

## Regularity of CR mappings of abstract CR structures

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We study the  $\mathcal{C}^\infty$  regularity problem for CR maps from an abstract CR manifold  $M$  into some complex Euclidean space  $\mathbb{C}^{N'}$ . We show that if  $M$  satisfies a certain condition called the microlocal extension property, then any  $\mathcal{C}^k$ -smooth CR map  $h: M \rightarrow \mathbb{C}^{N'}$ , for some integer  $k$ , which is nowhere  $\mathcal{C}^\infty$ -smooth on some open subset  $\Omega$  of  $M$ , has the following property: for a generic point  $q$  of  $\Omega$ , there must exist a formal complex subvariety through  $h(q)$ , tangent to  $h(M)$  to infinite order, and depending in a  $\mathcal{C}^1$  and CR manner on  $q$ . As a consequence, we obtain several  $\mathcal{C}^\infty$  regularity results generalizing earlier ones by Berhanu–Xiao and the authors (in the embedded case).

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### 1. Introduction and Results for CR Structures of Hypersurface Type

The purpose of this paper is to extend our recent study [16] of the regularity problem for CR mappings between smooth CR submanifolds of  $\mathbb{C}^N$  and  $\mathbb{C}^{N'}$ , respectively, to the case of maps from an abstract CR manifold with values in some complex Euclidean space  $\mathbb{C}^{N'}$ . More precisely, the question we are interested in is the following. Given an abstract CR manifold  $M$  and a  $\mathcal{C}^\infty$ -smooth CR manifold  $M' \subset \mathbb{C}^{N'}$ , under which conditions can we guarantee that there exists an integer  $k$  such that

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for any  $\mathcal{C}^k$ -smooth CR map  $h: M \rightarrow M'$  is actually  $\mathcal{C}^\infty$ -smooth on some open, dense, subset of  $M$ ? We refer the reader to the survey paper by Forstnerič [11] and to our recent paper [16] for a discussion explaining why this problem is a natural problem to study.

In this paper, we shall answer the question in a way similar to our approach for the regularity of CR maps between CR submanifolds, which built on our work on the convergence of formal CR maps [15]. This means that in order to understand under which circumstances we can expect regularity, we are actually going to study consequences of irregularity. We shall show that, under a condition on  $M$  called the *microlocal extension property*, if a  $\mathcal{C}^k$ -smooth CR map  $h: M \rightarrow M'$ , for some integer  $k$ , is nowhere  $\mathcal{C}^\infty$ -smooth on some open subset  $\Omega$  of  $M$ , then for a generic point  $q \in \Omega$ , there must exist a formal complex subvariety through  $h(q)$ , tangent to  $M'$  to infinite order, and depending in a  $\mathcal{C}^1$  and CR manner on  $q$ . In particular, the absence of such formal complex subvarieties provides an obstruction to irregularity, and therefore, one obtains that, under these circumstances, such a map  $h$  is necessarily smooth on an open, dense subset of  $M$ .

Our results generalize and recover those obtained in [16] in the embedded case, as well as those by Berhanu and Xiao [3, 4] in which they tackled the case of abstract CR manifolds  $M$  with Levi-nondegenerate real hypersurfaces  $M'$  as targets. Although the general approach taken in this paper follows the same philosophy as that of [16], we have to overcome some new difficulties as our source manifolds are general abstract CR structures, and therefore not necessarily embeddable into some complex Euclidean space (see e.g. [1, 2]). In particular, tools such as holomorphic extension of CR functions into wedges, which were crucial throughout the whole construction and proofs given in [16], are no longer available for non-embeddable CR manifolds. At the end of Sec. 2, we explain in more details how we carry over the strategy developed in [16] to this more general setting.

We now briefly recall some basic notions in order to state, at first, our results for abstract CR manifolds of hypersurface type.

An abstract CR manifold  $(M, \mathcal{V})$  is a  $\mathcal{C}^\infty$ -smooth manifold  $M$  together with a complex subbundle  $\mathcal{V} \subset \mathbb{C}TM$ , called the CR bundle, such that

$$\mathcal{V} \cap \bar{\mathcal{V}} = \{0\}, \quad [\mathcal{V}, \mathcal{V}] \subset \mathcal{V}. \tag{1.1}$$

We will assume that  $M$  is connected, and write  $\dim_{\mathbb{R}} M = 2n + d$ , where  $n = \dim_{\mathbb{C}} \mathcal{V} = \dim_{\mathbb{C}R} M$  and  $d = \text{codim}_{\mathbb{C}R} M$ . The second condition in (1.1) is often referred to as *formal integrability*, and is a shorthand notation for the fact that the Lie bracket of two CR vector fields (i.e.  $\mathcal{C}^\infty$ -sections of  $\mathcal{V}$ , the set of which we will denote by  $\Gamma(M, \mathcal{V})$ ) is again a CR vector field: For all  $X, Y \in \Gamma(M, \mathcal{V})$ , it holds that  $[X, Y] \in \Gamma(M, \mathcal{V})$ . One can, equivalently, introduce an abstract CR manifold by a subbundle  $T^c M \subset TM$  and a complex structure operator  $J: T^c M \rightarrow T^c M$ , and we shall use both descriptions in what follows.

