



Two decades of finite jet determination of CR mappings

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Abstract

This is a survey paper discussing the developments around the so-called finite jet determination problem for CR maps over the past twenty years.

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1 Introduction

It has been exactly 20 years since the survey by Zaitsev [79] and 15 years since the survey by Baouendi [10] of results surrounding the so-called *finite jet determination problem* for CR maps, appeared. The problem asks to determine whether, given CR manifolds M and M' , there exists a number k such that CR maps from M into M' are uniquely determined by their k -jets at any fixed point in M .

The subject has seen remarkable developments in different directions over the years, led to substantial achievements mostly in the equidimensional case, and more recently, in the positive codimensional case. There are fascinating and quite challenging problems left open, some of which we collect here. We hope that this article serves as a guide to the state-of-the-art in the field and points the reader to a wealth of references. We have strived to make our account as complete as possible, but it seems impossible not to miss some references; we apologize for any incompleteness.

Our survey starts with a short summary of the necessary geometric tools in Sect. 2. We then proceed to a thorough

discussion of the problem for CR diffeomorphisms of real-analytic CR manifolds in Sect. 3. It is followed by the corresponding discussion in the smooth category in Sect. 4, and results for more general (equidimensional) maps in Sect. 5. We then connect the local determination results with the more global results coming from boundary versions of H. Cartan's uniqueness theorems in Sect. 6. The most recent results in the subject addressing finite determination of maps in positive codimension are finally discussed in Sect. 7.

2 Nondegeneracy conditions for CR manifolds and maps

We recall here some nondegeneracy conditions for CR manifolds and CR maps which we will use throughout the paper. More details may be found in the monographs [3, 15, 25].

Let $M \subset \mathbb{C}^N$ be a \mathcal{C}^∞ -smooth CR submanifold of CR dimension n and CR codimension d , so that $\dim M = 2n + d$. When $n + d = N$, M is called *generic*. Denote by TM , $T^cM := TM \cap iTM \subset TM$, $\mathbb{C}TM = \mathbb{C} \otimes TM$ and $T^{0,1}M := T^{0,1}\mathbb{C}^N \cap (\mathbb{C}TM) \subset \mathbb{C}TM$ respectively, its real tangent bundle, complex tangent bundle, complexified tangent bundle and CR bundle. We recall that T^cM and $T^{0,1}M$ are canonically (anti)isomorphic as complex vector bundles and that $\mathbb{C}TM$ contains TM as a maximally totally real subspace. Nondegeneracy conditions are measurements of the nonintegrability of the distribution $T^cM \subset TM$, or equivalently, of the distribution $T^{1,0}M \oplus T^{0,1}M \subset \mathbb{C}TM$ where $T^{1,0}M = \overline{T^{0,1}M}$. A manifold M endowed with a formally integrable (i.e. closed under brackets) subbundle

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$T^{0,1}M \subset \overline{\mathbb{C}TM}$ is said to be an abstract CR manifold if $T^{0,1}M \cap T^{0,1}\overline{M} = \{0\}$.

The first important notion attached to M is its *Levi map* and is defined as follows. For every $p \in M$, the Levi map of M is the (vector-valued) Hermitian form

$$\mathcal{L}_p: T_p^{0,1}M \times T_p^{0,1}M \rightarrow \mathbb{C}T_pM / T_p^{0,1}M \oplus T_p^{1,0}M$$

given by

$$\mathcal{L}_p(X_p, Y_p) = \frac{1}{2i}[X, \bar{Y}]_p \pmod{T_p^{0,1}M \oplus T_p^{1,0}M}. \quad (2.1)$$

The definition of the Levi map \mathcal{L}_p is independent of the choice of the CR vector fields X and Y extending X_p and Y_p in a neighbourhood of p . We say that M is *Levi-nondegenerate* at $p \in M$, $\mathcal{L}_p(X_p, Y_p) = 0$ for all $Y_p \in T_p^{0,1}M$ implies that $X_p = 0$; we say that M is Levi-nondegenerate if it is Levi-nondegenerate at each of its points.

For each $p \in M$, the quadratic form associated to the Levi map \mathcal{L}_p is real-valued, and its image is contained in the maximally real subspace

$$T_pM / T_p^cM \subseteq \mathbb{C}T_pM / T_p^{0,1}M \oplus T_p^{1,0}M,$$

whose convex hull denoted by Υ_p is called the *Levi-cone* of M at p .

We denote by $T^0M \subset T^*M$ the characteristic bundle of M . The fibre T_p^0M consists of the set of (real) forms annihilating T_p^cM ; we often think about $T^*M \subset \mathbb{C}T^*M$, in which case T_p^0M is the set of real forms annihilating $T_p^{0,1}M$ and $T_p^{1,0}M$. We let $T'M := (T^{0,1}M)^\perp \subset \mathbb{C}T^*M$ be the holomorphic cotangent bundle. The manifold M is called *strongly pseudoconvex* at p if there exists $\theta_p \in T_p^0M$ such that the Hermitian form

$$T_p^{0,1}M \times T_p^{0,1}M \ni (X_p, Y_p) \mapsto \frac{1}{2i} \langle \theta_p, [X, \bar{Y}]_p \rangle$$

is positive definite, and M is said to be strictly pseudoconvex if it is strictly pseudoconvex at each of its points. Observe that the previous concepts are also well defined for abstract CR manifolds, or CR submanifolds of class \mathcal{C}^2 . When M is generic, an equivalent condition to M being strongly pseudoconvex at p is that a neighbourhood of p in M is contained in a strongly pseudoconvex real hypersurface of \mathbb{C}^N .

A finer nondegeneracy condition, still definable in the category of abstract \mathcal{C}^∞ CR manifolds and generalizing Levi-nondegeneracy, is that of *finite nondegeneracy* as introduced in [3,7]. For a point $p \in M$ and an integer k , M is called *k-finitely nondegenerate* (or *k-nondegenerate*) at a point p ,

if the Lie derivatives

$$\mathcal{L}_{\bar{K}_1} \cdots \mathcal{L}_{\bar{K}_j} \vartheta(p),$$

$$j \leq k, \quad \vartheta \in \Gamma(M, T^0M), \quad \bar{K}_v \in \Gamma(M, T^{0,1}M)$$

span T'_pM (and k is the smallest number with that property); here, $\Gamma(M, T^0M)$ and $\Gamma(M, T^{0,1}M)$ denote the set of smooth sections of the respective bundles. M is called (at most) *k-nondegenerate* if it is (at most) *k-nondegenerate* at each of its points. Observe that M is 1-nondegenerate if and only if it is Levi-nondegenerate (see [3] for more on this).

We note that a Levi-flat manifold (that is, one whose Levi form is constantly 0) is automatically foliated by complex manifolds, as the distribution $T^cM \subset TM$ is actually integrable in that case, and the leaves are complex manifolds by the Newlander-Nirenberg theorem. The “finest”, in some sense, nondegeneracy notion is that of *holomorphic nondegeneracy*, and it is designed as an obstruction to these kinds of trivial foliations by complex manifolds. We will only consider this notion for real-analytic CR submanifolds, as introduced by Stanton [74]. A (connected) real-analytic CR manifold $M \subset \mathbb{C}^N$ is *holomorphically degenerate* if for some (and equivalently, every) point $p \in M$, there exist a germ at p of a $(1, 0)$ holomorphic vector field that is tangent to M near p . It can be shown that a (connected) real-analytic CR submanifold M is holomorphically nondegenerate if and only if it is finitely nondegenerate on a Zariski open subset. We refer to [3] for details on this matter.

If $M \subset \mathbb{C}^N$ and $M' \subset \mathbb{C}^{N'}$ are two \mathcal{C}^∞ -smooth CR submanifolds, a \mathcal{C}^1 -smooth map $h: M \rightarrow M'$ is called CR if $h_*(T_p^{0,1}M) \subset T_{h(p)}^{0,1}M'$ for every $p \in M$. A CR map h is called *CR transversal* at $p \in M$ if

$$T_{h(p)}^{(1,0)}M' + T_{h(p)}^{(0,1)}M' + h_*(\mathbb{C}T_pM) = \mathbb{C}T_{h(p)}M'.$$

When M and M' have the same dimension and same CR dimension, a \mathcal{C}^k -smooth CR diffeomorphism $h: M \rightarrow M'$, $k \geq 1$, is a CR map of class \mathcal{C}^k , that is a diffeomorphism; note that h^{-1} is then automatically CR.

For real-analytic CR manifolds M and M' of the same dimension we will denote by $\mathcal{B}(M, M')$ the sheaf of biholomorphic maps sending M into M' , and $\mathcal{B}_p(M, M')$ its stalk at p . We note that this coincides with the real-analytic CR diffeomorphisms from M into M' .

We come now to two different notions of finite type. The first one is that of finite type in sense of Bloom-Graham [24]. Given a point $p \in M$, we say that M is of *finite type* if the Lie algebra generated by the CR vector fields (i.e. the smooth sections of $T^{0,1}M$) and their conjugates span $\mathbb{C}T_pM$. When M is furthermore real-analytic, this finite type condition is equivalent to the well-known condition of minimality (see [3]). A sufficient condition for M to be of finite type at

a point p is that its Levi-cone Υ_p has non-empty interior; this latter condition is also necessary when M is a quadric generic submanifold. But, in general, these two notions are not equivalent (see e.g. [43]).

On the other hand, following [31,32], we say that M is of *D’Angelo finite type* at a point $p \in M$ if the order of contact of (possibly singular) holomorphic curves with M at p is bounded. For a quantitative version of this definition we refer the reader to [31] for real hypersurfaces, or [65] where a definition is provided for arbitrary sets (that are not necessarily manifolds).

Let us conclude with some notation regarding jet spaces, that will be frequently used in the paper. Given a positive integer k and two (real) manifolds X and Y , and $x \in X$, we denote by $J_x^k(X, Y)$ the jet space at x of order k of smooth mappings from X to Y . For a smooth map $h: X \rightarrow Y$, we denote by $j_x^k h$ the k -jet of h at x . When $Y = X$, we simply write $J_x^k(X)$ for $J_x^k(X, X)$. For more details about jet spaces, see [44]. In particular, we emphasize that one can think about jets as polynomial maps, and that with that identification in mind, the jet of a holomorphic function (map) is just the truncation of its Taylor series at the point in question.

3 CR diffeomorphisms of real-analytic CR manifolds

In the present section, we shall only be considering real-analytic CR submanifolds embedded in \mathbb{C}^N with $N \geq 2$. In what follows, M, M' will denote real-analytic CR submanifolds of \mathbb{C}^N of the same dimension and same CR dimension. Let $\mathcal{F}^\omega(M, M')$ be the sheaf of real-analytic CR maps from M into M' and $\mathcal{B}(M, M')$ be the subsheaf of $\mathcal{F}^\omega(M, M')$ whose stalk at any point $p \in M$, denoted by $\mathcal{B}_p(M, M')$, consists of those germs of (real-analytic) local CR diffeomorphisms $(M, p) \rightarrow M'$. When $M' = M$, we simply write $\mathcal{B}_p(M)$ for $\mathcal{B}_p(M, M)$.

