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Formal versus analytic CR mappings

Bernhard Lamel (Wien) and Nordine Mir (Doha)

Dedicated to the memory of Józef Siciak

Abstract. We discuss the convergence/divergence problem for formal holomorphic mappings sending real-analytic CR submanifolds into real-analytic sets both lying in complex Euclidean spaces of arbitrary dimension. In particular, we survey the recent developments on this problem for minimal as well as nonminimal source submanifolds. We conclude by describing the results known up to date on comparing the notions of formal, biholomorphic and CR equivalence.

1. Introduction. It is usually a difficult task to trace back the exact place (or time) of birth of a field of research in mathematics. However, as far as CR geometry is concerned, the opposite is the case: Its origin clearly goes back to Poincaré's pioneering 1907 paper [49], where he initiated the study of the moduli space of real submanifolds in complex Euclidean spaces under the action of biholomorphic transformations. Poincaré discovered the striking fact that real hypersurfaces in complex Euclidean spaces $\mathbb{C}^N$ of dimension $N \geq 2$ have nontrivial local invariants under biholomorphic mappings: to be more precise, he showed (and we will recall his argument in §2.4.2) that if S and S' are real-analytic hypersurfaces in $\mathbb{C}^2$, or more generally in $\mathbb{C}^N$, $N \geq 2$, and $p \in S$, $p' \in S'$, then there are (in fact countably many) obstructions to the existence of a germ of biholomorphism $H\colon (\mathbb{C}^N, p) \to (\mathbb{C}^N, p')$ satisfying $H(S) \subset S'$. This fact highlights two important aspects of several complex variables and CR geometry. On the one hand, it shows the difference between one-dimensional complex analysis and several complex variables, as real-analytic curves in the complex domain are all locally equivalent to a piece of the real line. On the other hand, it also emphasizes the difference between

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real submanifolds and complex submanifolds, the latter being all locally biholomorphically equivalent to complex linear subspaces.

Poincaré's discovery was the starting point of the systematic study of the (local) biholomorphic equivalence problem for real hypersurfaces in complex spaces, starting with É. Cartan's work on strictly pseudoconvex hypersurfaces [11, 10], and brought to the conclusion for Levi-nondegenerate hypersurfaces in $\mathbb{C}^N$, $N \geq 2$, by the work of Tanaka [54, 53] and Chern and Moser [12]. In their seminal paper [12] Chern–Moser solved the equivalence problem for real-analytic Levi-nondegenerate hypersurfaces in $\mathbb{C}^N$, $N \geq 2$, by two independent approaches. One of these, the normal form approach, consists of solving the problem at the level of formal power series and then deriving the convergence of the formal objects. Looking at formal obstructions and then establishing the necessary convergence results is a natural line of thought that has been applied in many other classification problems in mathematics such as e.g. the conjugacy problem for germs of planar biholomorphisms, or the normalization of singular holomorphic vector fields, etc., to name but a few. In the context of CR geometry, such an approach has led to the study of some remarkable properties of formal holomorphic mappings between real-analytic submanifolds in complex space.

We are going to give a more thorough discussion of guiding examples below in §3.1, but in order to whet the reader's appetite, let us indicate two basic situations already here. First, there are ample examples of divergent formal holomorphic maps sending the real hyperplane $\operatorname{Im} w = 0$ in $\mathbb{C}^2$ into itself, as any divergent formal power series $\varphi(z, w)$ gives rise to such a map $H_\varphi(z, w) = (z + \varphi(z, w), w)$. However, if one considers (invertible) formal holomorphic maps mapping the real hypersurface $\operatorname{Im} w = |z|^2$ into itself, one can show that any such map is a linear fractional map.

It turned out over the years that formal holomorphic transformations have an uncanny tendency to exhibit unexpected convergence properties, so that a systematic study of such properties in their own right was initiated in the 1990s. One of the facets of this theory is that for real-analytic manifolds without so-called CR singularities, that is, whose CR structure is determined by a bundle, there are very natural geometric obstructions to convergence of formal holomorphic maps (which actually explain the examples already given).

In this survey, we will encounter many of the stepping points of this story, in the case of real analytic CR manifolds. For real manifolds having CR singularities, the convergence problem for formal holomorphic transformations has its own story too. In the pioneering work of Moser–Webster [48], convergence of formal holomorphic invertible transformations for so-called elliptic two-dimensional Bishop surfaces was established and divergence was shown to hold in some hyperbolic cases. The convergence properties of the maps in such a setting appear to be intimately related to the dynamical properties of an associated pair of involutions, and hence present completely different features compared to the CR case. Since the main focus of the present paper is on the CR category, we refer the interested reader to e.g. [48, 28, 23, 24, 26, 29, 30], and in particular to the recent survey of Huang [27], for a complete account of the CR singular case.

Our discussion will not only tackle the convergence problem for formal biholomorphic mappings but also for general formal mappings whose source and target manifolds are allowed to lie in complex Euclidean spaces of arbitrary and possibly different dimensions. The path we have chosen is to describe the most recent convergence as well as divergence results, highlighting some ideas and tools as well as some connections with questions related to other fields such as algebraic and analytic geometry or singular differential equations. We do have the advantage of hindsight: over the years, very natural and neat geometric obstructions to the convergence of formal maps between real-analytic CR submanifolds have been identified, and a rather complete picture has emerged by now. We will also highlight some remaining open questions.

The paper is organized as follows. We first recall some basic facts about formal and convergent power series in §2. We then fully discuss the convergence/divergence problem for arbitrary formal maps in §3, mostly driven by our recent work [40]. §4 addresses recent divergence results for invertible formal maps in $\mathbb{C}^2$ by Kossovskiy and Shafikov. §5 provides some elements of the proof of some of the results highlighted in §3. In the last section, we tackle the closely related question of comparing different notions of local equivalence for real-analytic CR manifolds.

2. Formal and convergent power series

2.1. Basic definitions: Formal power series maps. In this part, we shall introduce elementary notions about formal power series starting with formal *holomorphic* power series, depending on an N-dimensional complex variable z, and then introduce in the picture those power series depending on $\operatorname{Re} z$ and $\operatorname{Im} z$ and not necessarily holomorphic, that one can view as power series in z and $\bar{z}$.

We define a *formal* (*holomorphic*) *power series* $f(z)$ of the complex variables $z = (z_1, \dots, z_N)$ at the point $p \in \mathbb{C}^N$ as an expression of the form

$$f(z) = \sum_{\alpha \in \mathbb{N}^N} f_\alpha (z - p)^\alpha, \qquad f_\alpha \in \mathbb{C}.$$

Even though one cannot evaluate such series at actual values $z \in \mathbb{C}^N$, we define $f(p) = f_0$. The collection of all formal power series centred at p is denoted by $\mathbb{C}[\![z-p]\!]$. For a nonzero power series $f \in \mathbb{C}[\![z-p]\!]$, its *order at* p is defined to be $\operatorname{ord}_p f = \min\{|\alpha| \colon f_\alpha \neq 0\}$. By convention we set $\operatorname{ord}_p 0 = \infty$. If we consider, for $\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbb{N}^N$, the monomial $(z-p)^\alpha = (z_1 - p_1)^{\alpha_1} \dots (z_N - p_N)^{\alpha_N}$ as a basis vector, we obtain a topological vector space structure on the space of all formal power series, with the topology defined by the metric

$$d(f,g) = 2^{-\operatorname{ord}(f-g)}.$$

With this topology, $\mathbb{C}[\![z-p]\!]$ becomes a Fréchet space. It becomes a Fréchet algebra when one defines the product of the basis monomials in the usual way as $(z-p)^\alpha (z-p)^\beta = (z-p)^{\alpha+\beta}$. This algebra is local, with maximal ideal defined by $\mathfrak{m} = \{f \in \mathbb{C}[\![z-p]\!] \colon \operatorname{ord}_p f \geq 1\}$. An explicit (and rather efficient) way to compute the multiplicative inverse of $f \in \mathbb{C}[\![z-p]\!]$ with $f(p) \neq 0$ is given by the geometric series

$$f(z)^{-1} = (f(p) - \tilde{f}(z))^{-1} = \frac{1}{f(p)} \sum_{j=0}^{\infty} \left(\frac{\tilde{f}(z)}{f(p)} \right)^j.$$

A *formal* (*holomorphic*) *power series map* $H \colon (\mathbb{C}^N, p) \to \mathbb{C}^{N'}$ is an N'-tuple $H = (H_1, \dots, H_{N'}) \in \mathbb{C}[\![z-p]\!]^{N'}$. If $H(p) = (H_1(p), \dots, H_{N'}(p)) = p' \in \mathbb{C}^{N'}$, we write $H \colon (\mathbb{C}^N, p) \to (\mathbb{C}^{N'}, p')$. The set of all such maps is in the obvious way a (Fréchet) module over $\mathbb{C}[\![z-p]\!]$, and we define $\operatorname{ord}_p H = \min_j \operatorname{ord}_p H_j$. Given formal maps $H \colon (\mathbb{C}^N, p) \to (\mathbb{C}^{N'}, p')$ and $G \colon (\mathbb{C}^{N'}, p') \to (\mathbb{C}^m, q)$, their composition $G \circ H \colon (\mathbb{C}^N, p) \to (\mathbb{C}^m, q)$ is well defined as the formal power series map

$$G \circ H(z) = \sum_{\beta \in \mathbb{N}^{N'}} G_\beta (H(z) - p')^\beta.$$

(Note that the right hand side is well defined as a power series because $H(p) = p'$.) In addition to $H(p)$, for a formal power series map as above we also define $H'(p)$ to be the linear map $H'(p) \colon \mathbb{C}^N \to \mathbb{C}^{N'}$ given by $H'(p)(u) = \sum_{|\alpha|=1} H_\alpha u^\alpha$. One has $(G \circ H)'(p) = G'(p) \circ H'(p)$.

In particular, we note that the composition of two formal power series maps $H \colon (\mathbb{C}^N, p) \to (\mathbb{C}^N, p)$ and $G \colon (\mathbb{C}^N, p) \to (\mathbb{C}^N, p)$ again gives rise to a formal map $H \circ G \colon (\mathbb{C}^N, p) \to (\mathbb{C}^N, p)$. The neutral element with respect to composition is $\operatorname{id}(z) = (p_1 + (z_1 - p_1), \dots, p_N + (z_N - p_N))$. Let us now assume for simplicity that $p = 0$. If H has an inverse map, i.e. if there exists a map G which satisfies $G \circ H = H \circ G = \operatorname{id}$, then necessarily $H'(0)$ is invertible. On the other hand, this condition is also sufficient, as the formal inverse mapping theorem says. A simple proof is as follows: If we want to solve $H \circ G = \operatorname{id}$ for G, after expanding into homogeneous terms G_k, we see that the G_k are inductively determined by $H'(0) G_k(z) = p_k(z, G_1, \dots, G_{k-1})$ for some polynomial map p_k depending only on H. An important consequence of the formal inverse function theorem is the formal implicit function theorem: If $A(z,w) \in (\mathbb{C}[\![z,w]\!])^k$ where $w = (w_1,\dots,w_k)$, $A(0,0) = 0$, and the $k \times k$ matrix $A_w(0,0)$ is invertible, then there exists a formal (holomorphic) map $\varphi\colon (\mathbb{C}^k,0) \to (\mathbb{C}^k,0)$ such that $A(z,\varphi(z)) = 0$. Indeed, the formal map $\widetilde{A}(z,w) = (z, A(z,w))$ has an invertible derivative at $(0,0)$, and so there exists a formal map $B(z,w) = (B_1(z,w), B_2(z,w))$ such that $A \circ B = \mathrm{id}$, i.e. $B_1(z,w) = z$ and $A(z, B_2(z,w)) = w$. Hence, $\varphi(z) := B_2(z,0)$ solves the implicit function problem.

We define the *rank* (or *generic rank*) $\operatorname{rk} H$ of a formal power series map $H(z) \in \mathbb{C}[\![z-p]\!]^{N'}$ as the rank of the Jacobian matrix

$$H'(z) = \begin{pmatrix} \frac{\partial H_1}{\partial z_1}(z) & \dots & \frac{\partial H_1}{\partial z_N}(z) \\ \vdots & & \vdots \\ \frac{\partial H_{N'}}{\partial z_1}(z) & \dots & \frac{\partial H_{N'}}{\partial z_N}(z) \end{pmatrix}$$

over the field of fractions of $\mathbb{C}[\![z-p]\!]$, or equivalently the largest number k such that a minor of size k of $H'(z)$ is nonvanishing (as a formal power series in $\mathbb{C}[\![z-p]\!]$).

