at most k CR vector fields $\bar{L}_1,\dots,\bar{L}_a$, for some $a \le k$, on M gives an expression of the form

$$\begin{aligned}
&\bar{L}_1\dots\bar{L}_a\tilde{\rho}^j_z(z,\bar{z})\\
&= \left(\bar{L}_1\dots\bar{L}_a\rho^j_w\left(\hat{h}(z),\overline{\hat{h}(z)}\right)\right)\frac{\partial \hat{h}}{\partial z}(z,\bar{z}) + \rho^j_w\left(\hat{h}(z),\overline{\hat{h}(z)}\right)\mu(z,\bar{z})\\
&+\sum_{\gamma=1}^{a-1}\sum_{1\le i_1<\dots<i_\gamma\le a}\bar{L}_{i_1}\dots\bar{L}_{i_\gamma}\rho^j_w\left(\hat{h}(z),\overline{\hat{h}(z)}\right)\lambda_{i_1\dots i_\gamma}(z,\bar{z}) + O(k+1-a).
\end{aligned}$$

Taking all these equations together (for all possible choices of $\bar{L}_1,\dots,\bar{L}_a$ and $a \le k$), we infer that, as claimed, $\hat{r}_{k,\tilde{M}'}(p) \ge r^M_k(p)$. Summing up everything we have proved so far, we get the desired result. □

We conclude this section by the following useful and elementary property of the open subsets constructed in §4.

Proposition 6.4. *Let $M \subset \mathbb{C}^N$ be a $\mathscr{C}^\infty$-smooth generic submanifold, $h\colon M \to \mathbb{C}^{N'}$ a continuous CR map, and fix $\ell, k \in \mathbb{N}$ such that $0 \le \ell \le N'$. If the open subset $M^k_\ell := \{z \in M : \mathscr{S}^{N'-\ell+k}_k(M,z) \ge \ell\}$ is dense in M, then the open subset of M given by $\bigcup_{\nu=k}^{N'-\ell+k}\Omega^\ell_{k,\nu}$ is dense in M, where the open subsets $\Omega^\ell_{k,\nu}$ are given by (4.5) and (4.6).*

Proof. Since by assumption M^k_ℓ is dense in M, we only need to prove that $\bigcup_{\nu=k}^{N'-\ell+k}(\Omega^\ell_{k,\nu}\cap M^k_\ell)$ is dense in M^k_ℓ. For every ν with $k \le \nu \le N'-\ell+k$, consider the open subset of M^k_ℓ given by

$$M_\nu := \{z \in M^k_\ell : \mathscr{S}^{N'-\ell+k}_\nu(M,\xi) = \mathscr{S}^{N'-\ell+k}_\nu(M,z) \text{ for } \xi \text{ near } z\}.$$

As each mapping $M^k_\ell \ni z \mapsto \mathscr{S}^{N'-\ell+k}_\nu(M,z)$ is integer valued and lower semi-continuous, each M_ν is dense in M^k_ℓ and hence so is their intersection $\bigcap_{\nu=k}^{N'-\ell+k} M_\nu$. We now observe that since for $z \in M^k_\ell$ $\ell \le \mathscr{S}^{N'-\ell+k}_\nu(M,z) \le N'$ for all ν with $k \le \nu \le N'-\ell+k$, we have that

$$\bigcap_{\nu=k}^{N'-\ell+k} M_\nu \subset \bigcup_{\nu=k}^{N'-\ell+k}\left(\Omega^\ell_{k,\nu}\cap M^k_\ell\right)$$

which proves the proposition. □

7. Proof of Theorems 2.2 and 1.1, Corollaries 1.3, 2.3, 2.4, 2.5, 2.6 and 2.9

7.1. Proof of Theorem 2.2

Since any smooth CR submanifold in $\mathbb{C}^N$ is locally smoothly CR diffeomorphic to a generic submanifold in a lower dimensional complex space, see e.g. [2], we may assume without loss of generality that M itself is generic in $\mathbb{C}^N$. We first note that by definition, if h is of class $\mathscr{C}^{N'-\ell+k}$, then for $\xi \in M$, we have $\mathscr{S}_k^{N'-\ell+k}(M,\xi) \geq r_k(\xi)$ since, by Lemma 4.5, all of the $\bar{L}_1 \cdots \bar{L}_j \varrho(h(z), \overline{h(z)}) \in \mathscr{A}_\xi^{j,N'-\ell+k}$ for $\bar{L}_1, \ldots, \bar{L}_j \in \Gamma_\xi(M)$ and $\varrho \in \mathscr{I}_{h(M)}(h(\xi))$, $0 \leq j \leq k$. Since we assume that $r_k \geq \ell$, we have that $\ell \leq r_k \leq \mathscr{S}_k^{N'-\ell+k}(M,\xi)$ for ξ on some dense open subset of M. Hence the set $M_\ell^k \subset M$ from Proposition 6.4 is actually dense, and we obtain from that Proposition and (4.8) that

$$\mathscr{O} := \bigcup_{\nu=k}^{N'-\ell+k} \bigcup_{m=\ell}^{N'} \widehat{\Omega}_{k,\nu}^{\ell,m} \subset M$$