3.1 The finite jet determination property

In order to set the stage, we are going to start with a classical example.

Example 3.1 (The Heisenberg hypersurface) If one considers the so-called *Heisenberg hypersurface* $\mathbb{H}_N \subset \mathbb{C}_{(z,w)}^N = \mathbb{C}_z^n \times \mathbb{C}_w$ defined by

$$\text{Im } w = \|z\|^2,$$

then every holomorphic map $H: \mathbb{C}^N \rightarrow \mathbb{C}^N$ with $H(0) = 0$ and $\det H'(0) \neq 0$ which has the property that $H(\mathbb{H}_N \cap U) \subset$

\mathbb{H}_N is necessarily of the form

$$H(z, w) = (f(z, w), g(z, w)) \\ = \left(rU \frac{z + aw}{1 - 2i\langle z, a \rangle - (t + i\|a\|^2)w}, \frac{r^2 w}{1 - 2i\langle z, a \rangle - (t + i\|a\|^2)w} \right)$$

for some unitary $n \times n$ matrix $U, a \in \mathbb{C}^n, r \in \mathbb{R}_+,$ and $t \in \mathbb{R}$. This formula even holds if one replaces \mathbb{C}^N by a complex Hilbert space, see [62]. One sees that

$$r = \sqrt{g_w(0)} \\ U = \frac{f_z(0)}{\sqrt{g_w(0)}} \\ a = f_z(0)^{-1} f_w(0) \\ t = \text{Re} \frac{g_w^2(0)}{2g_w(0)}$$

so that every holomorphic map H as above is actually determined by its 2-jet $j_0^2 H$. The formulas above actually give rise to a *jet parametrization*, which reconstructs H from $j_0^2 H$; this is stronger than unique determination, but in many instances, a stepping stone for the proof of finite determination properties.

The earliest example of a *general* unique determination property comes as a consequence of the work of Cartan for strictly pseudoconvex hypersurfaces in \mathbb{C}^2 , Tanaka for strictly pseudoconvex hypersurfaces in \mathbb{C}^N and Chern–Moser for Levi-nondegenerate hypersurfaces in \mathbb{C}^N [27,29, 75]. The results of these authors actually give a complete solution of the biholomorphic equivalence problem in these classes of hypersurfaces, which as a consequence yield the following theorem.

Theorem 3.2 [27,29,75] *Let $M, M' \subset \mathbb{C}^N$ be real-analytic Levi-nondegenerate real hypersurfaces. Then, for any point $p \in M$, the mapping $j_p^2: \mathcal{B}_p(M, M') \rightarrow J_p^2(M, M')$ is injective.*

Theorem 3.2 is a remarkable result exhibiting how a nondegeneracy condition on the CR geometry of the hypersurfaces forces the 2-jet mapping to be injective. Let us give an example to show that this is not always the case: the elementary example of a real hyperplane in \mathbb{C}^N shows that for a real hypersurface $M \subset \mathbb{C}^N$, the mapping $j_p^k: \mathcal{B}_p(M) \rightarrow J_p^k(M)$ may fail to be injective for every $p \in M$ and any $k \in \mathbb{Z}_+$. After a linear change of coordinates, such a real hyperplane is given by

$$M = \{Z \in \mathbb{C}^N : \text{Im } Z_N = 0\}.$$

If $x \in \mathbb{R}$ and $h: (\mathbb{C}_\zeta, x) \rightarrow \mathbb{C}$ is a germ of a holomorphic function, such that $h(\zeta) = \sum_{j \geq 2} a_j \frac{(\zeta-x)^j}{j!}$ with all $a_j \in \mathbb{R}$, and furthermore $f: \mathbb{C}_Z^{N-1} \rightarrow \mathbb{C}^{N-1}$ is a biholomorphism, then the local holomorphic map

$$(Z', Z_N) \mapsto (f(Z'), Z_N + h(Z_N))$$

restricts to a germ of a real-analytic CR automorphism of M at any point $p = (z_1^0, \dots, z_{N-1}^0, x) \in M$ with $(z_1^0, \dots, z_{N-1}^0)$ in the domain of f . Hence for every integer $k \in \mathbb{Z}_+$, the mapping j_p^k is not injective. We will later see that this example actually fails the two important geometric conditions: It is degenerate (actually, Levi-flat, and so holomorphically degenerate) and everywhere nonminimal.

In view of this simple example and the breadth of the spectrum of possible conditions, a natural question that comes to mind is to characterize those real-analytic CR manifolds satisfying what we are going to call “the finite jet determination property”. The formal definition is as follows:

Definition 3.3 Let $M \subset \mathbb{C}^N$ be a real-analytic CR submanifold. We say that M has the finite jet determination property at $p \in M$ if there exists $k_p \in \mathbb{Z}_+$ such that, for every real-analytic CR manifold $M' \subset \mathbb{C}^N$, the mapping $j_p^{k_p}: \mathcal{B}_p(M, M') \rightarrow J_p^{k_p}(M, M')$ is injective. We say that M has the finite jet determination property if it has the finite jet determination property for every point $p \in M$.

Let us first note the following simple observation:

Lemma 3.4 If $M \subset \mathbb{C}^N$ is a real-analytic CR submanifold, then M has the finite jet determination property at p if and only if there exists $k_p \in \mathbb{Z}_+$ such that the mapping $j_p^{k_p}: \mathcal{B}_p(M) \rightarrow J_p^{k_p}(M)$ is injective. The k_p in this weaker condition and the k_p for the finite jet determination property coincide.

Indeed, if the weaker condition is satisfied, and we have two maps H_1, H_2 in $\mathcal{B}_p(M, M')$ with $j_p^{k_p} H_1 = j_p^{k_p} H_2$, then $H_1(p) = H_2(p)$, so $H_1^{-1} \circ H_2 \in \mathcal{B}_p(M)$ is defined and has $j_p^{k_p} (H_1^{-1} \circ H_2) = j_p^{k_p} \text{id}$, so $H_1^{-1} \circ H_2 = \text{id}$.

We also note that in the above definition we do not require specific bounds k_p for the jet order. We will discuss this point later.

By Theorem 3.2, any Levi-nondegenerate real-analytic real hypersurface satisfies the finite jet determination property. At the end of the 90’s, Baouendi, Ebenfelt and Rothschild [4–6] and Zaitsev [78] initiated the systematic study of the finite jet determination property for Levi-degenerate real-analytic CR manifolds and searched for optimal conditions on M for such a property to hold. There are essentially two optimal geometric conditions which are important for the finite jet determination property.

The first condition is that of *holomorphic nondegeneracy*, as introduced in Sect. 2. A germ of a holomorphic vector field X tangent to M actually generates a complex one-parameter group of biholomorphisms by its flow map $Z \mapsto e^{tX}Z$, and for any holomorphic function φ , we therefore obtain a biholomorphism $Z \mapsto e^{\varphi(Z)X}Z$ which cannot be uniquely determined from a finite jet. Actually a bit more concretely, at points q where we have such an X with $X(q) \neq 0$ (in particular, on a dense open subset of M), the germ of (M, q) is locally biholomorphically equivalent to a germ of a real-analytic CR manifold of the form $\tilde{M} \times \mathbb{C}$ for some real-analytic CR submanifold $\tilde{M} \subset \mathbb{C}^{N-1}$. Similarly to what we have highlighted in the real hyperplane case, one sees that any map of the form $(Z', Z_N) \mapsto (Z', Z_N + \varphi(Z))$ is a local biholomorphism if $\varphi_{Z_N}(0) \neq -1$, and so any manifold of the above form $\tilde{M} \times \mathbb{C}$ does not have the finite jet determination property, proving the necessity of the holomorphic nondegeneracy condition (these observations go back to [4]).

The second condition that is relevant in our problem is that of *finite type* in the sense of Bloom-Graham [24], as also defined in Sect. 2. When M is connected, it is easy to see, by unique continuation, that if M is of finite type at some point then it must be of finite type at all points of some Zariski open subset of M . On the other hand, when M is (connected and) nowhere of finite type, then near a generic point $p \in M$, M is foliated by its *CR orbits*, which means that the germ (M, p) is biholomorphically equivalent to a germ of a manifold of the form $\widehat{M} \times \mathbb{R}$ (see [2,3,15]) for some real submanifold $\widehat{M} \subset \mathbb{C}^{N-1}$. The fact that a real-analytic CR submanifold, nowhere of finite type, does not satisfy the finite jet determination property follows from adapting, again, the arguments used above in the hyperplane case.

In view of these two necessary conditions, one can formulate the following conjecture (see [8]):

Conjecture 3.5 Let $M \subset \mathbb{C}^N$ be a connected real-analytic CR submanifold that is holomorphically nondegenerate and of finite type somewhere. Then M satisfies the finite jet determination property.

Conjecture 3.5 has been settled in many important cases but is still open in full generality. Let us first discuss the interesting case of real hypersurfaces. In that situation, a (connected) real-analytic hypersurface that is holomorphically nondegenerate must automatically be somewhere of finite type. Indeed, a real hypersurface that is nowhere of finite type must be necessarily Levi-flat and hence holomorphically degenerate (see [2,3]). For real hypersurfaces, Conjecture 3.5 has been settled by Juhlin [53].

Theorem 3.6 [53] A connected real-analytic hypersurface $M \subset \mathbb{C}^N$ has the finite jet determination property if and only if it is holomorphically nondegenerate.

We should mention that in the two dimensional case ($N = 2$), Theorem 3.6 was proved earlier by Ebenfelt, Zaitsev and the first author [41]; that paper talked about Levi-nonflat hypersurfaces, but holomorphic nondegeneracy of a real hypersurface in \mathbb{C}^2 is equivalent to Levi-nonflatness.

The proof of Theorem 3.6 splits into two distinct parts as a consequence of the general structure of a holomorphically nondegenerate real-analytic hypersurface. Indeed, for such a manifold M , there is a Zariski open subset Ω such that M is of finite type at every $p \in \Omega$ and of infinite type at every point $p \in M \setminus \Omega$. To prove the injectivity of the k -jet mapping j_p^k , for some k , at points $p \in M \setminus \Omega$, Juhlin, building on previous work by Ebenfelt [37], reduces the problem to the finite determinacy of solutions of singular ODEs proved in [41]. On the other hand, the injectivity of the mapping j_p^k for $p \in \Omega$ (and for sufficiently large k large) is known from a previous result of Baouendi, Rothschild and the second author [8], valid for CR submanifolds of arbitrary codimension.

Theorem 3.7 [8] *If a real-analytic CR submanifold $M \subset \mathbb{C}^N$ is holomorphically nondegenerate and of finite type, then it has the finite jet determination property.*

A different proof of this Theorem providing a jet parametrization was later given by Juhlin and the first author [54].