More generally, we also define the rank of a matrix-valued formal power series map $A(z)$, denoted $\operatorname{rk} A$, as the rank of the matrix A over the field of fractions of $\mathbb{C}[\![z-p]\!]$, or equivalently as the largest number k such that a minor of size k of $A(z)$ is nonvanishing (as a formal power series in $\mathbb{C}[\![z-p]\!]$).

The *space of k-jets* of formal maps $(\mathbb{C}^N,p) \to (\mathbb{C}^{N'},p')$ is the quotient

$$J^k((\mathbb{C}^N,p),(\mathbb{C}^{N'},p')) = \mathbb{C}[\![z-p]\!]^{N'}/\mathfrak{m}^{k+1}\mathbb{C}[\![z-p]\!]^{N'}.$$

The natural map

$$j_p^k\colon \mathbb{C}[\![z-p]\!]^{N'} \to \mathbb{C}[\![z-p]\!]^{N'}/\mathfrak{m}^{k+1}\mathbb{C}[\![z-p]\!]^{N'}$$

is called the *k-jet map*. The k-jet of the formal map $H\colon (\mathbb{C}^N,p) \to (\mathbb{C}^{N'},p')$ can be identified with the polynomial map

$$j_p^k H(z) = \sum_{|\alpha| \le k} H_\alpha (z-p)^\alpha.$$

All of the usual algebraic operations descend to the quotient. The composition of k-jets is defined by the k-jet of the composition of any representatives: If $\Lambda_1 = j_p^k H \in J^k((\mathbb{C}^N,p),(\mathbb{C}^{N'},p'))$ and $\Lambda_2 = j_{p'}^k G \in J^k((\mathbb{C}^{N'},p'),(\mathbb{C}^m,q))$, then $j_p^k(G \circ H)$ is independent of the choice of representatives G and H for Λ_1 and Λ_2, respectively, and we define $\Lambda_2 \circ \Lambda_1 = j_p^k(G \circ H)$.

Many of the conditions that we will encounter deal with mappings of real objects embedded in complex spaces. Hence we will also be dealing with nonholomorphic or real-valued formal power series. We therefore recall the basic related notions and the interplay between holomorphic and nonholomorphic power series.

Denote by $\mathbb{C}[\![z-p,\overline{z-p}]\!]$ the ring of complex-valued formal power series whose underlying indeterminates are $\mathrm{Re}(z-p)$ and $\mathrm{Im}(z-p)$. This ring comes with a natural involution,

$$b\colon \mathbb{C}[\![z-p,\overline{z-p}]\!] \to \mathbb{C}[\![z-p,\overline{z-p}]\!], \qquad (b\varrho)(z,\bar z) = \bar\varrho(\bar z, z),$$

where for $\varrho(z,\bar z) = \sum_{\alpha,\beta} \varrho_{\alpha,\beta} z^\alpha \bar z^\beta$ we denote by

$$\bar\varrho(z,\bar z) = \sum_{\alpha,\beta} \bar\varrho_{\alpha,\beta} z^\alpha \bar z^\beta$$

the series with conjugate coefficients.

We say that an ideal $I \subset \mathbb{C}[\![z-p,\overline{z-p}]\!]$ is *real* if $bI \subset I$. If I is a real ideal, we call I a *real manifold ideal* or say that I defines a *formal real submanifold* $(M,p) \subset (\mathbb{C}^N,p)$ of real codimension d when I can be generated by d elements $\varrho^1,\dots,\varrho^d \in I$ whose differentials at p are independent.

The ring

$$\mathbb{C}[\![M]\!] = \mathbb{C}[\![z-p,\overline{z-p}]\!]/I$$

is called the *formal coordinate ring* of M; its field of fractions is denoted by $\mathbb{C}(\!(M)\!)$ and called the *formal function field* of M. The image of $\mathbb{C}[\![z-p]\!]$ in $\mathbb{C}[\![M]\!]$ is denoted by $\mathsf{CR}[\![M]\!]$, and its elements are called *formal CR functions* (on M); its field of fractions is denoted by $\mathsf{CR}(\!(M)\!)$ and its elements are called *formal CR meromorphic functions* (on M).

2.2. Basic definitions: Convergent power series maps. We say that a formal holomorphic power series map $H\colon (\mathbb{C}^N,p) \to (\mathbb{C}^{N'},q)$, given by

$$H(z) = \sum_{\alpha\in\mathbb{N}^N} H_\alpha (z-p)^\alpha, \qquad H_\alpha \in \mathbb{C}^{N'},$$

is *convergent* if for one (and hence any) fixed norm $\|\cdot\|$ on $\mathbb{C}^{N'}$, there exist constants $C,K>0$ such that $\|H_\alpha\| \le CK^{|\alpha|}$. The space of all convergent power series at p is denoted by $\mathbb{C}\{z-p\}$. If $f \in \mathbb{C}\{z-p\}$, then there exists a neighbourhood U of p such that the series $f(z)$ converges uniformly (and absolutely) on compact subsets of U. Formal power series which are not convergent are said to be *divergent*.

The space $\mathbb{C}\{z-p\}$ can be given an especially nice topological structure which exhibits it as a so-called DFS-space. One way to realize such a space is as an inductive limit of Fréchet spaces where the inclusion maps are compact. In our case, we first define for each $r>0$ the space

$$\mathbb{C}\{z-p\}_r = \left\{ f \in \mathbb{C}\{z-p\} \colon \limsup_{|\alpha|\to\infty} |f_\alpha| r^{|\alpha|} < \infty \right\},$$

and endow it with the seminorms $\varrho_s(f) = \sup\{s^{|\alpha|}|f_\alpha| \colon \alpha \in \mathbb{N}^N\}$ for $s < r$ in order to make it a Fréchet space. For $r < r'$, the inclusion map $\mathbb{C}\{z-p\}_{r'} \to \mathbb{C}\{z-p\}_r$ is compact. One can also obtain the same inductive limit topology as a limit of Banach spaces (for example, one can use the spaces $H^\infty(\Delta_r)$ of bounded holomorphic functions on the polydisk Δ_r).

All of the operations that we defined for formal power series in §2.1 become continuous when restricted to the space of convergent power series or the space of convergent power series maps. In particular, the space $(\mathbb{C}\{z-p\})^N$ of convergent power series is a topological algebra, and the open subset

$$\mathscr{B} = \{H \in (\mathbb{C}\{z-p\})^N \colon H(p) = p, \, \det H'(p) \neq 0\}$$

is a topological Lie group.

The topologies of these spaces have a number of useful properties. They are *sequential* (i.e. one can test for example continuity along sequences); and a linear map between spaces of this type is continuous if and only if it maps steps of the inductive limit boundedly into steps. To be precise, if $E = \varinjlim E_j$, where the E_j are Banach spaces, and $E_j \hookrightarrow E_{j+1}$ is compact, then one has the following facts:

(1) A sequence x_j converges to x in E if and only if there exists a k such that $\{x_j \colon j \in \mathbb{N}\} \cup \{x\} \subset E_k$ and $x_j \to x$ (as $j \to \infty$) in E_k (i.e. E is *sequentially retractive*).
(2) $B \subset E$ is bounded if and only if there exists a k such that $B \subset E_k$ and B is bounded in E_k.
(3) Every bounded set $B \subset E$ is relatively compact in E (i.e. E is a *Montel space*).
(4) $f \colon E \to Y$, where Y is some topological space, is continuous if and only if $f|_{E_k}$ is continuous for all k.

This means that even though the topology of germ spaces might look a bit complicated at first, in applications, they are flexible tools also from the functional-analytic point of view. We will, however, not use this viewpoint in this survey, and refer the reader for a rather complete discussion of this topic in the context of spaces of power series to the Ph.D. thesis of Woblistin [58].

2.3. Mapping properties of formal maps. We next define what it means for a formal map to map a subset of $\mathbb{C}^N$ into a subset of $\mathbb{C}^{N'}$.

Definition 2.1. Let $X \subset \mathbb{C}^N$, $X' \subset \mathbb{C}^{N'}$ be subsets, $p \in X$ and $H \colon (\mathbb{C}^N, p) \to \mathbb{C}^{N'}$ a formal holomorphic map. We say that H maps X into X' and write $H(X) \subset X'$ if for every integer k there exists a real-analytic map $H_k(z, \bar{z}) \in \mathbb{C}\{z-p, \overline{z-p}\}$ with $j_p^k H_k = j_p^k H$ which satisfies $H_k(U_k \cap X) \subset X'$ for some neighbourhood U_k of p in $\mathbb{C}^N$.

If X and X' are not just arbitrary subsets, but actually real-analytic, then there is at least one other reasonable definition. For this, we will identify a germ (X,p) of a real-analytic set $X\subset\mathbb{C}^N$ with its ideal $I\subset\mathbb{C}\{z-p,\overline{z-p}\}$, consisting of all germs (at p) of real-analytic functions ϱ which satisfy $\varrho|_X\equiv 0$ (near p). We write $\hat I=\mathbb{C}[\![z-p,\overline{z-p}]\!]I$ for the ideal generated by I in the ring of formal power series. Given a formal map $H\colon(\mathbb{C}^N,p)\to(\mathbb{C}^{N'},q)$, we denote by H^* the natural ring homomorphism $H^*\colon\mathbb{C}[\![w-q,\overline{w-q}]\!]\to\mathbb{C}[\![z-p,\overline{z-p}]\!]$ defined by $(H^*\varrho')(z,\bar z)=\varrho'(H(z),\overline{H(z)})$.

We now have the alternate

Definition 2.2. A formal (holomorphic) map $H\colon(\mathbb{C}^N,p)\to(\mathbb{C}^{N'},q)$ sends a germ of a real-analytic set $(X,p)\subset(\mathbb{C}^N_z,p)$ into $(X',q)\subset(\mathbb{C}^{N'}_w,q)$ if for the associated ideals I and I' we have $H^*I'\subset\hat I$.

The equivalence of the two seemingly different inclusion relations for real-analytic sets can be shown by using Artin's theorem over the field of real numbers (whose complex version is stated below, see Theorem 2.3), and another approximation theorem due to Wavrik [56]. The details are as follows.

First, assume that H satisfies Definition 2.2, i.e. $H^*I'\subset\hat I$, and also assume for simplicity $p=0$. We write $I=(\varrho_1(z,\bar z),\dots,\varrho_d(z,\bar z))$ and may assume that M' is given near $H(0)$ by the zero set of a real-analytic function ϱ'. Then by assumption $\varrho'(H(z),\overline{H(z)})=\sum_{j=1}^d f_j(z,\bar z)\,\varrho_j(z,\bar z)$ for some formal power series mapping $(f_1,\dots,f_d):=f$. This means that $(X,S)=(H(z),f(z,\bar z))$ is a formal solution of the system of real-analytic equations

$$\Phi(z,\bar z,X,\bar X,S)=\varrho'(X,\bar X)-\sum_{j=1}^d S_j\varrho_j(z,\bar z)=0.$$

By Artin's approximation theorem in the real category [1], for every nonnegative integer k there exists a convergent (real-analytic) solution $(H^k,g^k)\in(\mathbb{C}\{z,\bar z\})^{N'+d}$ which satisfies $j_0^kH^k=j_0^kH$. Hence for every k and for $z\in X$ sufficiently close to p, we have $\varrho'(H^k(z,\bar z),\overline{H^k(z,\bar z)})=0$, i.e. $H^k(z)\in X'$.