is dense in M. If h is nowhere $\mathscr{C}^\infty$ on some nonempty subset M_1 of M, then by Proposition 5.1 and (4.9), we have that

$$M_2 = M_1 \cap \mathscr{O} = M_1 \cap \left(\bigcup_{\nu=k}^{N'-\ell+k} \bigcup_{m=\ell}^{N'-1} \widehat{\Omega}_{k,\nu}^{\ell,m} \right) = \bigcup_{\nu=k}^{N'-\ell+k-1} \bigcup_{m=\ell}^{N'-1} (\widehat{\Omega}_{k,\nu}^{\ell,m} \cap M_1) \qquad (7.1)$$

is dense in M_1 and the conclusion of Theorem 2.2 follows now immediately from Proposition 5.2.

7.2. Proof of Theorem 1.1

First note that since M is strongly pseudoconvex the integer r_1^M defined in (6.2) must be equal to $n+1$. Because both M and M' are generic of codimension one in their respective complex space, we can use Remark 2.8 to see that we may apply Lemma 6.3, which tells us that $r_1 \geq n+1$ (because h is at least of class $\mathscr{C}^2$). We can therefore apply Theorem 2.2 with $k=1$ and $\ell = n+1$ and get that there exists a dense open subset ω of Ω such that $h(\omega) \subset \mathscr{E}_{h(M)} \subset \mathscr{E}_{M'}$. The inclusion $h(\Omega) \subset \mathscr{E}_{M'}$ now follows since the set $\mathscr{E}_{M'}$ is a closed subset of M' (see [8,9]).

7.3. Proof of Corollary 1.3

Corollary 1.3 is a direct consequence of Theorem 1.2, since in such a situation, the set of strongly pseudoconvex points in M is open and dense in M and the mapping h is automatically CR transversal at every point of M (see [2, Proposition 9.10.5] whose proof applies in our setting as well).

7.4. Proof of Corollary 2.3

Corollary 2.3 is an immediate consequence of Theorem 2.2 with $k = 0$, $\ell = N' - n'$ and Lemma 6.1.

7.5. Proof of Corollary 2.4

Corollary 2.4 is an immediate consequence of Theorem 2.2 with $k = 1$, $\ell = N' - n' + n$ and Lemma 6.2.

7.6. Proof of Corollary 2.5

Corollary 2.5 is a consequence of Theorem 2.2, Lemma 6.3 and the following result, whose proof can be obtained by adapting the arguments of [22, Proposition 3.1].

Proposition 7.1. *Let $M \subset \mathbb{C}^{n+1}$ and $M' \subset \mathbb{C}^{n'+1}$ be (connected) $\mathscr{C}^\infty$-smooth real hypersurfaces with M strongly pseudoconvex and M' Levi-nondegenerate of signature ℓ', $n' > n \geq 1$. Assume that there exists a point $p \in M$ and a germ at p of CR transversal map $h\colon (M, p) \to M'$ of class $\mathscr{C}^2$ satisfying the following: there exists a neighborhood $V \subset M$ of p , and for every $\xi \in V$, a smooth complex curve Υ_ξ containing $h(\xi)$, depending in a $\mathscr{C}^1$ manner on $\xi \in V$, such that the order of contact of Υ_ξ with M' at $h(\xi)$ is greater or equal to 3. Then necessarily $n < n' - \ell' < n'$.*

7.7. Proof of Corollary 2.6

Corollary 2.6 is a consequence of Theorem 2.2, Lemma 6.1 and the following result.

Proposition 7.2. *Let $M \subset \mathbb{C}^N$ be a $\mathscr{C}^\infty$- smooth minimal CR submanifold and $M' \subset \mathbb{C}^{N'}$ the tube over the light cone given by (2.3). Assume that there exists a point $p \in M$ and a germ at p of a continuous CR map $h\colon (M, p) \to M'$ satisfying the following: there exists a neighborhood $V \subset M$ of p, and for every $\xi \in V$, a smooth complex curve Υ_ξ containing $h(\xi)$, depending on a continuous and CR fashion on $\xi \in V$, such that the order of contact of Υ_ξ with M' at $h(\xi)$ is greater or equal to 3. Then there exists a germ at p of a continuous CR function g and real constants α_j, η_j, $1 \leq j \leq N' - 1$ with $\sum_{j=1}^{N'-1} \alpha_j^2 = 1$, such that for ξ near p*

$$h(\xi) = (\alpha_1 g(\xi) + i\eta_1, \dots, \alpha_{N'-1} g(\xi) + i\eta_{N'-1}, g(\xi)).$$

The proof of Proposition 7.2 consists of following the steps of the proof of [21, Proposition 6.6] and [22, Lemma 2.3] and using the well-known fact that a continuous real-valued CR function on a smooth minimal CR submanifold of $\mathbb{C}^N$ is necessarily constant. We leave the details to the reader.

7.8. Proof of Corollary 2.9

We apply Lemma 6.3 in conjunction with Theorem 2.2 with $k = \sigma$ and $\ell = N$.

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