For CR manifolds of arbitrary codimension, Theorem 3.7 is so far the most general result close to the solution of Conjecture 3.5. To complete such a solution, what remains to be understood is whether the k -jet mapping j_p^k remains injective (for some k) at every infinite type point p of a holomorphically nondegenerate CR submanifold (that is somewhere of finite type). Besides the case of real hypersurfaces (see above), the authors are not aware of any study in that direction for CR submanifolds of codimension ≥ 2 .

3.2 Bounds for jet order

Once one knows that a given CR manifold has the finite jet determination property, it is natural to ask whether one may find specific or universal bounds regarding the jet order k needed to have the injectivity of the k -jet mapping. For this question, Lemma 3.4 yields that it is enough to study germs of CR automorphisms of a given CR submanifold and look at the k -jet mapping $j_p^k: \mathcal{B}_p(M) \rightarrow J_p^k(M)$ for $p \in M$.

Let us first focus on the question for real hypersurfaces. Recall from Theorem 3.2 that for any real-analytic Levi-nondegenerate real hypersurface $M \subset \mathbb{C}^N$, the 2-jet at any fixed point uniquely determines local real-analytic CR automorphisms of M . In view of Theorem 3.6, one may wonder whether the mapping j_p^2 is also injective at any point p of a holomorphically nondegenerate real-analytic hypersurface $M \subset \mathbb{C}^N$; and if not whether there exists some universal integer ℓ (say depending only on N) such that injectivity of the mapping j_p^ℓ holds at each point p .

However, Kowalski [59] provided an example a real-analytic hypersurface $M_0 \subset \mathbb{C}^2$, satisfying the above conditions, showing that the answer to the first question is negative. More precisely, the hypersurface M_0 is Levi-nonflat (i.e. holomorphically nondegenerate) and of infinite type along some complex curve $\Sigma \subset M_0$; and for some point $p \in \Sigma$, the mapping j_p^3 is injective while j_p^2 is not. Building on [59], Zaitsev [79] even straightened the conclusion by showing that the answer to the second question is negative as well by proving the following:

Proposition 3.8 [79] *For every positive integer $k \geq 2$, there exists a real-analytic Levi-nonflat real hypersurface $M_k \subset \mathbb{C}^2$ and a point $p \in M_k$ (of infinite type) such that the mapping j_p^k is not injective while the mapping j_p^{k+1} is.*

Example 3.9 Let us give a quick summary of the construction of these types of counter examples (a very complete discussion is found in the work [56] of Kolar and the first author). We start with the Heisenberg hypersurface $\mathbb{H}_2 \subset \mathbb{C}_{(\zeta, \eta)}^2$ given by $\text{Im } \eta = |\zeta|^2$ and note that the germs of biholomorphisms H_t

$$H_t(\zeta, \eta) = \left(\frac{\zeta}{1 - t\eta}, \frac{\eta}{1 - t\eta} \right), \quad t \in \mathbb{R}$$

are determined by their 2-jets. We then introduce the weighted blowup $B(z, w) = (zw^2, w^{2k})$ and check that each of the maps

$$G_t = B^{-1} \circ H_t \circ B(z, w) = \left(\frac{z}{\sqrt[k]{1 - tw^{2k}}}, \frac{w}{\sqrt[k]{1 - tw^{2k}}} \right)$$

is determined by its $2k + 1$ jet but no lesser jet order will suffice. The weighing of the jet order guarantees that G_t is a germ of a biholomorphism mapping an infinite type real-analytic hypersurface $M_{2k} \subset B^{-1}(\mathbb{H}_2)$. Actually, the history of the construction of these counter examples is interesting: While Kowalski constructed his example from a complicated series argument, Zaitsev had the insight that it came from a blowup, and finally Kolar and the first author gave a complete list of possible counter examples explaining why the two different approaches of Kowalski and Zaitsev must yield a similar result.

As the reader sees, in Example 3.9 we create higher jet orders by introducing infinite type points via blowups. We therefore restrict the optimal jet order question to holomorphically nondegenerate real hypersurfaces which are *everywhere* of finite type. Interestingly, conclusions in the case $N = 2$ distinguish themselves from the case $N \geq 3$. Recall that, in \mathbb{C}^2 , a real-analytic hypersurface of finite type is automatically holomorphically nondegenerate. Ebenfelt, Zaitsev and the first author [41] proved the following remarkable result in \mathbb{C}^2 :

Theorem 3.10 [41] *Let $M \subset \mathbb{C}^2$ be a real-analytic hypersurface of finite type. Then for every $p \in M$, the mapping $j_p^2: \mathcal{B}_p(M) \rightarrow J_p^2(M)$ is injective.*

The following is the basic construction which yields higher jet orders in higher dimensions and shows that Theorem 3.10 does not generalize to holomorphically non-degenerate real-analytic hypersurfaces of finite type in \mathbb{C}^N for $N \geq 3$.

Example 3.11 Consider the Levi-nondegenerate hyperquadric $\mathbb{H}_{1,1}^3$ of signature (1, 1) defined by

$$\text{Im } \eta = \text{Re}(\zeta_1 \bar{\zeta}_2).$$

The elements of the family of biholomorphisms $L_t: (\zeta_1, \zeta_2, \eta) \mapsto (\zeta_1 + it\zeta_2, \zeta_2, \eta)$, defined for $t \in \mathbb{R}$ map $\mathbb{H}_{1,1}^3$ into itself. Now consider the modification map $B: (z_1, z_2, w) \mapsto (z_1, z_2^k, w)$ and define $M = B^{-1}(\mathbb{H}_{1,1}^3)$ (that is, M is defined by the equation $\text{Im } w = \text{Re } z_1 \bar{z}_2^k$). The family of maps $H_t = B^{-1} \circ L_t \circ B$ then yields the maps $(z_1, z_2, w) \mapsto (z_1 + itz_2^k, z_2, w)$ which are determined by their k -jet at 0 but by no jet of lower order. Note that the somewhat roundabout construction used in this example yields a flexible way to generate many more examples of this kind.

One can modify Example 3.11 to show that it is even possible to have arbitrarily high jet orders on a given fixed real hypersurface (see [64]):

Example 3.12 By the Weierstrass theorem, we may choose a nonzero entire function $\psi: \mathbb{C} \rightarrow \mathbb{C}$ such that, for every integer $n \in \mathbb{Z}_+$, $\psi^{(j)}(n) = 0$ for all $0 \leq j \leq n$. Let $M \subset \mathbb{C}_{(z_1, z_2, z_3)}^3$ be the real-analytic hypersurface given by

$$\text{Im } z_3 = \text{Re}(z_1 \overline{\psi(z_2)}).$$

Then M is holomorphically nondegenerate and (everywhere) of finite type. The entire biholomorphism $H(z_1, z_2, z_3) = (z_1 + i\psi(z_2), z_2, z_3)$ restricts to a (global) real-analytic CR diffeomorphism of M . For each integer $n \in \mathbb{Z}_+$, H agrees with the identity mapping up to order n at the point $(0, n, 0)$, but H is not the identity mapping.

So Example 3.12 shows that given a fixed real-analytic hypersurface $M \subset \mathbb{C}^N$, holomorphically nondegenerate and of finite type, with $N \geq 3$, there does not exist, in general, an integer $\ell(M)$ such that the mapping $j_p^{\ell(M)}: \mathcal{B}_p(M) \rightarrow J_p^{\ell(M)}(M)$ is injective for each $p \in M$. Looking a bit more closely to Example 3.12, we see that all chosen points $(0, n, 0)$ on M are of D’Angelo infinite type (see Sect. 2); indeed, the complex lines $z_2 = n, w = 0$ are entirely contained in M . This motivates the following:

Question 3.13 *Given a real-analytic hypersurface $M \subset \mathbb{C}^N$ of D’Angelo finite type (i.e. containing no complex-analytic subvariety of positive dimension), does there exist an integer $\ell(M)$ such that the mapping $j_p^{\ell(M)}: \mathcal{B}_p(M) \rightarrow J_p^{\ell(M)}(M)$ is injective for each $p \in M$? If yes, does there exist a universal bound $\ell(N)$ for $\ell(M)$ valid for all such hypersurfaces and depending only on N ?*

Theorem 3.10 obviously shows that the answer to both parts of Question 3.13 is affirmative for $N = 2$. For $N \geq 3$, the solution to Question 3.13 is unknown to the authors; it is even true that all of the known counter examples where $\ell(M)$ necessarily exceeds 2 are of D’Angelo infinite type. Furthermore, in some model cases, one can deduce upper bounds on the jet order see e.g. [57,60]. However, for real-analytic hypersurfaces that are compact (and hence of D’Angelo finite type as a consequence of [34]), the first part of the question has been settled by the affirmative:

Theorem 3.14 [63] *Let $M \subset \mathbb{C}^N$ be a compact real-analytic hypersurface, $N \geq 2$. Then there exists an integer $\ell = \ell(M)$ such that $j_p^\ell: \mathcal{B}_p(M) \rightarrow J_p^\ell(M)$ is injective for each $p \in M$.*

We should mention that Theorem 3.14 is even valid for compact real-analytic CR submanifolds (of arbitrary codimension) that are of finite type. It would be interesting to know whether one may find a universal bound ℓ valid for all compact real-analytic hypersurfaces in \mathbb{C}^N . Example 3.12 shows that for a real-analytic holomorphically nondegenerate hypersurface M of finite type in \mathbb{C}^N , the jet order required to get injectivity of the jet mapping may not be uniformly bounded over M . However, Juhlin and the first author have shown that it can be chosen to be uniformly bounded on compact subsets of M .

Theorem 3.15 [54] *Let $M \subset \mathbb{C}^N$ be a (connected) real-analytic holomorphically nondegenerate CR submanifold of finite type. Then for every $p \in M$, there exists an integer ℓ_p such that $j_p^{\ell_p}: \mathcal{B}_p(M) \rightarrow J_p^{\ell_p}(M)$ is injective and such that the mapping $M \ni p \mapsto \ell_p$ is bounded on compact subsets of M .*

Theorem 3.15 strengthens the result from [8] whose proof do not give any information on the local boundedness of the jet order. In view of Example 3.12, global boundedness of the jet order fails to hold in the setting of Theorem 3.15. Note that such an example involves a real-analytic hypersurface that is not real-algebraic. The authors are not aware of any analogous example with a real-algebraic hypersurface. In light of recent results in the mapping problems valid in the real-algebraic category but not in the real-analytic one [7,58], the following question might be worth being investigated:

Question 3.16 *Given a connected real-algebraic, holomorphically nondegenerate hypersurface of finite type $M \subset \mathbb{C}^N$,*

does there exist an integer $\ell(M)$ such that the mapping $j_p^{\ell(M)} : \mathcal{B}_p(M) \rightarrow J_p^{\ell(M)}(M)$ is injective for each $p \in M$?