On the other hand, consider a formal map H which satisfies Definition 2.1 for two real-analytic sets X, X'. We again assume that $p=0$ and write $I=(\varrho_1(z,\bar z),\dots,\varrho_d(z,\bar z))$ and pick any $\rho'\in I'$. We claim that there actually exist $f_j\in\mathbb{C}[\![z,\bar z]\!]$, $j=1,\dots,d$, such that

$$\varrho'(H(z),\overline{H(z)})=\sum_{j=1}^d f_j(z,\bar z)\varrho_j(z,\bar z).$$

We write $\hat{\mathfrak{m}}$ for the maximal ideal in $\mathbb{C}[\![z,\bar z]\!]$ and consider the system of formal real equations

$$\Psi(z,\bar z,S)=\varrho'(H(z),\overline{H(z)})-\sum_{j=1}^{d}S_j\,\varrho_j(z,\bar z)=0,$$

with $(S_1,\dots,S_d)$ being the unknown. Since H satisfies Definition 2.1, for every $k\in\mathbb{N}$ there exists a real-analytic map $(H^k(z,\bar z),g^k(z,\bar z))\in(\mathbb{C}\{z,\bar z\})^{N'+d}$ such that $j_0H^k=j_0^kH$ and $\varrho'(H^k(z,\bar z),\overline{H^k(z,\bar z)})=\sum_{j=1}^d g_j^k(z,\bar z)\,\varrho_j(z,\bar z)$, which shows that $\Psi(z,\bar z,g^k(z,\bar z))\in\hat{\mathfrak{m}}^{k+1}$. We may now apply Wavrik's theorem [56] to conclude that there exists an actual formal solution mapping $f(z,\bar z)$ satisfying $\Psi(z,\bar z,f(z,\bar z))=0$. This proves that $H^*I'\subset\hat I$, as required.

2.4. Standard simple instances where formal maps arise. As outlined in §1, formal maps are indispensable tools in the study of many types of analytic problems. Whenever one is looking for the solution of a problem involving analytic objects (i.e. objects defined by power series), one of the natural things to do is to try to figure out which type of relations the coefficients of these power series fulfill. This typically involves finding some kind of iterative scheme to determine the coefficients of a formal power series.

2.4.1. *The Poincaré–Dulac normal form.* Let us first illustrate this approach with the Poincaré–Dulac theorem, which provides a (formal) normal form for holomorphic vector fields. Consider such a vector field

$$X=\sum_{j=1}^{N}X_j(z)\frac{\partial}{\partial z_j}$$

as a map $X(z)=Mz+X^2(z)+\cdots$, where the decomposition is into terms X^j which are homogeneous of degree j in z. Let us assume for this discussion that $M=\operatorname{diag}(\lambda_1,\dots,\lambda_N)$ is actually diagonal. The action of a germ of a (formal) biholomorphism $H(z)=z+H^2(z)+\cdots$ on X is given by

$$(H,X)\mapsto Y(z)=(H^*X)(z)=(H'(z))^{-1}X(H(z)).$$

Therefore one checks that the homogeneous terms $Y(z)=Mz+Y^2(z)+\cdots$ can be calculated by

$$Y^k(z)=MH^k(z)-(H^k)'(z)Mz+\cdots,\qquad k\ge 2,$$

with the dots denoting terms which only depend on H^j for $j<k$. The role of the operator $N^k\colon H^k\mapsto MH^k(z)-(H^k)'(z)Mz$ is twofold: a normalization of the terms in Y^k is only possible up to a complement of its range, and unique up to its kernel. This is why *resonances* play a special role: Resonant monomials are those of the form z^α, where $\alpha=(\alpha_1,\dots,\alpha_N)$ with $|\alpha|\ge 2$, for which $\lambda_j=\sum_\ell\alpha_\ell\lambda_\ell$ for some j; and a basis for a complementary space of the image of N^k is given by certain (vector) multiples of resonant monomials.

The Poincaré–Dulac normal form states that there exists a formal biholomorphism H such that $Y = H^*X = Mz + R$, where R consists of resonant monomials only. The Poincaré–Dulac normal form is necessarily convergent if $0 \in \mathbb{C}$ is not in the convex hull of $\{\lambda_1, \ldots, \lambda_N\}$; such a collection of eigenvalues is said to belong to the *Poincaré domain*. A collection of eigenvalues not belonging to the Poincaré domain leads to subtle problems. But in any case, the existence of resonances is an obstruction to linearizing a vector field, which is found by using a formal map. For more on the Poincaré–Dulac normal form, we refer the reader to e.g. [31]; and to [41] for some modern developments on normal forms for vector fields which are perturbations of not necessarily linear ones.

2.4.2. *Poincaré's "problème local".* Let us now discuss another problem involving formal maps more related to the subject of CR geometry. In [49], Poincaré asked whether it is possible to find a biholomorphic map $H(z)$ taking a real hypersurface (in $\mathbb{C}^2$) into another such. Assuming that both real hypersurfaces, $M \subset \mathbb{C}^N$ and $M' \subset \mathbb{C}^N$, are real-analytic passing through 0, Poincaré's question can be formulated as follows: For fixed $\varrho(z,\bar z), \varrho'(z,\bar z) \in \mathbb{C}\{z,\bar z\}$, satisfying $d\varrho(0) \neq 0$ and $d\varrho'(0) \neq 0$, can we solve the equation

$$A(z,\bar z)\, \varrho'(H(z), \bar H(\bar z)) = \varrho(z,\bar z) \tag{2.1}$$

for some $A \in \mathbb{C}[\![z,\bar z]\!]$ and $H \in (\mathbb{C}[\![z]\!])^N$, $H(0) = 0$? To be precise, in addition to having a nonvanishing differential, ϱ and ϱ' satisfy the reality relations $\varrho = \bar\varrho$ and $\varrho' = \bar\varrho'$, and so will A. We also write

$$H(z) = \sum_\alpha H_\alpha z^\alpha, \qquad A(z,\bar z) = \sum_{\alpha,\beta} A_{\alpha,\beta} z^\alpha \bar z^\beta,$$
$$\varrho(z,\bar z) = \sum_{\alpha,\beta} \varrho_{\alpha,\beta} z^\alpha \bar z^\beta, \qquad \varrho'(z,\bar z) = \sum_{\alpha,\beta} \varrho'_{\alpha,\beta} z^\alpha \bar z^\beta.$$

In order to analyze (2.1), we use a procedure commonly referred to as *complexification* or *polarization*, which in our case just means that we treat z and $\bar z$ as independent variables (which they are). With this in mind, consider the left hand side of the equation above: we can write

$$\begin{aligned} A(z,\bar z)\varrho'(H(z),\bar H(\bar z)) &= \sum_{\alpha,\beta} P_{\alpha,\beta}(j_0^{|\alpha|} H, j_0^{|\beta|} \bar H, j_0^{|\alpha|+|\beta|-1} A) z^\alpha \bar z^\beta \\ &= \sum_{k=0}^{\infty} P_k(z,\bar z), \end{aligned}$$

with $P_{\alpha,\beta}$ as well as P_k polynomial in their variables, with P_k homogeneous of degree k in z and $\bar z$. We think of $\varrho'_{\alpha,\beta}$ as fixed, about H_α and $A_{\alpha,\beta}$ as independent variables, and about $\varrho_{\alpha,\beta}$ as dependent variables. We claim that there are many more dependent than independent variables in the sense that we count the number of variables appearing in the equations $j_0^k A(z,\bar z)\varrho'(H(z),\bar H(\bar z)) = j_0^k \varrho(z,\bar z)$ for large k. The reader who is not interested in the combinatorial details can safely skip the discussion, but still is encouraged to read on even if unfamiliar with the argument, as it reveals a fundamental difference between the case $N = 1$ and $N > 1$: the obstructions we count actually disappear if $N = 1$.

One checks that $P_{\alpha,\beta}$ depends on at most

$$2N \sum_{j=1}^{\max(|\alpha|,|\beta|)} \binom{j+N-1}{N-1} + \sum_{j=0}^{|\alpha|+|\beta|-1} \binom{j+2N-1}{2N-1}$$

real variables, and therefore $(P_1, \dots, P_k)$ depends on at most

$$2N \sum_{j=1}^{k} \binom{j+N-1}{N-1} + \sum_{j=0}^{k-1} \binom{j+2N-1}{2N-1}$$

real variables. However (at least as long as $N > 1$) the number of real variables appearing on the right hand side of our equation above, $\varrho_{\alpha,\beta}$ for $|\alpha| + |\beta| \le k$ with the reality condition $\varrho_{\alpha,\beta} = \overline{\varrho_{\beta,\alpha}}$, is given by

$$\sum_{j=0}^{k} \binom{j+2N-1}{2N-1}.$$

The difference between the number of dependent variables and of independent variables is therefore (at least approximately) equal to

$$\binom{k+2N-1}{2N-1} - 2N \sum_{j=1}^{k} \binom{j+N-1}{N-1}.$$

The first term, for large k, is asymptotic to $\frac{(k+2N-1)^{2N-1}}{(2N-1)!}$; the second term does not exceed

$$2N \sum_{j=1}^{k} \frac{(j+N-1)^{N-1}}{(N-1)!} = O(k^N).$$

Thus only in the case $2N - 1 = N$, i.e. $N = 1$, should one expect that there are no obstructions to the solution of Poincaré's *problème local.* (We will later see that this is actually the case, see §3.1). We note that these heuristical arguments, essentially going back to Poincaré, can also be used to provide a (rigorous) proof that generic real hypersurfaces are not embeddable into any sphere in higher dimension. For this, see Forstnerič [18, 20].

2.5. Convergence and approximation of formal power series solutions of analytic systems. In this section, we are going to recall two important theorems, of Artin and Gabrielov. Both are powerful tools in proving that formal power series solutions of certain analytic systems are convergent or can be approximated (in a suitable manner) by convergent solutions.

We start with (the complex version of) Artin's approximation theorem. In order to set up notation, assume that $A(z,w) \in (\mathbb{C}\{z_1,\dots,z_N,w_1,\dots,w_k\})^\ell$. A *formal solution* $\hat{w}(z) \in (\mathbb{C}[\![z]\!])^k$ of the system of equations $A(z,w)=0$ is a k-tuple $\hat{w}(z) = (\hat{w}_1(z),\dots,\hat{w}_k(z))$ of formal power series such that $\hat{w}(0)=0$ and $A(z,\hat{w}(z))=0$.

Theorem 2.3 (Artin's approximation theorem [1]). *Let $A \in (\mathbb{C}\{z,w\})^\ell$, and $\hat{w} \in \mathbb{C}[\![z]\!]^k$ a formal solution of $A(z,w)=0$ satisfying $\hat{w}(0)=0$. Then for every $m \in \mathbb{N}$ there exists a* convergent *solution $w(z) \in (\mathbb{C}\{z,w\})^\ell$ of $A(z,w)=0$ with $j_0^m w = j_0^m \hat{w}$.*

A particular instance where one can use Artin's theorem to derive a convergence result for formal power series is given in the following application:

Corollary 2.4. *Let $A \in (\mathbb{C}\{z,w\})^\ell$, and assume that $\hat{w} \in (\mathbb{C}[\![z]\!])^\ell$ is a formal solution of $A(z,w)=0$ satisfying* $\det A_w(z,\hat{w}(z)) \not\equiv 0$ *and $\hat{w}(0)=0$. Then $\hat{w} \in (\mathbb{C}\{z\})^\ell$.*

Proof. This follows from Artin's theorem because under the assumption that $\det A_w(z,\hat{w}(z)) \not\equiv 0$, we can actually find an integer k such that if a formal power series map $w(z)$ satisfies $A(z,w(z))=0$ and $j_0^k w = j_0^k \hat{w}$ then it must be the case that $w(z)=\hat{w}(z)$. To see this claim, write

$$A(z,X) - A(z,Y) = \int_0^1 \frac{\partial}{\partial t}\big(A(z,tX+(1-t)Y)\big)\,dt = B(z,X,Y)(X-Y).$$

Since $B(z,X,X) = A_w(z,X)$, we see that $\det B(z,\hat{w}(z),\hat{w}(z)) \not\equiv 0$ and therefore $\det B(z,w(z),\hat{w}(z)) \not\equiv 0$ if $j_0^k w = j_0^k \hat{w}$ for large enough k. ∎

The reader should note that if instead of $\det A_w(z,\hat{w}(z)) \not\equiv 0$ one assumes $\det A_w(0,0) \neq 0$, then the conclusion of the corollary follows as a straightforward application of the implicit function theorem. Hence Corollary 2.4 should be thought of as a convergence result for formal power series mappings satisfying singular systems of analytic equations.

The other theorem we would like to mention is Gabrielov's theorem on the analyticity of formal relations between analytic functions. One can think about it as a sort of dual version to Artin's theorem, in particular, it allows testing of convergence along arbitrary convergent generically submersive maps.

Theorem 2.5 (Gabrielov's theorem on formal relations [22], see also [16]). *Assume that $A(z) \in (\mathbb{C}\{z\})^k$, $A(0)=0$, is a generic submersion, i.e. $A'(z)$ is of full row rank on an open and dense subset of some neighborhood of*

$0 \in \mathbb{C}^N$. *If* $\varphi(w) \in \mathbb{C}[\![w]\!]$ *with* $\varphi(0) = 0$ *satisfies* $\varphi(A(z)) \in \mathbb{C}\{z\}$, *then* $\varphi(w) \in \mathbb{C}\{w\}$.