Let us now turn to discussing the currently known bounds for the jet determination problem for real-analytic CR submanifolds of higher codimension. The first natural question to tackle is to determine whether there is an analogous version of Theorem 3.2 in arbitrary codimension. And in view of Theorem 3.7, one may identify the class of real-analytic Levi-nondegenerate CR submanifolds of finite type to be a suitable class of CR submanifolds for which 2-jet determination of their CR automorphisms should be investigated. In [12], Beloshapka announced that 2-jets were indeed enough to uniquely determine CR automorphisms of real-analytic Levi-nondegenerate CR submanifolds whose Levi-cone has non-empty interior. Recall, as mentioned in §2, that the latter condition implies finite type. However, Gregorovič and Meylan [45] provided counter examples to such a statement. In fact, one of the results they prove is the following:

Theorem 3.17 [45] *For every integer $k \geq 4$, there exist an integer N and a quadric generic submanifold $M_k \subset \mathbb{C}^N$ through the origin, of codimension k , Levi-nondegenerate and of finite type, such that the mapping $j_0^2 : \mathcal{B}_0(M) \rightarrow J_0^2(M)$ is not injective (but such that $j_0^\ell : \mathcal{B}_0(M) \rightarrow J_0^\ell(M)$ is, for a suitable integer ℓ).*

It is even shown in [45] that for every integer n , there exist an integer N and a quadric submanifold $M_n \subset \mathbb{C}^N$, Levi-nondegenerate and of finite type, such that $j_0^n : \mathcal{B}_0(M) \rightarrow J_0^n(M)$ is injective but not j_0^{n-1} . On the other hand, Blanc-Centi and Meylan [22] have shown that 2-jet determination holds for real-analytic CR automorphisms of real-analytic (and even \mathcal{C}^∞ -smooth, see Sect. 4) Levi-nondegenerate generic submanifolds, of codimension two, whose Levi-cone has non-empty interior. Very recently, Beloshapka [13] proved that 2-jet determination holds for CR automorphisms of (analogous) generic submanifolds of codimension three in the real-analytic category. On the other hand, if one makes stronger assumptions on the manifolds (like strict pseudoconvexity), it is possible to get 2-jet determination for their CR automorphisms in any codimension. We will be discussing such results in §4 since they apply as well to smooth automorphisms of smooth generic submanifolds.

In view of the above discussion, one may wonder whether there even exist useful bounds for the required jet order on Levi-nondegenerate real-analytic CR submanifolds of finite type. The following result, following from the work of Baouendi, Ebenfelt, Rothschild [2] provides such a bound in arbitrary codimension:

Theorem 3.18 [2] *Let $M \subset \mathbb{C}^N$ be a real-analytic CR submanifold that is Levi-nondegenerate and of finite type, of CR codimension d . Then for every $p \in M$, the jet mapping $j_p^{d+1} : \mathcal{B}_p(M) \rightarrow J_p^{d+1}(M)$ is injective.*

It is shown in [2] that a more general result for finitely nondegenerate CR submanifolds holds:

Theorem 3.19 [2] *Let $M \subset \mathbb{C}^N$ be a real-analytic CR submanifold that is k_0 -finitely nondegenerate and of finite type, of CR codimension d . Then for every $p \in M$, the jet mapping $j_p^{k_0(d+1)} : \mathcal{B}_p(M) \rightarrow J_p^{k_0(d+1)}(M)$ is injective.*

The bound given in Theorem 3.19 improved the earlier bound $2k_0(d + 1)$ due to Zaitsev [78]. We should add that Theorem 3.19 applies as well to local biholomorphisms of \mathcal{C}^∞ -smooth CR submanifolds (see [5]).

The reason that the codimension appears in this bound is actually related to the fact that it is an *universal* bound on the so-called Segre number of a finite type manifold. Roughly speaking, in the proofs of Theorem 3.19, one shows that the jets of order $k_0\ell$, for every integer ℓ , suffice to determine mappings uniquely along the Segre set of order ℓ at p . First one defines the Segre varieties

$$S_q(U) := \{Z \in U : \varrho(Z, \bar{q}) = 0, \varrho \in \mathcal{I}(M)\} \subset U, \quad q \in U,$$

defined for small enough U . Here $\mathcal{I}(M)$ denotes the ideal of M in the sheaf of real-analytic functions on \mathbb{C}^N . For $p \in M \cap U$, one then defines inductively

$$\begin{aligned} S_p^1(U) &= S_p(U), \quad S_p^\ell(U) : \\ &= \bigcup_{q \in S_p^{\ell-1}(U)} S_q(U). \end{aligned}$$

One then notes that the germ S_p^ℓ of these Segre sets is well defined. It is known that M is of finite type at p if and only if there exists a number $\ell_0 \leq d + 1$, the smallest of which is called the Segre number of M , such that the germ $S_p^{\ell_0}$ contains an open set in \mathbb{C}^N . Thus the jets of order $k_0\ell_0$ suffice to determine elements of $\mathcal{B}_p(M)$ uniquely. Formal analogues of this construction are useful when trying to transfer jet determination results into the formal category, which is going to be of some importance below; for a summary of the approaches to the Segre maps in the formal context, the reader can consult e.g. [33].

4 CR diffeomorphisms of smooth CR manifolds

The finite jet determination problem for \mathcal{C}^k -smooth CR submanifolds, where $k \in \mathbb{Z}_+ \cup \{+\infty\}$ differs starkly from the real-analytic case discussed above. The main method used in the real-analytic setting, complexification, prolongation, and iteration along the Segre maps becomes unavailable in the smooth setting. Also the main determining factors, holomorphic nondegeneracy and finite type, are different in the smooth setting as we'll discuss below.

First we need to identify the CR automorphisms under study precisely as one may find in the literature different regularity assumptions on such maps. If M is \mathcal{C}^k -smooth, $k \in \mathbb{Z}_+ \cup \{\infty, \omega\}$ and $\ell \leq k$, then we denote by $\text{Aut}_p^\ell(M)$ the group of germs of CR automorphisms of M , fixing p and of class \mathcal{C}^ℓ . As usual, any integer is smaller than ∞ and that $\infty < \omega$. We shall also denote, for M as above and $p \in M$, by $\text{Aut}_p^\omega(M)$ the (pseudo)-group of local biholomorphisms near p , fixing M and p . Note that $\text{Aut}_p^\ell(M)$ is potentially much larger than $\text{Aut}_p^\omega(M)$ for $\ell \leq k$.

The techniques developed in [27,29,75] carry over to the case of \mathcal{C}^∞ -smooth Levi-nondegenerate real hypersurfaces, and therefore Theorem 3.2 holds for \mathcal{C}^∞ -smooth CR automorphisms between such hypersurfaces. The other results discussed in Sect. 3 also carry over in a formal setting, and it is important that the reader is aware about the distinction between the bona-fide finite determination conditions discussed below and these formal results.

Every \mathcal{C}^∞ smooth CR manifold $M \subset \mathbb{C}_Z^N$ at any of its points $p \in M$ gives rise to a formal CR manifold \hat{M}_p , which is in general not an embedded submanifold of some \mathbb{C}^N but rather defined by the ideal generated by the Taylor series of the defining equations of M at p in the formal power series ring $\mathbb{C}[[Z, \bar{Z}]]$. The notions of finite type and all of the non-degeneracy conditions discussed above naturally extend to this formal setting, and one can deduce finite jet determination results in this extended setting, but for formal mappings. Now for a smooth CR map h , the associated formal mapping \hat{h}_p at the point p is just the Taylor series of the smooth CR map h at the point p , that is, an element of $\mathbb{C}[[Z]]^N$. A determination result in the formal category therefore gives unique determination of \hat{h}_p from a jet $j_p^k h$ of fixed order, but not of the smooth map h . One might wonder whether there is a way to obtain the unique determination of h from some form of unique continuation property, but to date, this question seems to be out of reach.

The notion of k_0 -nondegeneracy has special prominence among the nondegeneracy conditions in the smooth category as it seems to be the only one which yields just as strong results as its formal counterpart. In this context, Ebenfelt [37] proved the first significant result by showing the following:

Theorem 4.1 [37] *Let $M \subset \mathbb{C}^N$ be a \mathcal{C}^∞ -smooth real hypersurface that is k_0 -nondegenerate. Then for every $p \in M$, the jet mapping $j_p^{2k_0} : \text{Aut}_p^\infty(M) \rightarrow J_p^{2k_0}(M)$ is injective.*

The general idea of the proof of Theorem 4.1 is to show that local CR automorphisms satisfy a certain type of “complete” system of differential equations of some fixed order, from which jet determination may be deduced. Such an approach was previously used by Han [47,48] to study diffeomorphisms of real-analytic hypersurfaces.

Later, Kim and Zaitsev [55] were able to extend Theorem 4.1 to abstract CR manifolds of arbitrary codimension as follows.

Theorem 4.2 [55] *Let M be an abstract \mathcal{C}^∞ -smooth CR manifold that is k_0 -nondegenerate and of finite type. Then for every $p \in M$, the jet mapping $j_p^{2k_0(d+1)} : \text{Aut}_p^\infty(M) \rightarrow J_p^{2k_0(d+1)}(M)$ is injective, where d is the CR codimension of M .*

Observe that the jet order needed to guarantee injectivity of the jet mapping in Theorem 4.2 is $2k_0(d + 1)$ and therefore higher than the one in Theorem 4.1 in the embedded hypersurface case. However, one can obtain the bound in Theorem 4.1 from Theorem 4.2 in the following way.

First, one associates to M and a fixed $p \in M$ the associated generic formal submanifold \hat{M}_p , which is obtained by finding a basis of formal integrals for the CR structure of M at p (for details, see [67] after Def. 6.4.). Any CR diffeomorphism h of M then gives rise to a formal CR automorphism \hat{h}_p of \hat{M}_p . These are uniquely determined by their $k_0(d + 1)$ jets by [5, Thm. 2.1.1.]. Hence we get that the jet of any order of \hat{h}_p is uniquely determined by $j_p^{k_0(d+1)} \hat{h}_p = j_p^{k_0(d+1)} h$.

Let us point out that Theorem 4.2 admits also versions for CR manifolds that are merely \mathcal{C}^k -smooth for some k . We refer to [55] for the exact statements of these results.

In fact recently there has been a lot of activity in trying to derive finite jet determination results for mappings and manifolds that are \mathcal{C}^k -smooth, with k as low as possible. The sharpest result in that direction extending Theorem 3.2 (for hypersurfaces) was proven by Bertrand and Blanc-Centi [17] in 2014:

Theorem 4.3 [17] *Let $M \subset \mathbb{C}^N$ be a \mathcal{C}^4 -smooth Levi-nondegenerate real hypersurface. Then for every $p \in M$, the mapping $j_p^2 : \text{Aut}_p^3(M) \rightarrow J_p^2(M)$ is injective.*

The method to derive Theorem 4.3 is different from previous ones and consists of attaching “stationary” discs to the hypersurface. Such invariants objects were first introduced by Lempert [69] in his celebrated work on the Kobayashi metric on strongly convex domains, and in a rather precise sense, stationary discs solve the Euler–Lagrange equations for the Kobayashi pseudometric. While strong convexity is not an invariant notion, some results also hold on strictly pseudoconvex domain (see Huang [50]). Let us expand a bit on the ideas behind this approach.

Let Δ be the open unit disc in the complex plane. A holomorphic disc $A : \Delta \rightarrow \mathbb{C}^N$ is said to be attached to M if it extends to a (sufficiently smooth, typically $\mathcal{C}^{k,\alpha}(\bar{\Delta})$) map $A : \bar{\Delta} \rightarrow \mathbb{C}^N$ with $A(\partial\Delta) \subset M$. It is a fact that any CR function extends to any holomorphic disc attached to M via the Cauchy transform, and so, the elements of $\text{Aut}_p^k(M)$ act on the set of attached discs in a natural way. The problem

for finite determination is that the set of attached discs is rather large (it is infinite dimensional). One is therefore led to look for smaller subsets of discs. A disc A is said to be *stationary* if it lifts to a map with a simple pole at the origin $\hat{A} = (A, \hat{A}): \bar{\Delta} \rightarrow T^*\mathbb{C}^N$ such that it is attached to the *conormal bundle* $N^*M = T^0M \subset T^*\mathbb{C}^N$. For Levi-nondegenerate real hypersurfaces M , the conormal bundle is a totally real subset of $T^*\mathbb{C}^N$. One checks that stationary discs are a finite dimensional family of discs attached to M and therefore are much more suitable for finite determination properties.

Let us describe the problem that the disc determination method runs into for general hypersurfaces in the context of \mathbb{C}^2 . Consider a real hypersurface $M \subset \mathbb{C}^2$ with a defining equation of the form

$$\operatorname{Im} w = \varphi(z, \bar{z}, \operatorname{Re} w).$$

If one decomposes the Taylor series $\hat{\varphi} \in \mathbb{C}[[z, \bar{z}, s]]$ in the following form:

$$\hat{\varphi}(z, \bar{z}, s) = \sum_{\alpha, \beta} \varphi_{\alpha, \beta}(z) \bar{z}^\alpha s^\beta,$$

then it turns out that for any (α, β) minimal with respect to the condition $\varphi_{\alpha, \beta} \neq 0$ in the partial ordering generated by the cone $\Gamma \subset \mathbb{R}^2$ given by $\Gamma = \{(x, y) : y \geq 0, x + y \geq 0\}$, the pair (α, β) is a biholomorphic invariant of M , called an *invariant pair*, see [38]. Actually, also the order $\gamma(\alpha, \beta) = \operatorname{ord}_z \varphi_{\alpha, \beta}$ is invariant in that case, and we write $\hat{\varphi}_{\alpha, \beta}(z) = c_{\alpha, \beta} z^\gamma + O(\gamma + 1)$.

If we are dealing with a strictly pseudoconvex boundary, then we trivially have exactly one invariant pair, namely $(1, 0)$, with order $\gamma(1, 0) = 1$. This order, it turns out, has to do with the fact that stationary discs are those which lift to a map with a *simple* pole at 0. Also turns out that there is a suitable replacement if $\gamma(1, 0) > 1$, namely k -stationary discs. Those are, geometrically speaking, solutions to the Euler–Lagrange equations of higher order variants of the Kobayashi metric [16].

The construction of k -stationary discs are based on solving a Riemann–Hilbert problem and go in two steps. First, one attaches particular stationary discs to model manifolds, in the case of a finite type manifold, that corresponds to the model manifold $\operatorname{Im} w = \operatorname{Re} c_{\alpha, 0} z^\gamma \bar{z}^\alpha$, where $(\alpha, 0)$ is the (unique) invariant pair on the line $\beta = 0$. One then shows that stationarity of a disc corresponds to solving a certain associated system of equations in a suitable Banach space and it turns out that the implicit function theorem allows for their solution also when the defining equation is deformed away from the model manifold. The allowable perturbations of the defining equation are limited, however; in the context of [20, 21] this means essentially that one is limited to manifolds

with *only one invariant pair*. A simple example of a real hypersurface in \mathbb{C}^2 with more than one invariant pair would be $\operatorname{Im} w = \operatorname{Re} w|z|^2 + \operatorname{Re} z\bar{z}^3$, and no unique determination result are known for smooth deformations of this manifold.

For \mathcal{C}^k -smooth CR submanifolds of higher codimension, the 2-jet determination property in Theorem 4.3 cannot hold in general for Levi-nondegenerate CR submanifolds of finite type, or even with non-empty Levi-cone interior as shown by Theorem 3.17. However, the approach of attaching stationary discs can be generalized to higher codimension, as Tumanov [77] showed. In [76] he has recently obtained the following result in the strongly pseudoconvex case:

Theorem 4.4 [76] *Let $M \subset \mathbb{C}^N$ be a \mathcal{C}^4 -smooth strongly pseudoconvex CR submanifold with non-empty Levi-cone interior. Then for every $p \in M$, the mapping $j_p^2: \operatorname{Aut}_p^3(M) \rightarrow J_p^2(M)$ is injective.*

For Levi-nondegenerate CR submanifolds that are not necessarily strongly pseudoconvex, sufficient conditions which guarantee that the conclusion of Theorem 4.4 still holds have been given in [18, 19]. These results use refinements of the Levi-nondegeneracy conditions and exploit the special structure of the set of stationary discs in those situations, which is closer to the hypersurface case than in general. The conditions introduced there are in particular satisfied by any \mathcal{C}^4 -smooth Levi-nondegenerate CR submanifold in \mathbb{C}^4 with non-empty Levi-cone interior. However, the following problem seems so far still open:

Problem 4.5 *Find necessary and sufficient conditions on any \mathcal{C}^4 -smooth Levi-nondegenerate CR submanifold with non-empty Levi-cone interior, guaranteeing that the mapping $j_p^2: \operatorname{Aut}_p^3(M) \rightarrow J_p^2(M)$ is injective, for every $p \in M$.*

Using different techniques, Blanc-Centi and Meylan [22] have shown that for any \mathcal{C}^∞ -smooth CR submanifold, Levi-nondegenerate, whose Levi-cone has non-empty interior, and of CR codimension 2, the mapping $j_p^2: \operatorname{Aut}_p^\omega(M) \rightarrow J_p^2(M)$ is injective, for every $p \in M$. It is unknown whether such a conclusion also holds with $\operatorname{Aut}_p^\omega(M)$ replaced by $\operatorname{Aut}_p^k(M)$ for $k \geq 2$.

Notice that for \mathcal{C}^k -smooth CR submanifolds (k finite) most attention has been devoted to Levi-nondegenerate manifolds and the 2-jet determination property. In the Levi-degenerate case, the method of stationary discs used in [17, 19, 76] fails to work. However, the replacement by k_0 stationary (where the simple pole appearing in the definition of stationarity becomes a pole of some order k_0), can be used to provide finite jet determination results for some classes of Levi-degenerate real hypersurfaces in \mathbb{C}^N . We refer the reader to the papers [20, 21] where such an approach has been carried out. However, even for \mathcal{C}^∞ real hypersurfaces, one is far from a complete understanding of the finite jet deter-

mination property for Levi-degenerate hypersurfaces. Hence the following problem is still widely open:

Problem 4.6 Find necessary and sufficient conditions on a \mathcal{C}^∞ -smooth real hypersurface $M \subset \mathbb{C}^N$ so that for every $p \in M$, the mapping $j_p^k: \text{Aut}_p^\infty M \rightarrow J_p^k(M)$ is injective for $k = k(p)$ large enough.

This is a fascinating question for the simple reason that we know a formal obstruction, but the formal obstruction is useless for general smooth hypersurfaces, like the hypersurface $\text{Im } w = e^{-\frac{1}{|z|^2}}$ in \mathbb{C}^2 which at the origin formally coincides with the Levi-flat hypersurface $\text{Im } w = 0$. The natural analogue to holomorphic nondegeneracy for a smooth real hypersurface $M = \{Z: \varrho(Z) = 0\} \subset \mathbb{C}^N$ is CR-nondegeneracy in the sense that there does not exist a vector field, defined on an open neighbourhood Ω of some $p \in M$, of the form

$$X = \sum_{j=1}^N X_j(Z) \frac{\partial}{\partial Z_j}$$

where the X_j are smooth functions on Ω whose restrictions to M are CR functions on M ; i.e. the equation is that $\sum_j X_j(Z) \varrho_{Z_j}(Z, \bar{Z}) = 0$ on $\Omega \cap M$. The existence of such a vector field is the correct obstruction to finite nondegeneracy on an open subset, and yield a large family of CR diffeomorphisms as in the case of holomorphic nondegeneracy. However, it remains to be seen whether one can prove finite determination under such a general condition.

5 Determination results for more general maps

So far, we have discussed the unique jet determination problem for CR automorphisms of real-analytic and for diffeomorphisms of smooth CR manifolds of the same dimension in \mathbb{C}^N . Here we want to address the same uniqueness questions for germs of maps that are only generically invertible. In such a situation, one may define two distinct notions of unique jet determination property that we formalize in the next definition. We also allow, in the next definition, our manifolds to belong to affine complex spaces of possibly different dimensions as this notion will be useful in §7 as well.

In what follows, if $M \subset \mathbb{C}^N$ and $M' \subset \mathbb{C}^{N'}$ are \mathcal{C}^ℓ -smooth CR manifolds, with $\ell \in \{\infty, \omega\}$, we denote by $\mathcal{F}^\ell(M, M')$ the sheaf of \mathcal{C}^ℓ -smooth CR maps from M into M' .

Definition 5.1 Let $M \subset \mathbb{C}^N$ and $M' \subset \mathbb{C}^{N'}$ be \mathcal{C}^ℓ -smooth CR submanifolds and \mathcal{S} be a subsheaf of $\mathcal{F}^\ell(M, M')$, $\ell \in \{\infty, \omega\}$. We say that :

- (a) \mathcal{S} has the weak finite jet determination property if for every open subset $U \subset\subset M$ and every $f_0 \in \mathcal{S}(U)$, there exists an integer $k = k(f_0, U)$ such that for every $q \in U$, if $g \in \mathcal{S}_q$ and $j_q^k g = j_q^k f_0$, then $g = f_0$.
- (b) \mathcal{S} has the strong finite jet determination property if there exists a locally bounded map $k: M \rightarrow \mathbb{Z}_+$ such that for every $p \in M$, the mapping $j_p^{k(p)}: \mathcal{S}_p \rightarrow J_p^{k(p)}(M, M')$ is injective.

Clearly, Property (a) is stronger than (b). This difference between the two properties does not appear in previous sections, since then we were considering the subsheaf of CR diffeomorphisms and the reader can easily check that the two properties are in fact equivalent for this sheaf.

Assume, in what follows, that $M, M' \subset \mathbb{C}^N$ are real-analytic CR submanifolds of the same dimension and same CR dimension. It is natural to ask whether the jet determination results mentioned for CR diffeomorphisms in Sect. 3 still hold for more general maps, such as maps of generic full rank. We denote by \mathcal{S} the subsheaf of $\mathcal{F}^\omega(M, M')$, consisting of such maps. The following result from [53] extends Theorem 3.7 to maps in the sheaf \mathcal{S} .