3. Convergence and divergence of formal CR transformations. We now start discussing the main question that is at the heart of the present survey. Given a real-analytic submanifold $M \subset \mathbb{C}^N$, a real-analytic submanifold (or subset) $M' \subset \mathbb{C}^{N'}$ and any fixed point $p \in M$, our aim is to describe the main results known up to date regarding the convergence or divergence of formal holomorphic mappings $H\colon (\mathbb{C}^N, p) \to \mathbb{C}^{N'}$ sending M into M'. As mentioned in the introduction, our focus in this paper will be on real submanifolds M that are CR (see below for the definition). We first discuss some basic and very simple mapping situations, then go on to describe some necessary and sufficient conditions for convergence in a general setting. The main convergence results in the subject, recently proved in [40], are then highlighted together with their ramifications. In the next section, we give a sketch of the proof of such statements.

3.1. Basic mapping examples. In order to put later general considerations into context, let us briefly review a number of basic guiding examples when it comes to the convergence or divergence of formal maps between real-analytic submanifolds.

The first example one might think about is the case of the real line $\mathbb{R} \subset \mathbb{C}$. In this case, a formal map $H\colon (\mathbb{C}, 0) \to \mathbb{C}$ sends $\mathbb{R}$ into itself if and only if the power series identity

$$H(z) - \bar{H}(\bar{z}) = A(z, \bar{z})(z - \bar{z})$$

holds for some formal power series $A \in \mathbb{R}[\![z, \bar{z}]\!]$. If we write $H(z) = \sum_j H_j z^j$ and evaluate the above equation at $t = z = \bar{z}$, we see that the previous identity holds if and only if $H_j = \bar{H}_j$ for all j, or equivalently, if and only if $H \in \mathbb{R}[\![z]\!]$. In particular, there are plenty of such formal maps H which diverge.

This observation actually generalizes right away to the case of real-analytic arcs $\Gamma, \tilde{\Gamma} \subset \mathbb{C}$. If $p \in \Gamma$, then there exists a neighborhood U of p and a biholomorphic map $\varphi\colon U \to V = \varphi(U)$ such that $\varphi(\Gamma \cap U) = \mathbb{R} \cap V$ and $\varphi(p) = 0$. Similarly, if $q \in \tilde{\Gamma}$, then we can find a biholomorphic map $\tilde{\varphi}$ identifying a neighborhood of q with $\mathbb{R}$. Therefore, $H\colon (\mathbb{C}, p) \to (\mathbb{C}, q)$ sends Γ to $\tilde{\Gamma}$ if and only if $\tilde{\varphi} \circ H \circ \varphi^{-1}$ is a formal map taking $(\mathbb{R}, 0)$ to itself, if and only if $\tilde{\varphi} \circ H \circ \varphi^{-1} \in \mathbb{R}[\![z]\!]$.

We can also generalize this observation to arbitrary *totally real submanifolds* $M \subset \mathbb{C}^N$.

DEFINITION 3.1. We say that a real-analytic submanifold $M \subset \mathbb{C}^N$ is *totally real* if for each $p \in M$ there exist holomorphic coordinates z in $\mathbb{C}^N$

in which $p = 0$ and for which the defining equation of M can be written as $z = \varphi(\bar{z})$ for some $\varphi \in (\mathbb{C}\{\bar{z}\})^N$.

If we perform an additional coordinate change, we can actually assume that $z = (z^1, z^2)$, with $z^1 \in \mathbb{C}^{N-e}$ and $z^2 \in \mathbb{C}^e$, and that in these coordinates the defining equations of M are $z^1 = 0$, $\operatorname{Im} z^2 = 0$. If $e = N$, then we say that M is *maximally* (*totally*) *real.*

Analogously to the case $M = \mathbb{R}$, the reader can check that the formal holomorphic self-maps of a totally real manifold M, given in these coordinates, are of the form

$$H(z^1, z^2) = (z^1 f(z^1, z^2), z^1 g(z^1, z^2) + h(z^2))$$

with $h(z^2) \in (\mathbb{R}[\![z^2]\!])^e$. Again, there is no reason to expect convergence of any such maps. More generally, one can consider arbitrary real-analytic target sets M' instead of M itself: as long as they are not points, we are going to find plenty of divergent maps by mapping our M divergently into a real arc $\Gamma \subset M'$ (see §3.3 below).

However, if one moves away from the totally real setting and considers for example real hypersurfaces, the simple examples yield rather different answers. Let us for instance consider the *Lewy hypersurface* $M \subset \mathbb{C}^N$, given by $\operatorname{Im} z_N = |z_1|^2 + \cdots + |z_{N-1}|^2 =: \langle \tilde{z}, \tilde{z} \rangle$. If $H\colon (\mathbb{C}^N, 0) \to (\mathbb{C}^N, 0)$ is a nonconstant formal holomorphic self-map of such a hypersurface, then one can show that H is actually of the form

$$H(\tilde{z}, z_N) = \left(rU \frac{\tilde{z} + az}{1 - 2i\langle \tilde{z}, a\rangle - (\langle a, a\rangle + it) z_N}, \frac{r^2 z}{1 - 2i\langle \tilde{z}, a\rangle - (\langle a, a\rangle + it) z_N} \right)$$

for some unitary matrix U, $r \in \mathbb{R} \setminus \{0\}$, $a \in \mathbb{C}^{N-1}$, and $t \in \mathbb{R}$. In particular, the map H is convergent. This important special case actually generalizes, using the Chern–Moser normal form [12], to arbitrary *Levi-nondegenerate hypersurfaces*, i.e. hypersurfaces defined by a real-analytic equation in $\mathbb{C}^N_{\tilde{z}, z_N}$ of the form

$$\operatorname{Im} z_N = (\tilde{z})^* A \tilde{z} + O(3) \tag{3.1}$$

with an invertible hermitian matrix A. We shall next discuss these examples in more detail and highlight several general necessary conditions for convergence of formal maps.

3.2. Basic facts on CR submanifolds. Let $M \subset \mathbb{C}^N_z$ be a real-analytic submanifold, of codimension d, and $J\colon \mathbb{C}^N \to \mathbb{C}^N$ the standard complex structure map. The submanifold M is called *CR* if the distribution of *complex tangent spaces* $T_p^c M := T_p M \cap J(T_p M)$ is of constant rank as p varies in M, in which case the rank of such a distribution is called the *CR dimension* of M and will be denoted by n. When $n = 0$, M is a totally real submanifold (as discussed in §3.1). On the other hand, if $\dim_{\mathbb{R}} M = 2n$, then M must be a complex submanifold (see e.g. [5, Proposition 1.3.14]). If, in addition, M satisfies $T_pM + J(T_pM) = T_p\mathbb{C}^N$ for every $p \in M$, we call M a *generic* submanifold. Note that this last condition is equivalent to requiring that $N = n + d$.

We also define $T^{(1,0)}M$ and $T^{(0,1)}M$ as the eigenspaces of $+i$ and $-i$, respectively, of the extension of J to the complexification of the tangent bundle $\mathbb{C}TM$. Equivalently, we can define $T^{(1,0)}M = \mathbb{C}TM \cap T^{(1,0)}\mathbb{C}^N$ and $T^{(0,1)}M = \mathbb{C}TM \cap T^{(0,1)}\mathbb{C}^N$.

Let us also recall that the *Levi form* of M is the (vector-valued) hermitian form

$$\mathscr{L} \colon T^{(1,0)}M \times T^{(1,0)}M \to \mathbb{C}TM/T^{(1,0)}M \oplus T^{(0,1)}M$$

defined by

$$\mathscr{L}_p(X_p, Y_p) = [X, \bar{Y}]_p \bmod T^{(1,0)}M \oplus T^{(0,1)}M. \tag{3.2}$$

This definition is independent of the choice of the vectors fields X and Y extending X_p and Y_p in a neighborhood of p. If M is a hypersurface, then the matrix A in (3.1) is a matrix representation of $\mathscr{L}_0$, and we say that M is (*strictly*) *pseudoconvex* if the Levi form is nonnegative (resp. positive). We say that M is *Levi-nondegenerate* if $\mathscr{L}$ is nondegenerate (as a hermitian form).

For more on CR manifolds, the reader is referred to the monographs [5, 8, 9].

3.3. Some general necessary conditions for convergence. Let $N, N' \in \mathbb{N}$ and suppose now that M is a real-analytic CR submanifold, $M' \subset \mathbb{C}_w^{N'}$ is a real-analytic set and that $p \in M$. If $H \colon (\mathbb{C}^N, p) \to \mathbb{C}^{N'}$ is a formal holomorphic map sending M into M', we write $H \colon (M, p) \to M'$ and say that H is a *formal CR map*. Our first main goal is to understand under which optimal conditions on M and M' every formal CR map $H \colon (M, p) \to M'$ converges at every point $p \in M$. We will then study the convergence problem for a fixed (or for a family of) CR map(s) and show how the geometric behavior of the map interplays with its convergence/divergence properties.

In order to guarantee that any formal CR map H as above must be convergent, genericity of M is a basic condition to assume. Indeed, if M is not generic, then for every $p \in M$, the germ of M at p is biholomorphically equivalent to a germ (at 0) of a manifold of the form $\widetilde{M} \times \{0\} \subset \mathbb{C}^{N-k} \times \mathbb{C}^k$ for some real-analytic generic submanifold $\widetilde{M}$ in $\mathbb{C}^{N-k}$ and some integer $1 \le k \le N' - 1$ (see [5, Corollary 1.8.10]). The reader may then see that it is easy to construct plenty of divergent formal CR maps $(M, p) \to M'$.

We shall therefore assume, from now on, that M is a generic real-analytic submanifold.

If we seek for geometric conditions implying divergence/convergence of formal CR maps, the first natural condition that arises is the presence of analytic discs in the target set M'. Indeed, suppose that M' contains some analytic disc Γ. We parametrize Γ by a nonconstant holomorphic map $\gamma\colon U \to \Gamma \subset \mathbb{C}^{N'}$, defined on some connected open neighborhood of 0 in the complex domain. Pick an arbitrary point $p \in M$ and let $h = h(z)$ be any divergent formal holomorphic power series, vanishing at p. Then $\gamma \circ h$ clearly defines a formal CR map from M into M'. Because h cannot be constant, we must have $\gamma'(h(z)) \not\equiv 0$. Since h is divergent, it follows from Corollary 2.4 that $\gamma \circ h$ must be divergent too. We summarize this first observation in the following statement:

FACT 3.2. *Suppose that $M \subset \mathbb{C}^N$ is a generic real-analytic submanifold and that $M' \subset \mathbb{C}^{N'}$ is a real-analytic set. If M' contains some analytic disc, then, for every point $p \in M$, there exists a divergent formal CR map $H\colon (M,p) \to M'$.*

A second natural condition, on the manifold M this time, under which one can easily construct divergent maps is when M is *nowhere minimal*. Nowhere minimality of M means that each point $p \in M$ is *nonminimal* (in the sense of Tumanov), i.e. that there exists a real-analytic CR submanifold S_p, passing through p, of the same CR dimension as that of M but with $\dim_{\mathbb{R}} S_p < \dim_{\mathbb{R}} M$ (see [55]). On the other hand, if M is somewhere minimal and is furthermore connected, then one may show that it must be minimal at each point of some Zariski open subset of M (see [5]). Another possible, and useful for our purpose here, characterization of nowhere minimality is to say that near a generic point $p \in M$, there exists a germ of a nonconstant holomorphic function at p whose restriction to (a neighborhood of) p in M is real-valued (see [5, Lemma 13.3.2]). Let p be such a point and consider such a germ h of a holomorphic function, which may, without loss of generality, be assumed to satisfy $h(p) = 0$. Pick any divergent power series $\beta(t) \in \mathbb{R}[\![t]\!]$ such that β sends $\mathbb{R}$ into M' (there are plenty of such maps, as already observed in §3.1). Then $\beta \circ h$ clearly defines a formal CR map from (M,p) into M' and must be divergent. Indeed, if not, we could write $\beta \circ h(z) = d(z)$ for some convergent (holomorphic) power series $d(z)$, and since h is not constant, it would follow from Theorem 2.5 that β has to be convergent, a contradiction. Hence we may summarize this second observation in the following statement:

FACT 3.3. *Suppose that $M \subset \mathbb{C}^N$ is a generic real-analytic submanifold and that $M' \subset \mathbb{C}^{N'}$ is a real-analytic set. If M is nowhere minimal, then, for a generic point $p \in M$, there exists a divergent formal CR map $H\colon (M,p) \to M'$.*

3.4. Sufficiency of the conditions. Putting Facts 3.2 and 3.3 together, one naturally wonders whether the assumptions of somewhere minimality of M and nonexistence of analytic discs in M' are sufficient conditions to guarantee the convergence of all formal maps sending M into M', i.e.:

PROBLEM 3.4. *Let $M \subset \mathbb{C}^N$ be a generic real-analytic connected submanifold and $M' \subset \mathbb{C}^{N'}$ be a real-analytic set. Assume M is somewhere minimal and that M' does not contain any analytic disc. Is it true that, for every point $p \in M$, all formal CR maps $H\colon (M,p) \to M'$ are convergent?*

In her survey [50], Rothschild conjectured that the answer to Problem 3.4 is affirmative for any (everywhere) minimal source manifold M. Such a conjecture has been verified to be true only in a very limited number of special cases. Baouendi, Ebenfelt and Rothschild [3] solved the case where M and M' are real hypersurfaces in the same complex space (i.e. $N = N'$) with the additional assumption that M does not contain any complex-analytic disc. Meylan, Zaitsev and the second author [42] settled the case where M' is a real-algebraic subset of $\mathbb{C}^{N'}$. Lately, in [39], the authors answered in the affirmative the case where M' is a strongly pseudoconvex CR manifold (which automatically does not contain any analytic disc). The final solution to Rothschild's conjecture, and hence solution to Problem 3.4 for everywhere minimal generic submanifolds M, has been very recently obtained by the authors in [40].