Theorem 5.2 [53] Suppose that $M, M' \subset \mathbb{C}^N$ are real-analytic CR submanifolds of the same dimension and same CR dimension. Assume furthermore that M is holomorphically nondegenerate and of finite type. Then \mathcal{S} satisfies the weak finite jet determination property. When M and M' are real hypersurfaces in \mathbb{C}^N , the same conclusion holds without the finite type assumption.

On the other hand, it is not known, whether in the situation of Theorem 5.2, the strong finite jet determination property holds. This latter property is known to hold in a more restrictive case, namely for essentially finite manifolds. We have:

Theorem 5.3 [64] Suppose that $M, M' \subset \mathbb{C}^N$ are real-analytic CR submanifolds of the same dimension and same CR dimension. Assume furthermore that M is essentially finite and of finite type. Then \mathcal{S} satisfies the strong finite jet determination property.

The notion of essential finiteness lies in between that of finite nondegeneracy and holomorphic nondegeneracy: one requires that for $p \in M$, the so-called essential variety

$$V_p = \bigcap_{q \in S_p} S_q$$

reduces to the point p (for details, see [3]).

Theorem 5.3, as stated above, follows from [64, Theorem 9] and a CR transversality result due to Ebenfelt and Son [39]. Theorem 5.3 has a number of interesting consequences, including the following one regarding the finite jet determination property for the full sheaf of arbitrary CR maps.

Corollary 5.4 [64] *Let $M, M' \subset \mathbb{C}^N$ be real-analytic hypersurfaces of D'Angelo finite type. Then $\mathcal{F}^\omega(M, M')$ has the strong finite jet determination property.*

As the reader may have noticed, the question whether a subsheaf of maps (say between real-analytic CR submanifolds) satisfies the weak/strong finite jet determination depends on the manifolds M, M' under study. If one is interested in such a property for the full sheaf $\mathcal{F}^\omega(M, M')$, then one has to impose stronger nondegeneracy conditions on the manifolds compared to the case of a specific “smaller” subsheaf (such as e.g. the subsheaf of CR diffeomorphisms). The problem of determining which triples (M, M', \mathcal{S}) satisfy the weak/strong finite jet determination remains not fully understood for real-analytic CR manifolds, and even less, for \mathcal{C}^k -smooth ($k \in \mathbb{Z}_+ \cup \{\infty\}$) CR manifolds where, to our knowledge, there is no single result dealing with sheaves other than the sheaf of CR diffeomorphisms, as discussed earlier in this section. In Sect. 7, we will discuss recent tools to tackle this problem for maps between CR submanifolds of arbitrary dimension and CR dimension, provided that the source manifold is real-analytic and the target manifold is Nash.

6 Boundary versions of H. Cartan's uniqueness theorem

Let us start by recalling H. Cartan's uniqueness theorem for holomorphic self-maps.

Theorem 6.1 [28] *Let $\Omega \subset \mathbb{C}^N$ be a bounded domain and $h: \Omega \rightarrow \Omega$ be a holomorphic map. If there exists $p \in \Omega$ such $f(z) = z + O(|z - p|^2)$, then necessarily $h = \text{Id}$.*

A similar boundary uniqueness property does not hold, in general, for points $p \in \partial\Omega$. Indeed, the simplest example of the unit ball in \mathbb{C}^N , where all holomorphic automorphisms are known to extend holomorphically (rationally) through the boundary (see e.g. [73]), shows that one needs to assume that $f(z) = z + O(|z - p|^3)$ for the conclusion to hold. Hence, in order to get boundary uniqueness versions of Theorem 6.1, one must allow a higher order tangency condition on the map at the boundary point. There are a number of results providing sufficient conditions for a boundary version of Theorem 6.1 to hold. Such conditions are mostly assumed on the boundary geometry of the domain Ω .

We start by indicating the boundary versions of Theorem 6.1 that follow as a direct application of the results mentioned in previous sections. In all such applications, one needs to assume that the maps under study are biholomorphic or proper holomorphic to start with. We note that the statements below actually allow for the comparison of arbitrary pairs of maps. For instance, as a consequence of Theorem 5.3, one has:

Theorem 6.2 *Let $\Omega \subset \mathbb{C}^N$ be a bounded domain with smooth real-analytic boundary. Then there exists an integer ℓ , depending only on the boundary $\partial\Omega$, such that for every other bounded domain Ω' with smooth real-analytic boundary, if $h_1, h_2: \Omega \rightarrow \Omega'$ are two proper holomorphic maps extending smoothly up to $\partial\Omega$ near some point $p \in \partial\Omega$ which satisfy $h_1(z) = h_2(z) + o(|z - p|^\ell)$, then necessarily $h_1 = h_2$.*

Note that the smooth extension to the boundary of all proper holomorphic maps in Theorem 6.2 is known to automatically hold when e.g. Ω and Ω' are pseudoconvex (see [14,35]). In the special case of dimension two, we can mention the following result, obtained by combining the results of [36,41,52].

Theorem 6.3 *Let $\Omega \subset \mathbb{C}^2$ be a bounded domain with smooth real-analytic boundary and $h_1, h_2: \Omega \rightarrow \Omega$ two proper holomorphic self-maps. If there exists $p \in \partial\Omega$ such that $h_1(z) = h_2(z) + o(|z - p|^2)$, then $h_1 = h_2$.*

Further results are known when one puts some (pseudo)-convexity assumptions on the domain Ω . Firstly, in the strictly pseudoconvex case, the following is an immediate consequence of Theorem 4.3 and the extension result from [30]:

Theorem 6.4 *Let $\Omega, \Omega' \subset \mathbb{C}^N$ be two bounded strictly pseudoconvex domains with \mathcal{C}^4 -smooth boundary. If $h_1, h_2: \Omega \rightarrow \Omega'$ are two proper holomorphic maps such $h_1(z) = h_2(z) + o(|z - p|^2)$ for some $p \in \partial\Omega$, then $h_1 = h_2$.*

Theorem 6.4 should be compared with the following, earlier, result for general holomorphic maps due to Burns-Krantz [26]:

Theorem 6.5 [26] *Let $\Omega \subset \mathbb{C}^N$ be a bounded strictly pseudoconvex domain with \mathcal{C}^6 boundary and $h: \Omega \rightarrow \Omega$ be a holomorphic self-map. If there exists $p \in \partial\Omega$ such that $h(z) = z + O(|z - p|^4)$, then $h = \text{Id}$.*

While Theorem 6.5 compares arbitrary holomorphic maps with the identity mapping at a boundary point, Theorem 6.4 compares arbitrary pairs of proper holomorphic maps at a boundary point. Hence, despite of having the same flavour, Theorems 6.4 and 6.5 are independent from each other. Theorem 6.5 is usually referred to as a boundary Schwarz lemma result. Along these lines, Huang [49] later extended Theorem 6.5 to smoothly bounded convex domains of finite type (see defined in Sect. 2). He proved the following :

Theorem 6.6 [49] *Let $\Omega \subset \mathbb{C}^N$ be a \mathcal{C}^∞ -smoothly bounded convex domain of finite type and $h: \Omega \rightarrow \Omega$ be a holomorphic self-map. If there exists $p \in \partial\Omega$ such that $h(z) = z + o(|z - p|^m)$ for some $m > L(p)$, then $h = \text{Id}$. The number $L(p)$ depends only on the geometry of $\partial\Omega$ near p .*

Very recently, Zimmer [80] significantly strengthened Theorem 6.6 in different directions by showing the following:

Theorem 6.7 [80] *Let $\Omega \subset \mathbb{C}^N$ be a bounded convex domain with \mathcal{C}^2 -smooth boundary and $h: \Omega \rightarrow \Omega$ be a holomorphic self-map. If there exists $p \in \partial\Omega$ such that $f(z) = z + O(|z - p|^5)$, then $f = \text{Id}$.*

It is unknown whether one can lower the vanishing order condition in Theorem 6.7; with more assumptions on the boundary point in question, the vanishing order can be lowered; see Zimmer [80], where the reader can furthermore find related results for biholomorphisms for certain classes of domains, with no regularity assumption on the boundary. We also refer the reader to [11,51,80] for further results and related work on the boundary Schwarz lemma in Several Complex Variables and the references therein.

There are a number of questions left open regarding the boundary versions of Theorem 6.1 discussed in this section. For instance, it would be interesting to know whether versions of Theorem 6.5 remain valid for merely pseudoconvex domains, even with \mathcal{C}^∞ -smooth boundaries.

7 CR maps of positive codimension

This last section discusses the finite jet determination problem for CR maps between real submanifolds embedded in complex spaces of possibly different dimension. In contrast to all results mentioned in Sects. 3–4, we now allow our CR manifolds $M \subset \mathbb{C}^N$ and $M' \subset \mathbb{C}^{N'}$ to lie in complex spaces where N and N' might be different and where the CR dimensions of M and M' might be different as well. There are extremely few existing results in such a setting, even when the manifolds are real-analytic, which we will assume from now on.

We fix a subsheaf $\mathcal{S} \subset \mathcal{F}^\omega(M, M')$ and are interested in deciding under which conditions on the triple (M, M', \mathcal{S}) the sheaf \mathcal{S} has the weak/strong finite jet determination property (see Definition 5.1).

Let us first discuss the case of spheres $M = \mathbb{S}^{2N-1}$ and $M' = \mathbb{S}^{2N'-1}$ and the full sheaf of maps $\mathcal{S} = \mathcal{F}^\omega(\mathbb{S}^{2N-1}, \mathbb{S}^{2N'-1})$. In such a situation, Forstnerič has shown in [42] that any map $f \in \mathcal{S}$ extends to a global rational map from $\mathbb{C}^N \rightarrow \mathbb{C}^{N'}$ and the degree of these maps is uniformly bounded in terms of N and N' . As an easy consequence of these facts, one immediately gets the following:

Theorem 7.1 $\mathcal{F}^\omega(\mathbb{S}^{2N-1}, \mathbb{S}^{2N'-1})$ has the strong finite jet determination property.

The proof of Theorem 7.1 is heavily based on the fact that $\mathcal{F}^\omega(\mathbb{S}^{2N-1}, \mathbb{S}^{2N'-1})$ is a sheaf of rational maps of uniformly

bounded degree. Such a property does not hold even if one replaces \mathbb{S}^{2N-1} by an arbitrary real-analytic hypersurface.

In positive codimension, there were only a few results up to very recently which guaranteed finite determination, and most of them were restricted to the weak finite determination property. In our current language, the first author [61] showed that the sheaf of so-called *constantly degenerate* maps has the weak finite determination property in the real-analytic setting, and Ebenfelt and the first author [40] later generalized this to the smooth setting.

Theorem 7.2 [61] *Assume that $M \subset \mathbb{C}^N$ is a generic real-analytic submanifold of finite type and $M' \subset \mathbb{C}^{N'}$ is a real-analytic hypersurface. Denote by $\mathcal{S} \subset \mathcal{F}^\omega(M, M')$ the sheaf of real-analytic CR maps of constant degeneracy. Then \mathcal{S} has the weak finite jet determination property in the following situations:*

- M' is strictly pseudoconvex;
- M' is Levi-nondegenerate and $N' = N + 1$.

The problem with the sheaf \mathcal{S} is that the membership of maps in this sheaf cannot be a priori finitely determined. The first result that bypasses this difficulty was obtained recently by Zaitsev and the second author [71] who were able to generalize Theorem 7.1 as follows:

Theorem 7.3 [71] *Let $M \subset \mathbb{C}^N$ be a real-analytic CR submanifold of finite type. Then $\mathcal{F}^\omega(M, \mathbb{S}^{2N'-1})$ has the strong finite jet determination property.*

On the other hand, one cannot expect that Theorem 7.3 to hold for the full sheaf $\mathcal{F}^\omega(M, M')$ for an arbitrary target manifold M' . Indeed, it is easy to see that whenever M' contains a complex curve, $\mathcal{F}^\omega(M, M')$ does not even satisfy the weak finite jet determination property. This motivates the following question about the full sheaf $\mathcal{F}^\omega(M, M')$.

Problem 7.4 *Let $M \subset \mathbb{C}^N$ be a real-analytic CR submanifold of finite type and $M' \subset \mathbb{C}^{N'}$ be a real-analytic CR submanifold of D'Angelo finite type. Does the sheaf $\mathcal{F}^\omega(M, M')$ satisfy the weak/strong finite jet determination property?*

In what follows, we shall describe some results from [68] generalizing Theorem 7.3 to arbitrary Nash submanifolds, i.e. semi-algebraic subsets which are also real-analytic submanifolds (see [23]). Such results not only provide affirmative answers to Problem 7.4 in a number of new situations but also address the possibility of changing the subsheaf of maps under study. Let us illustrate this with the following simple example.

Example 7.5 Let $M \subset \mathbb{C}_{z_1, z_2}^2$ be the Lewy hypersurface given by $\text{Im } z_2 = |z_1|^2$ and $M' \subset \mathbb{C}_{\xi_1, \xi_2, \xi_3}^3$ the hyperquadric with

positive signature given by $\text{Im } \zeta_3 = |\zeta_1|^2 - |\zeta_2|^2$. Because M' contains the complex line $\zeta_1 = \zeta_2, \zeta_3 = 0$, the full sheaf $\mathcal{F}^\omega(M, M')$ does not satisfy the weak finite jet determination property as mentioned above. However, considering the subsheaf $\mathcal{S} \subset \mathcal{F}^\omega(M, M')$ of CR transversal maps (see Sect. 2), we shall prove that such a sheaf has the strong finite jet determination property. Hence for a fixed given pair of manifolds (M, M') , there might be subsheaves of $\mathcal{F}^\omega(M, M')$ that have or have not the (weak/strong) finite jet determination property.

The main tool used in [68] to tackle the above mentioned situation relies on the notion of so-called 2-approximation CR \mathcal{S} -deformation. This condition takes its origin from the previous works [66,70].

Definition 7.6 Let $M \subset \mathbb{C}^N$ and $M' \subset \mathbb{C}^{N'}$ be, respectively, a real-analytic CR submanifold and a real-analytic submanifold, and \mathcal{S} a subsheaf of $\mathcal{F}^\omega(M, M')$. We say that a germ of a real-analytic CR map $B : (M \times \mathbb{C}^k, (p, 0)) \rightarrow \mathbb{C}^{N'}$, for some point $p \in M$ and some integer $k \geq 1$, is a germ of a 2-approximate CR \mathcal{S} -deformation from M into M' if it satisfies the following properties :

- (i) $B|_{t=0} \in \mathcal{S}_p$;
- (ii) $\text{rk} \frac{\partial B}{\partial t}(p, 0) = k$;
- (iii) For every $\rho : (M', B(p, 0)) \rightarrow \mathbb{R}$ germ of a real-analytic function vanishing on M' , we have

$$\rho(B(\xi, t), \overline{B(\xi, t)}) = O(|t|^3),$$

for $\xi \in M$ near p and $t \in \mathbb{C}^k$ close to 0.

If a germ of a CR map $B : (M \times \mathbb{C}^k, (p, 0)) \rightarrow \mathbb{C}^{N'}$ only satisfies (ii) and (iii), we simply say that B is a (germ of a) 2-approximation CR deformation from M into M' .

Definition 7.6 slightly differs from the one given in [68] since the results mentioned there hold for sheaves of \mathcal{C}^∞ -smooth CR maps and therefore the deformations are allowed to be \mathcal{C}^∞ -smooth CR maps in that setting. The definition given here will be sufficient since we are considering only real-analytic maps.

For a given triple (M, M', \mathcal{S}) as above, the existence of a 2-approximate \mathcal{S} -approximate deformation should be regarded a strong degeneracy condition. In [68], we relate the strong finite jet determination property to the existence of such deformations. The exact statement is as follows.

Theorem 7.7 Let $M \subset \mathbb{C}^N$ be a real-analytic CR submanifold, $M' \subset \mathbb{C}^{N'}$ a Nash submanifold, and \mathcal{S} a subsheaf of $\mathcal{F}^\omega(M, M')$. Assume that M is of finite type and there is no germ of a 2-approximate CR \mathcal{S} -deformation from M into

M' . Then (M, M', \mathcal{S}) satisfies the strong finite jet determination property.

Theorem 7.7 translates the finite jet determination property to the non-existence of 2-approximate CR \mathcal{S} -deformations. In [68], it is shown that such deformations do not exist in the following situations:

- M is any real-analytic CR submanifold in \mathbb{C}^N , M' is any strongly pseudoconvex real-analytic CR submanifold in $\mathbb{C}^{N'}$ and $\mathcal{S} = \mathcal{F}^\omega(M, M')$;
- M is any real-analytic CR submanifold in \mathbb{C}^N , M' is any weakly pseudoconvex real-analytic hypersurface in $\mathbb{C}^{N'}$ and \mathcal{S} is the subsheaf of real-analytic CR maps mapping no open subset of M into the Levi-degenerate set of M' ;
- M is any (connected) real-analytic Levi-nondegenerate hypersurface in \mathbb{C}^N of signature ℓ , M' is any (connected) real-analytic hypersurface in $\mathbb{C}^{N'}$ of signature either ℓ or $\ell + N' - N$, and \mathcal{S} is the subsheaf of real-analytic CR transversal maps.

Hence, Theorem 7.7 allows a unified treatment of a number of target manifolds, including everywhere Levi-degenerate targets such as boundaries of the classical domains. In the latter case, the non-existence of approximate CR deformations has been thoroughly studied by Greilhuber and the first author in [46] and yields in conjunction with Theorem 7.7 a number of unique jet determination results that are even new in that special setting. We refer the reader to [68] for the exact statements.

To conclude, we mention the following approach, inspired by [66], that would lead to the solution of Problem 7.4 using approximate deformations. For a given integer k_0 , one may easily define, building on Definition 7.6, the notion of k_0 -approximate CR- \mathcal{S} deformation. The following conjecture seems plausible:

Conjecture 7.8 Let $M \subset \mathbb{C}^N$ be a real-analytic CR submanifold, $M' \subset \mathbb{C}^{N'}$ a real-analytic submanifold, and \mathcal{S} a subsheaf of $\mathcal{F}^\omega(M, M')$. Assume that M is of finite type and there is no germ of a k_0 -approximate CR \mathcal{S} -deformation from M into M' for some $k_0 \in \mathbb{Z}_+$. Then (M, M', \mathcal{S}) satisfies the strong finite jet determination property.