THEOREM 3.5 ([40]). *Let $M \subset \mathbb{C}^N$ be a generic real-analytic submanifold and $M' \subset \mathbb{C}^{N'}$ be a real-analytic set. Assume M is minimal and that M' does not contain any analytic disc. Then, for every point $p \in M$, all formal CR maps $H\colon (M,p) \to M'$ are convergent.*

As already mentioned, a connected generic (real-analytic) submanifold M that is somewhere minimal must be minimal at all points of some Zariski open subset Ω of M. Hence Theorem 3.5 also implies a positive solution to Problem 3.4 for all formal CR maps $H\colon (M,p) \to M'$ at all points $p \in \Omega$ in such a Zariski open subset. The remaining open question is whether the same conclusion remains valid for all points $p \in M \setminus \Omega$.

3.5. Finer results involving the behavior of the map. Theorem 3.5 provides (almost) optimal conditions on a given pair (M, M') as above to guarantee that all formal CR maps $H\colon (M,p) \to M'$ must converge for every $p \in M$. If we assume additional conditions on the mappings, then one may drastically relax the geometric condition of nonexistence of an analytic disc in the target set M'. To illustrate this, we need to recall the notion of *infinite type points* in a real-analytic set in the sense of D'Angelo [13, 15]. A point p' in the real-analytic set M' is of *infinite D'Angelo type* if there exists an analytic disc contained in M' passing through p'. In what follows, we denote by $\mathscr{E}_{M'}$ the collection of all points $p' \in M'$ that are of infinite D'Angelo type, which is a closed subset of M' by [14, 15].

Going back to the convergence problem and recalling the discussion leading to Fact 3.2, if M' contains an analytic disc Γ, then one easily constructs divergent formal CR maps $H\colon (M,p) \to M'$ for every $p \in M$. Furthermore, the construction is done in such a way that the maps always satisfy $H(M) \subset \Gamma$ and therefore the formal inclusion $H(M) \subset \mathscr{E}_{M'}$, as given in Definition 2.1, holds. One may wonder whether mapping (formally) the manifold M into the set $\mathscr{E}_{M'}$ of D'Angelo infinite type points might be the only way to produce divergent maps. The next result from [40] shows that this is indeed the case, and in particular implies Theorem 3.5:

Theorem 3.6 ([40]). *Let $M \subset \mathbb{C}^N$ be a (connected) generic real-analytic minimal submanifold and $M' \subset \mathbb{C}^{N'}$ be a real-analytic set. Then for every $p \in M$, every divergent formal CR map $H\colon (M,p) \to M'$ must satisfy $H(M) \subset \mathscr{E}_{M'}$.*

Another remarkable feature of Theorem 3.6 is that it does not hold if one only assumes that M is somewhere minimal because of the following result due to Kossovskiy and Shafikov:

Theorem 3.7 ([38]). *There exists a connected real-analytic Levi-nonflat hypersurface $M \subset \mathbb{C}^2$ and $p \in M$ such that there are divergent invertible formal CR maps $H\colon (M,p) \to (M,p)$.*

Let us point out here that Levi-nonflatness of M is equivalent, for real hypersurfaces, to somewhere minimality (see e.g. [5]). As the reader may notice, Theorem 3.6 is much stronger than Theorem 3.5. An earlier version of Theorem 3.6, valid only for real-algebraic targets M', was obtained by Meylan, Zaitsev and the second author [42].

Theorem 3.6 provides a sufficient condition for a given mapping H to be convergent even in the presence of analytic discs in M'. On the other hand, if $M' = \mathscr{E}_{M'}$, which happens in a number of interesting cases such as e.g. when M' is any everywhere Levi-degenerate real hypersurface (see e.g. [21]), Theorem 3.6 does not and cannot provide any conclusion regarding the convergence/divergence properties of formal CR maps unless further geometric assumptions are made on the mappings under consideration. The next result establishes a general necessary condition for the divergence of a formal CR map; it is a most appropriate tool when one wants to study the divergence/convergence properties for a given fixed map, or, for suitably prescribed classes of formal CR maps, typically stable under small deformations (such as e.g. classes of maps defined through rank conditions).

In order to describe the necessary condition, we need to define the following important notion:

Definition 3.8. Let $M \subset \mathbb{C}^N$ be a generic real-analytic submanifold, $M' \subset \mathbb{C}^{N'}$ be a real-analytic set and $p \in M$. Given a positive integer k, a *k-approximate formal (holomorphic) deformation of* (M, M') at p is a formal holomorphic map $B^k\colon (\mathbb{C}^N_z \times \mathbb{C}^r_t, (p,0)) \to \mathbb{C}^{N'}$ for some integer $r \geq 1$, with $B^k(p,0) \in M'$, satisfying the following conditions:

(i) $\operatorname{rk} \frac{\partial B^k}{\partial t}(z,0) = r$;

(ii) for every germ of a real-analytic function $\varrho\colon (\mathbb{C}^{N'}, B^k(p,0)) \to \mathbb{R}$, vanishing on M' near $B^k(p,0)$,

$$\varrho(B^k(z,t), \overline{B^k(z,t)})|_{z\in M} = O(|t|^{k+1}).$$

If, in addition, $H\colon (M,p) \to M'$ is a formal CR map, we say that H admits a *k-approximate formal deformation* if there exists a k-approximate formal deformation B^k of (M, M') at p as above satisfying $B^k(z,0) = H(z)$. In that case, we also say that H *admits B^k as a k-approximate formal deformation.*

Note that if B^k is a k-approximate formal deformation, we may assume, after truncating B^k up to order k with respect to t, that $B^k \in (\mathbb{C}[\![z-p]\!][t])^{N'}$. One may therefore view such an object as a holomorphic family of formal holomorphic maps $(B^k_t)_{t\in\mathbb{C}^k}$, deforming the map B^k_0 and mapping approximately M into M'.

The next theorem shows that any divergent formal CR map $H\colon (M,p) \to M'$ necessarily generates approximate formal deformations of any order. As we shall see, such a statement has strong implications on the geometry of the set M'.

Theorem 3.9 ([40]). *Let $M \subset \mathbb{C}^N$ be a generic real-analytic minimal submanifold, $p \in M$, and $M' \subset \mathbb{C}^{N'}$ be a real-analytic subset. If $H\colon (M,p) \to M'$ is a divergent formal CR map, then there exists an integer $r \in \{1, \ldots, N'\}$, and for every $k \in \mathbb{N}$, a formal holomorphic map $B^k\colon (\mathbb{C}^N \times \mathbb{C}^r, (p,0)) \to \mathbb{C}^{N'}$ such that H admits B^k as a k-approximate formal deformation.*

One may get a pretty good idea of the impact of the existence of such approximate formal deformations on the CR geometry of M' by looking at the picture if all mappings under consideration, i.e. H and B^k, were convergent. For every $k \in \mathbb{N}$, a mapping B^k as in Theorem 3.9 gives rise to a family $(\Gamma_z)_{z\in M_k}$ of r-dimensional complex submanifolds, where M_k is some dense open subset of a neighborhood of p in M, depending in a CR manner on $z \in M_k$, such that each submanifold Γ_z passes through $H(z)$ and has order of contact at least k with M' at $H(z)$. Of course, the picture we indeed have in Theorem 3.9 is the formal analog of the one just described above, and some nontrivial obstacles have to be overcome in order to get a good geometric picture. In particular, it can be shown that Theorem 3.6 follows from Theorem 3.9, but for this one needs to prove that from the formal picture described it is possible to find some families of complex-analytic subvarieties entirely contained in the target set M' and suitably related to the mapping. For more details on this we refer to §5.2 and [40]. Moreover, and this is one of the main points of the present discussion, Theorem 3.9 may be applied to situations where Theorem 3.6 happens to be inconclusive, yielding at the same time a number of optimal convergence results for suitable classes of formal CR maps, some of which will be described below.

3.5.1. *The invertible case.* As a first instance, consider the class of formal CR maps of rank N between real-analytic generic submanifolds M, M' in the same complex space, i.e. $N = N'$. In that situation, it can be shown that if H is any formal CR map in the above class, then H has a 1-approximate formal deformation if and only if M is holomorphically degenerate in the sense of Stanton (see [51, 5]). Hence, applying Theorem 3.9 to this specific context, one recovers the following convergence result:

COROLLARY 3.10 ([6, 52]). *Let $M, M' \subset \mathbb{C}^N$ be (connected) generic real-analytic submanifolds with M minimal and holomorphically nondegenerate. Then for every $p \in M$, every formal CR map $H\colon (M,p) \to M'$ of rank N is convergent.*

The geometric conditions imposed on the manifolds in Corollary 3.10 are essentially optimal. Indeed, Baouendi, Ebenfelt and Rothschild [4] recognized the necessity of the holomorphic nondegeneracy assumption in Corollary 3.10 by observing that for any holomorphically degenerate (connected) generic real-analytic submanifold $M \subset \mathbb{C}^N$, and every $p \in M$, there always exist divergent formal CR invertible self-maps $H\colon (M,p) \to (M,p)$. Furthermore, Theorem 3.7 implies that the minimality assumption on M in Corollary 3.10 cannot be replaced by the weaker assumption of somewhere minimality.

At this point, we should mention that, historically, the convergence problem for invertible (or full rank) formal CR maps is the one that has been studied the most. In their work on the biholomorphic equivalence problem for Levi-nondegenerate hypersurfaces [12], Chern–Moser were the first to show that any formal CR invertible mapping between Levi-nondegenerate hypersurfaces must necessarily converge. The result was later extended to more general classes of hypersurfaces and manifolds by Baouendi, Ebenfelt and Rothschild [4, 3], and the second author [43, 44], culminating with Corollary 3.10. This result provided a complete solution to the convergence problem for formal CR invertible maps between minimal generic submanifolds. The remaining question according to [6] was to decide whether Corollary 3.10 holds for (connected) generic submanifolds that are merely somewhere minimal. The question has been open for some time, even for real hypersurfaces in $\mathbb{C}^2$. Juhlin [32] and Juhlin and the first author [33] proved the convergence of the maps between some classes of Levi-nonflat hypersurfaces in $\mathbb{C}^2$. However, the final solution to this question was obtained only recently by Kossovskiy and Shafikov [38] who proved Theorem 3.7, thereby answering the question in the negative. Looking further into [38], the reader will notice that the hypersurfaces constructed there are not algebraic. Here we recall that a real hypersurface is *algebraic* if it is locally defined by the vanishing of a real polynomial. This leads us to:

CONJECTURE 3.11. *Let $M, M' \subset \mathbb{C}^N$ be (connected) generic real-algebraic submanifolds with M somewhere minimal and holomorphically nondegenerate. Then for every $p \in M$, every formal CR invertible map $H\colon (M,p) \to M'$ is convergent.*

To our knowledge, Conjecture 3.11 is open even for real hypersurfaces M, M' in $\mathbb{C}^N$ with $N \geq 3$, but has been proved very recently by Kossovskiy, Stolovitch and the first author [37] in dimension $N = 2$ (see §6 for further discussion on this).

3.5.2. *Levi-nondegenerate hypersurfaces with positive signature.* We note that the class of holomorphically nondegenerate generic submanifolds contains plenty of examples of hypersurfaces that are everywhere Levi-degenerate, including, for instance, boundaries of bounded symmetric domains (see e.g. [34, 59]). Since such hypersurfaces must be entirely foliated by complex-analytic submanifolds (see [21]), Theorem 3.6 is of no help in the convergence problem for such manifolds. In addition, this foliation property by complex manifolds may also hold even for Levi-nondegenerate real-analytic hypersurfaces: an example is given by the hyperquadric in $\mathbb{C}^3$ of positive signature given by

$$\operatorname{Im} z_3 = |z_1|^2 - |z_2|^2,$$

which is foliated by the complex lines

$$\zeta \mapsto \big(a_1 + \zeta, a_2 + e^{i\theta}\zeta, s + i(|a_1|^2 - |a_2|^2) + 2i\zeta(\bar{a}_1 + e^{i\theta}\bar{a}_2)\big), \qquad a_1, a_2 \in \mathbb{C},\ s, \theta \in \mathbb{R}.$$

In the setting of formal maps between Levi-nondegenerate hypersurfaces with a possible foliation by complex varieties, we will now discuss how Theorem 3.9 can be used to prove convergence. For this, we shall consider the class of so-called formal CR transversal mappings between real hypersurfaces. If M, M' are real hypersurfaces in $\mathbb{C}^N, \mathbb{C}^{N'}$ respectively, and $p \in M$, a formal CR map $H\colon (M,p) \to M'$ is called *CR transversal* if

$$T^{1,0}_{H(p)}M' + dH(T^{1,0}_p(\mathbb{C}^N)) = T^{1,0}_{H(p)}\mathbb{C}^{N'}.$$

Note that H being CR transversal also means that its normal component has a nonvanishing derivative along the normal direction (to M) at p. If, furthermore, one assumes M (resp. M') to be connected and Levi-nondegenerate, the minimum and maximum number among the positive and negative eigenvalues of the Levi form are the same at each point of M (resp. M') and are called the *signature* and *cosignature* of M (resp. M').

COROLLARY 3.12 ([40]). *Let $M \subset \mathbb{C}^N$ and $M' \subset \mathbb{C}^{N'}$ be (connected) real-analytic Levi-nondegenerate hypersurfaces, of signature ℓ and ℓ' respectively. Assume that M and M' have either the same signature (i.e. $\ell = \ell'$) or cosignature (i.e. $N - \ell = N' - \ell'$). Then, for every $p \in M$, any formal CR transversal mapping $H\colon (M,p) \to M'$ is convergent.*

The corollary follows as an application of Theorem 3.9 by showing that if M and M' have either the same signature or the same cosignature, then no CR transversal map admits a 2-approximate formal deformation [40, Proposition 6.4]. Furthermore, the assumption on the signature of the hypersurfaces is optimal to yield convergence (see [40, Remarks 6.5 and 6.6]).

For further applications of Theorem 3.9 going beyond those of Theorem 3.6, we again refer the reader to [40].

4. Convergence and divergence of formal invertible CR maps in the nonminimal case. As we already mentioned above, the minimality assumption made in the convergence Theorem 3.6 is actually necessary in view of the examples provided by Theorem 3.7. Kossovskiy and Shafikov [38] proved that there exist Levi-nonflat hypersurfaces M in $\mathbb{C}^2$ (actually, their examples are strictly pseudoconvex away from the nonminimal locus) which allow for divergent formal automorphisms $H\colon (M,p) \to (M,p)$, where p is a (necessarily) nonminimal point. These examples came as a bit of surprise, as the only nonminimal cases which had been settled before (by Juhlin [32] and Juhlin and the first author [33]) were positive: Formal automorphisms of so-called 1-nonminimal real hypersurfaces actually do converge. This divergence phenomenon was established in [38] by developing a new effective method to study formal invertible mappings between nonminimal hypersurfaces in $\mathbb{C}^2$. The method was further extended by Kossovskiy and Lamel [36] to study smooth mappings between such hypersurfaces as well. Yet a closer look at the technique that produced these examples explains this difference very nicely.

Let us therefore discuss some of the background. We will restrict ourselves to the case of a nonminimal real-analytic (Levi-nonflat) hypersurface $M \subset \mathbb{C}^2$. Locally, the set of nonminimal points in M forms a complex hypersurface $X \subset M$.

If one looks at a (small) neighborhood U of a point $p \in M$, it is possible to define the family of *Segre varieties* $S_q \subset U$, for $q \in U$, as follows:

$$S_q := \{z \in U : \rho(z, \bar{q}) = 0\},$$

where ρ is any real-analytic defining function of $M \cap U$, defined all over U (and where, in the definition of S_q, ρ has been complexified). Segre varieties were introduced in 1977 by Webster [57] and have since been extremely useful in studying the mapping problems (see e.g. [19]). The Segre varieties form a (finite-dimensional) family of complex submanifolds of U. An important result going back to Baouendi, Ebenfelt, and Rothschild [2, 5] states that the minimality of M at p is equivalent to the fact that one of the *Segre sets* S_p^k, defined inductively by

$$S_p^1 = S_p, \qquad S_p^k = \bigcup_{q \in S_p^{k-1}} S_q, \quad k \geq 2,$$

covers an open neighborhood of p. This process actually stops at the first step if M is nonminimal at p: It turns out that for any point p in the nonminimal locus X, one has $S_p = X \cap U$. Thus, $S_p^k = X \cap U$ for all $k \geq 1$. Therefore, the Segre iteration does not work in the nonminimal setting.

In order to overcome this problem, one studies rather how the Segre varieties "degenerate" as one approaches the locus of nonminimal points. Indeed, from a heuristic point of view, one can think about the Segre varieties as solutions to a complete system of PDEs, and about the degeneration mentioned above as a singularity in this complete system of PDEs; let us describe this briefly, in the setting of a real-analytic hypersurface $M \subset \mathbb{C}^2$ as already done above. The reader is referred to [38, 36] for references on this classical construction and the details of the corresponding construction in the nonminimal setting. If $\varrho(z, w, \bar{z}, \bar{w})$ is the defining function of $M \subset \mathbb{C}^2_{z,w}$ near say the origin, then for (a, b) close to 0, the Segre variety $S_{(\bar{a},\bar{b})}$ is given by $\{(z, w) \approx 0 : \varrho(z, w, a, b) = 0\}$. If we assume that $\varrho_w(0, 0) \neq 0$, we can solve this equation for w and obtain a function $w(z, a, b)$ satisfying $\varrho(z, w(z, a, b), a, b) = 0$. We now claim that the Levi-nondegeneracy of M at 0 implies that we can solve for (a, b) as functions of $w(z)$ and $w'(z)$. Indeed, consider the derivative (in z) of the equation $\varrho(z, w(z), a, b) = 0$, i.e. $\varrho_z + \varrho_w w' = 0$. The derivatives with respect to a and b of those two equations are

$$\begin{pmatrix} \varrho_{\bar{z}} & \varrho_{\bar{w}} \\ \varrho_{z,\bar{z}} + \varrho_{w,\bar{z}} w' & \varrho_{z,\bar{w}} + \varrho_{w,\bar{w}} w' \end{pmatrix}.$$

Assuming furthermore that $T_0^c M = \{w = 0\}$, we have $w(z, 0, 0) = 0$, and so $w(0) = w'(0) = 0$; thus the matrix above has nonzero determinant for small (w, w', a, b) by Levi-nondegeneracy of M (which implies that $(\varrho_{\bar{z}}, \varrho_{\bar{w}})$ and $(\varrho_{z,\bar{z}}, \varrho_{z,\bar{w}})$ are linearly independent as vectors in $\mathbb{C}^2$).

It thus turns out that if M is Levi-nondegenerate at 0, then the Segre varieties S_q for q near 0 are the graphs of the solutions of the second order system of ordinary differential equations

$$w'' = -\frac{\varrho_{z^2} + 2\varrho_{z,w} w' + {w'}^2 \varrho_{w^2}}{\varrho_w^2},$$

which is obtained by differentiating the identity $\varrho(z, w(z,a,b), a, b) = 0$ twice with respect to z and replacing (a,b) by the functions of w and w' obtained above. If $p \in X$ and M is Levi-nondegenerate at points $q \in M \setminus X$ (near p), to be precise, if its defining function can be written in the form

$$\operatorname{Im} w = (\operatorname{Re} w)^m (|z|^2 + O(z^2\bar z, \bar z^2 z)) + O((\operatorname{Re} w)^{m+1})$$

for some local holomorphic coordinate system (z, w) and for some integer m, then the varieties S_q are the graphs of the solutions of a second order *singular* ordinary differential equation of the form

$$w'' = w^m \Phi\left(z, w, \frac{w'}{w^m}\right).$$

The order m of the singularity of this second order equation is precisely the integer m appearing in the defining equation of the hypersurface. As is well known classically, m can alter the behavior of the solutions of an ODE drastically: if $m = 1$, the singularity is Fuchsian, and in particular regular; if $m > 1$, the singularity is (usually) irregular. Formal solutions of regular singular equations converge, while typically, solutions of irregular singular equations diverge.

The correspondence between the Segre varieties of our nonminimal real-analytic hypersurface and the singular ordinary differential equation does not stop with the fact that one can associate a differential equation to a hypersurface: it actually also encompasses (formal) CR invertible maps of the hypersurface, which give rise to (formal) invertible maps taking (formal) solutions to solutions of the singular ODE. Kossovskiy and Shafikov used this correspondence in the other way, by characterizing which ODEs actually give rise to a real hypersurface. They then constructed a family of singular ODEs which satisfy this criterion and which have divergent formal automorphisms. They proceeded to show that those automorphisms give rise to divergent self-maps of the associated real hypersurfaces.

It is not clear at the present stage when one should expect the existence of divergent formal CR invertible maps between nonminimal hypersurfaces. Kossovskiy and the first author [35, Section 4.3] have shown that if the underlying singular ODEs are regular singular (in particular, if they are Fuchsian), then one should expect convergence. In the irregular singular case, even though one might expect that, in analogy with the singular ODEs, divergence is more common than convergence, such results are hard to come by.

We should note that not only do formal automorphisms of nonminimal hypersurfaces not necessarily converge, but also there exist C^∞ CR automorphisms whose Taylor series not necessarily converge, again by [36]. Actually, in $\mathbb{C}^2$, there is a rather complete picture of what formal biholomorphisms of nonminimal hypersurfaces are, as will be explained in §6: each gives rise to a C^∞ CR diffeomorphism. Thus, in the nonminimal setting, even though one cannot expect a formal map to converge, there are circumstances under which we can still give "real" meaning as alluded to in the introduction to a formal map as arising from an associated actual map.

5. Elements of the proof of Theorems 3.9 and 3.6. We briefly sketch here the main ideas of the proof of Theorems 3.9 and 3.6.

5.1. Constructing k-approximate deformations. The rough general idea of constructing approximate formal deformations of any order for any given divergent formal CR map is the following. For every germ at p of a generic real-analytic submanifold $M \subset \mathbb{C}^N$ and for every formal holomorphic map $H\colon (\mathbb{C}^N, p) \to \mathbb{C}^{N'}$, we introduce a new numerical invariant attached to (M, H), called the *divergence rank* of H. This invariant measures the lack of convergence of H when M is minimal (at p). A fundamental point in the definition of the divergence rank is to look at the collection of *all* identities of a certain type satisfied by H, including the one coming from the basic mapping identity $H(M) \subset M'$. Then, using the properties of the divergence rank, we identify the "directions" in which one may deform H. Such directions can be chosen to be formally meromorphic and then from these we may construct the desired approximate formal deformations of any order.