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References

- Baouendi, M.S., Ebenfelt, P., Rothschild, L.P.: Algebraicity of holomorphic mappings between real algebraic in \mathbb{C}^n . *Acta Math.* **177**, 225–273 (1996)
- Baouendi, M.S., Ebenfelt, P., Rothschild, L.P.: CR automorphisms of real analytic manifolds in complex space. *Commun. Anal. Geom.* **6**, 291–315 (1998)
- Baouendi, M.S., Ebenfelt, P., Rothschild, L.P.: *Real Submanifolds in Complex Space and Their Mappings*. Princeton Math. Series vol. 47. Princeton Univ. Press, Princeton (1999)
- Baouendi, M.S., Ebenfelt, P., Rothschild, L.P.: Parametrization of local biholomorphisms of real analytic hypersurfaces. *Asian J. Math.* **1**, 1–16 (1997)
- Baouendi, M.S., Ebenfelt, P., Rothschild, L.P.: Rational dependence of smooth and analytic CR mappings on their jets. *Math. Ann.* **315**, 205–249 (1999)
- Baouendi, M.S., Ebenfelt, P., Rothschild, L.P.: Convergence and finite determination of formal CR mappings. *J. Am. Math. Soc.* **13**, 697–723 (2000)
- Baouendi, M.S., Huang, X., Rothschild, L.P.: Regularity of CR mappings between algebraic hypersurfaces. *Invent. Math.* **125**, 13–36 (1996)
- Baouendi, M.S., Mir, N., Rothschild, L.P.: Reflection ideals and mappings between generic submanifolds in complex space. *J. Geom. Anal.* **12**(4), 543–580 (2002)
- Baouendi, M.S., Rothschild, L.P., Zaitsev, D.: Equivalences of real submanifolds in complex space. *J. Differ. Geom.* **59**, 301–351 (2001)
- Baouendi, M.S.: Strong unique continuation and finite jet determination for Cauchy-Riemann mappings Phase space analysis of partial differential equations. *Progr. Nonlinear Differ. Equ. Appl.* **69**, 17–28 (2006)
- Baracco, L., Zaitsev, D., Zampieri, G.: A Burns-Krantz type theorem for domains with corners. *Math. Ann.* **336**, 491–504 (2005)
- Beloshapka, V.K.: A uniqueness theorem for automorphism of a nondegenerate surface in complex space. (Russian) *Mat. Zametki*, 47: 17–22; transl. *Math. Notes* **47**(1990), 239–243 (1990)
- Beloshapka, V.K.: On exceptional quadrics. *Russ. J. Math. Phys.* **29**, 17–27 (2022)
- Bell, S., Catlin, D.: Boundary regularity of proper holomorphic mappings. *Duke Math. J.* **49**, 385–396 (1982)
- Berhanu, S., Cordaro, P., Hounie J.: *An Introduction to Involutive Structures*. New Mathematical Monographs, vol. 6. Cambridge University Press, Cambridge (2008)
- Bertrand, F., Della Sala, G., Joo, J.-C.: Extremal and stationary discs for the Kobayashi k-metric. *Michigan Math. J.* **69**(3), 451–463 (2020)
- Bertrand, F., Blanc-Centi, L.: Stationary holomorphic discs and finite jet determination problems. *Math. Ann.* **358**, 477–509 (2014)
- Bertrand, F., Meylan, F.: Nondefective stationary discs and 2-jet determination in higher codimension. *J. Geom. Anal.* **31**, 6292–6306 (2021)
- Bertrand, F., Blanc-Centi, L., Meylan, F.: Stationary discs and finite jet determination for non-degenerate generic real submanifolds. *Adv. Math.* **343**, 910–934 (2019). Addendum in: *Adv. Math.* **383** (2021), Paper No. 107701, 3pp
- Bertrand, F., Della Sala, G.: Stationary discs for smooth hypersurfaces of finite type and finite jet determination. *J. Geom. Anal.* **25**, 2516–2545 (2015)
- Bertrand, F., Della Sala, G., Lamel, B.: Jet determination of smooth CR automorphisms and generalized stationary discs. *Math. Z.* **294**, 1611–1634 (2020)
- Blanc-Centi, L., Meylan, F.: Chern-Moser operators and weighted jet determination problems in higher codimension. *Int. J. Math.* **33**(6) (2022)
- Bochnak, J., Coste, M., Roy, M.-F.: *Real Algebraic Geometry*. *Ergebnisse der Mathematik und ihrer Grenzgebiete*, Springer, Berlin (1998)
- Bloom, T., Graham, I.: On ‘type’ conditions for generic real submanifolds of \mathbb{C}^n . *Invent. Math.* **40**, 217–243 (1977)
- Bogges, A.: *CR Manifolds and the Tangential Cauchy-Riemann Complex*. *Studies in Advanced Mathematics*, CRC Press, Boca Raton (1991)
- Burns, D.M., Krantz, S.G.: Rigidity of holomorphic mappings and a new Schwarz lemma at the boundary. *J. Am. Math. Soc.* **7**, 661–676 (1994)
- Cartan, E.: Sur la géométrie pseudo-conforme des hypersurfaces de deux variables complexes, I. *Ann. Mat. Pura Appl.* **11**, 17–90; Part II. *Ann. Sc. Norm. Sup. Pisa* **1**(1932), 333–354 (1932)
- Cartan, H.: *Sur les groupes de transformations analytiques*, Act. Sc. et Int., Hermann, Paris (1935)
- Chern, S.S., Moser, J.K.: Real hypersurfaces in complex manifolds. *Acta Math.* **133**, 219–271 (1974)
- Coupet, B.: Precise regularity up to the boundary of proper holomorphic mappings. *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* **20**, 461–482 (1993)
- D’Angelo, J.-P.: Real hypersurfaces, orders of contact, and applications. *Ann. Math.* **115**, 615–637 (1982)
- D’Angelo, J.-P.: *Several Complex Variables and the Geometry of Real Hypersurfaces*. *Studies in Advanced Mathematics*. CRC Press, Boca Raton (1993)
- Della Sala, G., Juhlin, R., Lamel, B.: Formal theory of Segre varieties. *Illinois J. Math.* **56**(1), 101–125 (2013)
- Diederich, K., Fornæss, J.E.: Pseudoconvex domains with real-analytic boundary. *Ann. Math.* **107**(3), 371–384 (1978)
- Diederich, K., Fornæss, J.E.: Boundary regularity of proper holomorphic mappings. *Invent. Math.* **67**, 363–384 (1982)
- Diederich, K., Pinchuk, S.: Proper holomorphic maps in dimension 2 extend. *Indiana Univ. Math. J.* **44**, 1089–1126 (1995)
- Ebenfelt, P.: Finite jet determination of holomorphic mappings at the boundary. *Asian J. Math.* **5**, 637–662 (2001)
- Ebenfelt, P., Lamel, B., Zaitsev, D.: Degenerate real hypersurfaces in \mathbb{C}^2 with few automorphisms. *Trans. Am. Math. Soc.* **361**(6), 3241–3267 (2009)
- Ebenfelt, P., Son, D.N.: CR transversality of holomorphic mappings between generic submanifolds in complex spaces. *Proc. Am. Math. Soc.* **140**, 1729–1748 (2012)
- Ebenfelt, P., Lamel, B.: Finite jet determination of CR embeddings. *J. Geom. Anal.* **14**, 241–265 (2004)
- Ebenfelt, P., Lamel, B., Zaitsev, D.: Finite jet determination of local analytic CR automorphisms and their parametrization by 2-jets in the finite type case. *Geom. Funct. Anal.* **13**, 546–573 (2003)
- Forstnerič, F.: Extending proper holomorphic mappings of positive codimension. *Invent. Math.* **95**, 31–62 (1989)
- Forstnerič, F.: Mappings of quadric Cauchy-Riemann manifolds. *Math. Ann.* **292**, 163–180 (1992)
- Golubitsky, M., Guillemin, V.: *Stable Mappings and Their Singularities*, Graduate Texts in Mathematics, vol. 14. Springer, New-York (1973)
- Gregorovič, J., Meylan, F.: Construction of counterexamples to the 2-jet determination Chern-Moser theorem in higher codimension. *Math. Res. Lett.* (to appear)

46. Greilhuber, J., Lamel, B.: Regularity of CR maps into uniformly pseudoconvex hypersurfaces and applications to proper holomorphic maps. *Ann. Scuola Norm. Sup. Pisa* (to appear)
47. Han, C.-K.: Analyticity of CR equivalences between real hypersurfaces in \mathbb{C}^n with degenerate Levi form. *Invent. Math.* **73**, 51–69 (1983)
48. Han, C.-K.: Complete system for the mappings of CR manifolds of nondegenerate Levi forms. *Math. Ann.* **309**, 401–409 (1997)
49. Huang, X.: A boundary rigidity problem for holomorphic mappings on some weakly pseudoconvex domains. *Can. J. Math.* **47**, 405–420 (1995)
50. Huang, X.: A preservation principle of extremal mappings near a strongly pseudoconvex point and its applications. *Ill. J. Math.* **38**(2), 283–302 (1994)
51. Huang, X.: Some applications of Bell’s theorem to weakly pseudoconvex domains. *Pac. J. Math.* **158**, 305–315 (1993)
52. Huang, X., Pan, Y.: Proper holomorphic mappings between real analytic domains in \mathbb{C}^n . *Duke Math. J.* **82**, 437–446 (1996)
53. Juhlin, R.: Determination of formal CR mappings by a finite jet. *Adv. Math.* **222**, 1611–1648 (2009)
54. Juhlin, R., Lamel, B.: Automorphism groups of minimal real-analytic CR manifolds. *J. Eur. Math. Soc.* **15**, 509–537 (2013)
55. Kim, S.-Y., Zaitsev, D.: Equivalence and embedding problems for CR-structures of any codimension. *Topology* **44**, 557–584 (2005)
56. Kolar, M., Lamel, B.: Holomorphic Equivalence and Nonlinear Symmetries of Ruled Hypersurfaces in \mathbb{C}^2 . *J. Geom. Anal.* **25**(2), 1240–1281 (2015)
57. Kolar, M., Meylan, F., Zaitsev, D.: Chern-Moser operators and polynomial models in CR geometry. *Adv. Math.* **263**, 321–356 (2014)
58. Kossovskiy, I., Lamel, B., Stolovitch, L.: Equivalence of three-dimensional Cauchy-Riemann manifolds and multisummability theory. *Adv. Math.* **397**, (2022), Paper no. 108117, 42 pp
59. Kowalski, R.T.: A hypersurface in \mathbb{C}^2 whose stability group is not determined by 2-jets. *Proc. Am. Math. Soc.* **130**, 3679–3686 (2002)
60. Lamel, B.: Explicit bounds for the finite jet determination problem. *Trans. Am. Math. Soc.* **358**, 3447–3457 (2006)
61. Lamel, B.: Holomorphic maps of real submanifolds in complex spaces of different dimensions. *Pac. J. Math.* **201**(2), 357–387 (2001)
62. Lamel, B.: Local automorphisms of the Hilbert ball. *Proc. Am. Math. Soc.* **136**(8), 2815–2822 (2008)
63. Lamel, B., Mir, N.: Parametrization of local CR automorphisms by finite jets and applications. *J. Am. Math. Soc.* **20**, 519–572 (2007)
64. Lamel, B., Mir, N.: Finite jet determination of CR mappings. *Adv. Math.* **216**, 153–177 (2007)
65. Lamel, B., Mir, N.: On the \mathbb{C}^∞ regularity of CR mappings of positive codimension. *Adv. Math.* **335**, 696–734 (2018)
66. Lamel, B., Mir, N.: Convergence and divergence of formal CR mappings. *Acta Math.* **220**, 367–406 (2018)
67. Lamel, B., Mir, N.: Regularity of CR mappings of abstract CR structures. *Intern. J. Math.* **31**(01), (2020)
68. Lamel, B., Mir, N.: The finite jet determination problem for CR maps of positive codimension into Nash manifolds. *Proc. Lond. Math. Soc.* **124**(3), 737–771 (2022)
69. Lempert, L.: La métrique de Kobayashi et la représentation des domaines sur la boule. *Bull. Soc. Math. Fr.* **109**, 427–474 (1981)
70. Mir, N.: Holomorphic deformations of real-analytic CR maps and analytic regularity of CR mappings. *J. Geom. Anal.* **27**, 1920–1939 (2017)
71. Mir, N., Zaitsev, D.: Unique jet determination and extension of germs of CR maps into spheres. *Trans. Am. Math. Soc.* **374**, 2149–2166 (2021)
72. Poincaré, H.: Les fonctions analytiques de deux variables et la représentation conforme. *Rend. Circ. Mat. Palermo* **II**(23), 185–220 (1907)
73. Rudin, W.: *Function Theory in the Unit Ball of \mathbb{C}^n* . Reprint of the 1980 edition. *Classics in Mathematics*. Springer, Berlin (2008)
74. Stanton, N.: Infinitesimal CR automorphisms of real hypersurfaces. *Am. J. Math.* **118**, 209–233 (1996)
75. Tanaka, N.: On generalized graded Lie algebras and geometric structures I. *J. Math. Soc. Jpn.* **19**, 215–254 (1967)
76. Tumanov, A.: Stationary discs and finite jet determination of CR mappings in higher codimension. *Adv. Math.* **371**, 107254 (2020)
77. Tumanov, A.: Extremal discs and the regularity of CR mappings in higher codimension. *Am. J. Math.* **123**(3), 445–473 (2001)
78. Zaitsev, D.: Germs of local automorphisms of real-analytic CR structures and analytic dependence on k -jets. *Math. Res. Lett.* **4**, 823–842 (1997)
79. Zaitsev, D.: Unique determination of local CR-maps by their jets: a survey. In: *Harmonic Analysis on Complex Homogeneous Domains and Lie Groups* (Rome, 2001), *Atti. Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei* (9) *Mat. Appl.*, vol. 13, pp. 295–305 (2002)
80. Zimmer, A.: Two boundary rigidity results for holomorphic maps. *Am. J. Math.* **144**, 119–168 (2022)