More precisely, we are given (M, H) as above and equip $\mathbb{C}^N$ with coordinates $z = (z_1, \ldots, z_N)$, and $\mathbb{C}^{N'}$ with coordinates $w = (w_1, \ldots, w_{N'})$. We write

$$H = H(z) = \big(H_1(z), \ldots, H_{N'}(z)\big) \in (\mathbb{C}[\![z - p]\!])^{N'}.$$

Let $\mathscr{A}_H$ be the set of all pairs $(\varDelta, S)$ of power series maps such that $\varDelta = \varDelta(z) \in (\mathbb{C}[\![z - p]\!])^m$ for some m and

$$S = S(z, \bar z, \lambda, w) \in \mathbb{C}\{z - p, \bar z - \bar p, \lambda - \overline{\varDelta(p)}, w - H(p)\}, \tag{5.1}$$

where $\lambda \in \mathbb{C}^m$. For every $(\varDelta, S) \in \mathscr{A}_H$, we set

$$\begin{aligned} S^{\varDelta} &:= S(z, \bar z, \overline{\varDelta(z)}, H(z))|_M \in \mathbb{C}[\![M]\!], \\ S^{\varDelta}_{w_j} &:= \frac{\partial S}{\partial w_j}(z, \bar z, \overline{\varDelta(z)}, H(z))|_M \in \mathbb{C}[\![M]\!], \qquad j = 1, \ldots, N', \\ S^{\varDelta}_{w} &:= (S^{\varDelta}_{w_1}, \ldots, S^{\varDelta}_{w_{N'}}) \in (\mathbb{C}[\![M]\!])^{N'}. \end{aligned}$$

Consider the subring $\mathscr{S}_H(M) \subset \mathbb{C}[\![M]\!]$ consisting of those power series of the form $S^{\varDelta}$ for some $(\varDelta, S) \in \mathscr{A}_H$, and let $\mathbb{K}^M_H$ denote the quotient field of $\mathscr{S}_H(M)$.

Now denote by $\mathscr{A}_H^0(M)$ the subset of $\mathscr{A}_H$ consisting of all pairs (Δ, S) satisfying $S^\Delta = 0$ and define

$$\operatorname{rank} \mathscr{A}_H^0(M) := \dim_{\mathbb{K}_H^M} \operatorname{span}\{S_w^\Delta \colon (\Delta, S) \in \mathscr{A}_H^0(M)\},$$

where the dimension is computed over the field $\mathbb{K}_H^M$, and where every S_w^Δ is considered as a vector in $(\mathscr{S}_H(M))^{N'} \subset (\mathbb{K}_H^M)^{N'}$.

Definition 5.1. Let M and H be as above. We define the *divergence rank* of H by

$$\operatorname{divrk}_M H = N' - \operatorname{rank} \mathscr{A}_H^0(M).$$

The following shows that the divergence rank measures the lack of convergence of H.

Proposition 5.2. *Let $M \subset \mathbb{C}^N$ be a generic real-analytic submanifold, $p \in M$ and let $H \colon (\mathbb{C}^N, p) \to \mathbb{C}^{N'}$ be a formal holomorphic map. Then:*

(a) $\operatorname{divrk}_M H \le \delta$, *where δ is the number of divergent components of H.*
(b) *If M is minimal at p, then* $\operatorname{divrk}_M H = 0$ *if and only if H is convergent.*

The inequality in (a) may be strict. The proof of the left to right implication in (b) is the main nontrivial part in the above proposition. It relies on the following convergence result for formal power series mappings satisfying some singular analytic systems of equations with formal parameters appearing in a very specific way.

Proposition 5.3 ([39, Proposition 3.1]). *Let $M \subset \mathbb{C}^N$ be a real-analytic generic submanifold through the origin and let $\Theta = (\Theta_1, \ldots, \Theta_{N'})$ be a convergent power series mapping with components in $\mathbb{C}\{z, \bar z, \lambda, w\}$ where $z \in \mathbb{C}^N$, $w \in \mathbb{C}^{N'}$, $\lambda \in \mathbb{C}^r$, $N', N, r \ge 1$. Let $h \colon (\mathbb{C}^N, 0) \to \mathbb{C}^{N'}$, $g \colon (\mathbb{C}^N, 0) \to \mathbb{C}^r$ be formal holomorphic power series mappings, vanishing at 0, satisfying*

$$\Theta(z, \bar z, \overline{g(z)}, h(z))|_M = 0, \qquad \det \frac{\partial \Theta}{\partial w}(z, \bar z, \overline{g(z)}, h(z))\bigg|_M \not\equiv 0.$$

If M is minimal at 0, then h is a convergent holomorphic map.

The proof of Proposition 5.3 involves a number of ingredients, including the Segre sets characterization of minimality due to Baouendi, Ebenfelt and Rothschild [2, 5] and Artin's approximation theorem [1]. Furthermore, the minimality assumption in Proposition 5.3 is crucial since there exist germs of real-analytic nonminimal hypersurfaces (M, p) in $\mathbb{C}^N$ and divergent formal holomorphic maps $H \colon (\mathbb{C}^N, p) \to \mathbb{C}^N$ such that $\operatorname{divrk}_M H = 0$ (see [40] for details).

Next, define the following vector subspace of $(\mathbb{K}_H^M)^{N'}$:

$$\begin{aligned} \mathscr{V}_H^M &:= \{V = (V_1, \ldots, V_{N'}) \in (\mathbb{K}_H^M)^{N'} : V \cdot S_w^\Delta = 0, \, \forall (\Delta, S) \in \mathscr{A}_H^0(M)\} \qquad (5.2) \\ &= \operatorname{Ann}\{S_w^\Delta : (\Delta, S) \in \mathscr{A}_H^0(M)\}, \end{aligned}$$

where $V\cdot S^{\Delta}_w=\sum_{j=1}^{N'}V_jS^{\Delta}_{w_j}$. Then $\mathscr{V}_H^M$ will indicate the directions in which the map H will be formally deformed. The key result regarding this subspace is given by the following:

Proposition 5.4. *Let $M\subset\mathbb{C}^N$ be a generic real-analytic submanifold, $p\in M$ and $H\colon(\mathbb{C}^N,p)\to\mathbb{C}^{N'}$ be a formal holomorphic map. For $\mathscr{A}^0_H(M)$, $\mathscr{V}_H^M$, $\operatorname{divrk}_M H$ defined as above we have*

$$\dim_{\mathbb{K}_H^M}\mathscr{V}_H^M=\operatorname{divrk}_M H, \tag{5.3}$$

and there exists a basis of $\mathscr{V}_H^M$ that consists of ℓ CR vectors *in $(\mathbb{K}_H^M)^{N'}$ which are linearly independent over $\mathbb{K}_H^M$ with $\ell:=\operatorname{divrk}_M H$.*

Given $\eta\in\Bbbk_H^M$, by definition, there exist $(\Delta,S),(\Delta,T)\in\mathscr{A}_H(M)$ with $(\Delta,T)\notin\mathscr{A}^0_H(M)$ such that $\eta=S^{\Delta}/T^{\Delta}$. We define the gradient of η with respect to w by setting $\eta_w:=(S/T)^{\Delta}_w$. This definition of η_w depends on the choice of a representative for η. However, in what follows, all expressions involving a gradient with respect to w of any element in $\Bbbk_H^M$ will be independent of such choices. In the same vein, given a polynomial map $P(t,\bar t)=\sum_{\alpha,\beta\in\mathbb{N}^k}P_{\alpha,\beta}t^\alpha\bar t^\beta\in(\mathbb{K}_H^M[t,\bar t])^c$, $t=(t_1,\dots,t_k)$, $k,c\geq 1$, we define, for $V\in\mathscr{V}_H^M$,

$$V\cdot P_w(t,\bar t):=\sum_{\alpha,\beta\in\mathbb{N}^k}V\cdot P_{\alpha,\beta;w}t^\alpha\bar t^\beta\in(\mathbb{K}_H^M[t,\bar t])^c,$$

where we write $P_{\alpha,\beta;w}=(P_{\alpha,\beta})_w$.

The crucial step of the proof is to construct a formal deformation of a (divergent) map with formal meromorphic coefficients. This is done through the following construction of "exponential map type".

Theorem 5.5. *Let $M\subset\mathbb{C}^N$ be a generic real-analytic submanifold and $p\in M$. Let $H\colon(\mathbb{C}^N,p)\to\mathbb{C}^{N'}$ be a formal holomorphic map with $r=\operatorname{divrk}_M H\in\{1,\dots,N'\}$, and let $\mathbb{V}=(V^1,\dots,V^r)$ be a basis, containing only CR vectors, of $\mathscr{V}_H^M$ over $\mathbb{K}_H^M$ (as given by Proposition 5.4). For $t=(t_1,\dots,t_r)\in\mathbb{C}^r$, set $t\cdot\mathbb{V}=\sum_{i=1}^r t_iV^i$ and define, for every $\ell\in\mathbb{Z}_+$, a homogeneous polynomial map D of degree ℓ in $(\mathbb{K}_H^M[t])^{N'}$ inductively as follows:*

$$D^1(t):=t\cdot\mathbb{V},\qquad D^{\ell+1}(t)=\frac{1}{\ell+1}(t\cdot\mathbb{V})\cdot D^\ell_w(t), \tag{5.4}$$

and set $D(t)=\sum_{\ell=1}^\infty D^\ell(t)\in(\mathbb{K}_H^M[\![t]\!])^{N'}$. Then:

(i) *$D(t)\in(\mathsf{CR}(\!(M)\!)[\![t]\!])^{N'}$.*
(ii) *If $\varrho\in\mathbb{C}\{w-H(p),\bar w-\overline{H(p)}\}$ satisfies $\varrho(H(z),\overline{H(z)})|_M=0$ then*

$$\varrho(H+D(t),\overline{H+D(t)})=0\quad\text{in } \mathbb{C}(\!(M)\!)[\![t,\bar t]\!].$$

The proof of Theorem 5.5 relies on the following items:

(a) the appropriate (and in fact unique) choice of D;
(b) the property that each V^i belongs to $\mathscr{V}_H^M$, i.e. that $V^i \cdot S_w^\Delta = 0$ for every $(\Delta, S) \in \mathscr{A}_H^0(M)$;
(c) the fact that for each V^i its conjugate $\overline{V^i}$ still lies in $\mathbb{K}_H^M$ (since V^i is CR too);
(d) an appropriate use of the chain rule.

Once Theorem 5.5 is proven, it is not difficult to deduce Theorem 3.9, by simply truncating D to any fixed order k and clearing denominators.

5.2. Generating complex-analytic submanifolds from approximate deformations of any order. In order to deduce Theorem 3.6 from Theorem 3.9, one needs to infer from the formal conclusion provided by Theorem 3.9 the existence of families of complex-analytic submanifolds entirely contained in M' and closely related to the formal map H. This is done by establishing the following result:

Corollary 5.6. *Let $M \subset \mathbb{C}^N$ be a generic real-analytic minimal submanifold, $M' \subset \mathbb{C}^{N'}$ be a real-analytic set and $p \in M$. If $H\colon (M,p) \to M'$ is a divergent formal CR map, there exists an integer $r \in \{1,\dots,N'\}$ and, for any positive integer k, a neighborhood U_k of p in $\mathbb{C}^N$ and a real-analytic map $h_k\colon U_k \to \mathbb{C}^{N'}$ such that:*

(a) *$h_k(M \cap U_k) \subset M'$ and h_k agrees with H at p up to order k.*
(b) *$h_k(M \cap U_k) \subset \mathscr{E}_{M'}$ for every positive integer k.*

We immediately see that Theorem 3.6 follows from Corollary 5.6. In order to derive the latter, three main ingredients are used: the closedness of the set $\mathscr{E}_{M'}$ (proved by D'Angelo [14, 15]), Artin's approximation theorem [1] to pass from formal maps to convergent ones, and the following parameter version of a strong approximation theorem due to Hickel and Rond [25].

Theorem 5.7. *Let $R_1,\dots,R_m \in \mathbb{C}\{u-q, \bar u - \bar q, t, \bar t, \zeta, \bar\zeta\}$, where $u \in \mathbb{C}^{n_1}$, $t \in \mathbb{C}^{n_2}$, $\zeta \in \mathbb{C}^{n_3}$ and $q \in \mathbb{C}^{n_1}$ is fixed. Then there exists an open neighborhood V of q in $\mathbb{C}^{n_1}$ and a function $\mathscr{L}\colon \mathbb{N} \to \mathbb{N}$ such that the following holds: For every $u \in V$, if $S(t) \in (\mathbb{C}\{t\})^{n_3}$ satisfies $S(0) = 0$ and*

$$R_j(u, \bar u, t, \bar t, S(t), \overline{S(t)}) = O(|t|^{\mathscr{L}(k)+1}), \qquad j = 1,\dots,m,$$

for some $k \in \mathbb{N}$, then there exists $\widetilde S(t) \in (\mathbb{C}\{t\})^{n_3}$ such that

$$R_j(u, \bar u, t, \bar t, \widetilde S(t), \overline{\widetilde S(t)}) = 0, \qquad j = 1,\dots,m,$$

and $S(t) - \widetilde S(t) = O(|t|^{k+1})$.

Theorem 5.7, in conjunction with Artin's approximation, allows us, roughly speaking, to derive, from the existence of families of formal holomorphic manifolds tangent to M' along $H(M)$ to any prescribed fixed order, the existence of families of complex-analytic submanifolds entirely contained in M' (with suitable properties related to the map H as in Corollary 5.6).

6. Various notions of equivalence for CR manifolds. Two germs (M,p) and (M',p') of real-analytic generic submanifolds of the same dimension in $\mathbb{C}^N$ are called *biholomorphically equivalent* if there exists a local biholomorphim $h\colon U \to \mathbb{C}^{N'}$ defined in a neighborhood U of p with $h(p) = p'$ such that $h(U \cap M) \subset M'$. In this case we shall write $(M,p) \sim_\omega (M',p')$. The germs are called *formally equivalent* if there exists a formal CR invertible map $H\colon (M,p) \to (M',p')$, in which case we write $(M,p) \sim_f (M',p')$. Obviously, if $(M,p) \sim_\omega (M',p')$ then $(M,p) \sim_f (M',p')$. Whether the converse also holds is not obvious and an important question in its own right. Indeed, an affirmative answer for a given class of submanifolds M and M' is an important step in solving the biholomorphic equivalence problem between them since the problem may then be reduced to an existence question for formal power series where convergence issues can be avoided. At this point, we should mention that the problem that we have just raised is also meaningful in the context of general real-analytic submanifolds in $\mathbb{C}^N$, but in what follows we shall focus on CR manifolds. For submanifolds with CR singularities, the interested reader is referred to the surveys [26, 29, 27] for a further discussion of this matter.

Given (M,p) and (M',p') as above, if every formal CR invertible $H\colon (M,p) \to (M',p')$ is automatically convergent, then the convergent power series mapping obtained defines a local biholomorphism between (M,p) and (M',p'), and therefore $(M,p) \sim_\omega (M',p')$. Hence, the convergence results discussed in the previous section (for formal CR invertible maps) immediately show that $(M,p) \sim_f (M',p') \Rightarrow (M,p) \sim_\omega (M',p')$ for all germs of generic submanifolds M, M' that are minimal and holomorphically nondegenerate (by applying Corollary 3.10).

However, the reader should note that two germs of generic submanifolds could be formally equivalent through a divergent map and still be biholomorphically equivalent. In fact, even if $(M,p) = (M',p')$ is simply any holomorphically degenerate generic submanifold, one knows from §3.5.1 that (M,p) admits divergent invertible formal CR self-maps, while obviously it is biholomorphically equivalent to itself. Hence, understanding the implication

$$(M,p) \sim_f (M',p') \implies (M,p) \sim_\omega (M',p')$$

goes far beyond determining whether all formal CR invertible maps between such germs converge. It rather asks whether the existence of a formal CR invertible map between such germs forces the existence of a (possibly different) (holomorphic) convergent power series mapping between them.

We now define a third notion of equivalence between germs (M,p) and (M',p') of generic submanifolds. We say that they are *CR equivalent* and write $(M,p)\sim_{\mathrm{CR}}(M',p')$ if there exists a germ at p of a C^∞ CR diffeomorphism $g\colon (M,p)\to(M',p')$. It is well known that the power series expansion of any germ of such a C^∞ map induces a formal CR map $G\colon (M,p)\to(M',p')$ (see [5]). Hence we have the following obvious implication:

$$(M,p)\sim_{\mathrm{CR}}(M',p') \implies (M,p)\sim_f(M',p'), \tag{6.1}$$

while the converse is still far from being understood. We may now summarize in the following diagram the obvious implications between the three (a priori) distinct notions of equivalence already introduced:

$$\begin{tikzcd}
(M,p)\sim_\omega(M',p') \arrow[r, Rightarrow] \arrow[dr, Rightarrow] & (M,p)\sim_{\mathrm{CR}}(M',p') \arrow[d, Rightarrow] \\
 & (M,p)\sim_f(M',p')
\end{tikzcd}$$

Regarding the other implications, Baouendi, Rothschild and Zaitsev [7] have shown that they also hold at a generic point p of any real-analytic generic submanifold M. More precisely, they proved the following:

Theorem 6.1 ([7]). *For every real-analytic generic submanifold $M\subset\mathbb{C}^N$, there exists a Zariski open subset Ω of M such that, for every $p\in\Omega$, if there exists a real-analytic generic submanifold $M'\subset\mathbb{C}^N$ such that $(M,p)\sim_f(M',p')$ for some point $p'\in M'$, then necessarily $(M,p)\sim_\omega(M',p')$.*

In what follows, we rather seek to characterize those generic submanifolds for which the reverse implications hold (or do not hold) for arbitrary points p,p'. Regarding the implication "formal implies biholomorphic", the following positive result was proved by Baouendi, Rothschild and the second author:

Theorem 6.2 ([6]). *Two germs of generic real-analytic minimal submanifolds in $\mathbb{C}^N$ are formally equivalent if and only if they are biholomorphically equivalent.*

For some time, it has been thought that Theorem 6.2 could be true for arbitrary generic submanifolds (without the minimality assumption), until finally Kossovskiy and Shafikov came up with a negative answer to such a guess by showing the following:

Theorem 6.3 ([38]). *There exist germs (M,p) and (M',p') of real-analytic (Levi-nonflat) hypersurfaces in $\mathbb{C}^2$ that are formally equivalent but not biholomorphically equivalent.*

Theorem 6.3 shows more: the minimality assumption in Theorem 6.2 cannot be replaced by somewhere minimality (for connected submanifolds).

Regarding the implication $(M,p) \sim_{\mathrm{CR}} (M',p') \Rightarrow (M,p) \sim_{\omega} (M',p')$, it obviously holds for generic real-analytic minimal submanifolds as a consequence of (6.1) and Theorem 6.2. In the nonmimal case, Kossovskiy and the first author were able to show that a similar phenomenon to that of Theorem 6.3 holds in the C^{∞} category as well, namely:

THEOREM 6.4 ([36]). *There exist germs (M,p) and (M',p') of real-analytic (Levi-nonflat) hypersurfaces in $\mathbb{C}^2$ that are CR equivalent but not biholomorphically equivalent.*

A noteworthy consequence of Theorem 6.4 is that it also provides a negative answer to a conjecture of Ebenfelt–Huang [17] regarding the analyticity of C^{∞} CR diffeomorphisms between Levi-nonflat real-analytic hypersurfaces in $\mathbb{C}^2$.

Of course, one could deduce Theorem 6.4 from Theorem 6.3 if one knew that the notions of CR equivalence and formal equivalence were equivalent. However, this question, which amounts to establishing a Borel type result for CR maps is also far from being solved, except in complex dimension 2 which has recently been settled by Kossovskiy, Stolovitch and the first author:

THEOREM 6.5 ([37]). *Two germs of real-analytic hypersurfaces in $\mathbb{C}^2$ are formally equivalent if and only if they are CR equivalent.*

To summarize the previous discussion, the notions of formal, CR and biholomorphic equivalence always coincide for minimal real-analytic generic submanifolds in $\mathbb{C}^N$. For nonminimal generic submanifolds, they do not coincide in general. However, the notions of formal and CR equivalence are equivalent for real hypersurfaces in $\mathbb{C}^2$. We therefore make the following:

CONJECTURE 6.6. *Two germs of real-analytic generic submanifolds in $\mathbb{C}^N$ are formally equivalent if and only if they are CR equivalent.*

Regarding Theorems 6.2 and 6.3, the reader should not think that these results are the last word on the subject. In fact, looking more closely into the geometric structure of the examples from [38], one sees that the real hypersurfaces constructed are Levi-nonflat and that the points where formal equivalence does not imply biholomorphic equivalence happen to be nonminimal points (as discussed in §4). Furthermore, away from those points, the notions of formal equivalence and biholomorphic equivalence coincide (by Theorem 6.2). The nonminimal points on a real-analytic Levi-nonflat hypersurface in $\mathbb{C}^2$ correspond, geometrically, to singularities of the foliation of the hypersurface by so-called CR orbits, which we now discuss.

For any generic real-analytic and connected submanifold $M \subset \mathbb{C}^N$, recall that T^cM denotes the complex tangent bundle of M. For every $p \in M$, there exists a unique germ of a real-analytic CR submanifold Σ_p through p with the property that every point $q \in \Sigma_p$ can be reached from p by following a piecewise differentiable curve in M whose tangent vectors are in T^cM (see [5, 8]). We call this germ the (*local*) *CR orbit* of M at p. Note that M is minimal at p precisely when this CR orbit is a neighborhood of p in M. Furthermore, the dimension of Σ_p coincides with the dimension of the Lie algebra generated by the sections of T^cM, evaluated at p. Since M is real-analytic, by analytic continuation, $\dim_p \Sigma_p = e$ is constant and maximal outside a proper real-analytic subvariety V of M. When $V = \emptyset$, we say that M is *of constant orbit dimension.* Hence from the definition, $M \setminus V$ is of constant orbit dimension, i.e. is foliated by real-analytic CR manifolds (of the same dimension), the local CR orbits. On the other hand, the points in V are those points where the CR orbits are of lower dimension and, in the case of a Levi-nonflat hypersurface in $\mathbb{C}^2$, those points coincide with the nonminimal points of M. In view of Theorem 6.3, it is natural to believe that such a set V in any generic submanifold might be the locus where one may find points where formal equivalence fails to imply biholomorphic equivalence. We therefore formulate the following:

Conjecture 6.7. *Two germs of real-analytic generic submanifolds of constant orbit dimension in $\mathbb{C}^N$ are formally equivalent if and only if they are biholomorphically equivalent.*

This conjecture is known to be true in the following two main instances. Firstly, it is satisfied, again by Theorem 6.2, for all generic minimal submanifolds of $\mathbb{C}^N$, which correspond to constant-orbit-dimension generic submanifolds having CR orbits of the highest possible dimension (by definition). The second class of submanifolds for which Conjecture 6.7 has been verifed is that of real-algebraic generic submanifolds (of constant orbit dimension) as a consequence of [45] (noticing that the main result of [45] applies in the formal category as well by making use of [47, Proposition 2.4]). We note that in this latter setting, from formal equivalence one gets a stronger conclusion than that of biholomorphic equivalence, namely algebraic equivalence (see [45, 46] for the details). Finally, the reader should observe that Conjecture 6.7 also holds when $N = 2$ since any real-analytic hypersurface in $\mathbb{C}^2$ of constant orbit dimension is either (everywhere) minimal or Levi-flat.

The proofs of Theorems 6.3–6.5 heavily rely on the technique already discussed in §4 based on the theory of singular complex ODEs associated to nonminimal hypersurfaces in $\mathbb{C}^2$.

Let us conclude by mentioning the common general philosophy behind Theorems 6.1 and 6.2. In order to produce, from a given formal CR invertible map between two germs (M, p) and (M', p'), a convergent one, the strategy is to prove a stronger statement, namely to produce a sequence of convergent holomorphic maps sending the germs into each other and furthermore approximating the original formal map in the Krull topology. This is done through nontrivial reductions of the original problem to a suitable application of Artin's approximation theorem [1] (or other deep variants of such a result). We will not discuss the details here but refer the interested reader to [7, 6] or [46] for a more complete account. When following the strategy just described, one may even formulate more general questions than the ones addressed at the beginning of this section. Indeed, it then becomes natural to ask when it is possible to approximate any formal CR map between arbitrary real-analytic generic submanifolds (lying in complex spaces of possibly different dimensions) by a sequence of convergent ones. Again, this type of questions is fully discussed in [46].

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Bernhard Lamel
Fakultät für Mathematik
Universität Wien
Oskar-Morgenstern-Platz 1
A-1090 Wien, Austria
E-mail: bernhard.lamel@univie.ac.at

Nordine Mir
Science Program
Texas A&M University at Qatar
P.O. Box 23874
Education City
Doha, Qatar
E-mail: nordine.mir@qatar.tamu.edu