When  $d = 1$ ,  $M$  is an abstract CR manifold of *hypersurface type*. We recall that an abstract CR manifold  $M$  of hypersurface type is *strongly pseudoconvex* if

its Levi form (see Sec. 2 for the definition) is either positive definite at all points or negative definite at all points.

We define the characteristic bundle  $T^0M \subset T^*M$  as the set of (real) forms annihilating  $\mathcal{V}$  and  $\bar{\mathcal{V}}$ , and the holomorphic cotangent bundle  $T'M := \mathcal{V}^\perp \subset \mathbb{C}T^*M$ . We usually write  $N = \dim_{\mathbb{C}} T'M = n + d$ . If  $(M, \mathcal{V})$  and  $(M', \mathcal{V}')$  are abstract CR manifolds, and  $h : M \rightarrow M'$  is a CR map of class  $\mathcal{C}^1$ , we say that  $h$  is *strictly noncharacteristic* if for every  $p \in M$ ,

$$h^*(T_p^0M') = T_p^0M.$$

We should mention that the notion of strictly noncharacteristic map coincides with the well-known condition of CR transversality (see e.g. [10, 13]) when  $M$  and  $M'$  have the same CR codimension (see [16]). If  $h$  is as above, the singular support of  $h$ , denoted  $\text{SingSupp } h$ , is the locus of points  $p$  in  $M$  such that  $h$  is not  $\mathcal{C}^\infty$ -smooth in any neighborhood of  $p$ .

In order to state our first main result, we shall briefly recall an extension of the notion of finite type which was used in [16], building on the original concept introduced by D'Angelo [6]. If  $X \subset \mathbb{C}^{N'}$  is a set, we denote the ideal of germs at  $q \in X$  of smooth functions vanishing along  $X$  by  $\mathcal{I}_q(X) = \{\varphi \in \mathcal{C}^\infty(\mathbb{C}^{N'}, q) : \varphi|_X = 0\}$ . For  $\psi \in \mathcal{C}^\infty(\mathbb{C}, 0)$ , denote by  $\nu_0\psi$  the order of vanishing of  $\psi$  at 0. For  $p \in X$ , we define the 1-type of  $X$  at  $p$  as

$$\Delta(X, p) = \sup_{\substack{\gamma : \Delta \rightarrow \mathbb{C}^{N'} \\ \gamma(0) = p}} \left( \inf_{\rho \in \mathcal{I}_q(X)} \frac{\nu_0(\rho \circ \gamma)}{\nu_0(\gamma)} \right) \in [0, \infty], \tag{1.2}$$

where the supremum is taken over all *holomorphic curves*  $\gamma : \Delta = \{\zeta \in \mathbb{C} : |\zeta| < 1\} \rightarrow \mathbb{C}^{N'}$ . We say that  $p$  is a *D'Angelo finite-type point* of  $X$  if  $\Delta(X, p) < \infty$ , and a *D'Angelo infinite-type point* of  $X$  otherwise. We denote the set of infinite-type points in  $X$  by  $\mathcal{E}_X$  and recall that, in the case where  $X$  is a smooth real hypersurface, then  $\mathcal{E}_X$  is closed in  $X$  by [7, 8].

Our main result for abstract CR manifolds of hypersurface type is the following.

**Theorem 1.1.** *Let  $M$  be an abstract strongly pseudoconvex CR manifold of hypersurface type, of CR dimension  $n$ ,  $M' \subset \mathbb{C}^{n'+1}$  be a  $\mathcal{C}^\infty$ -smooth real hypersurface,  $n' > n \geq 1$ , and let  $h : M \rightarrow M'$  be a strictly noncharacteristic CR map of class  $\mathcal{C}^{n'-n+1}$ . Then  $h((\text{SingSupp } h)^\circ) \subset \mathcal{E}_{M'}$ .*

When  $\mathcal{E}_{M'} = \emptyset$ , we obtain as an immediate consequence the following regularity result.

**Theorem 1.2.** *Let  $M$  be an abstract strongly pseudoconvex CR manifold of hypersurface type, of CR dimension  $n$ , and  $M' \subset \mathbb{C}^{n'+1}$  be a  $\mathcal{C}^\infty$ -smooth real hypersurface of D'Angelo finite type,  $n' > n \geq 1$ . If  $h : M \rightarrow M'$  is a strictly noncharacteristic CR map of class  $\mathcal{C}^{n'-n+1}$ , then  $(\text{SingSupp } h)^\circ = \emptyset$ , i.e.  $h$  is  $\mathcal{C}^\infty$ -smooth on a dense open subset of  $M$ .*

Note that it follows from the proof that any map as in Theorem 1.1 must be automatically CR immersive and hence Theorem 1.2 appears also as a regularity

result of CR embeddings. Our approach in this paper will allow us to get more general versions of Theorems 1.1 and 1.2 for CR manifolds of arbitrary CR codimension (see Sec. 2).

## 2. Statement of Results for Abstract CR Manifolds of Any CR Codimension

The results in Sec. 1 follow from more general results that we shall now describe. To this end, we first introduce some notation to be used throughout the paper, and define a number of notions (some of them are not necessarily standard).

Let  $(M, \mathcal{V})$  be an abstract CR manifold of CR dimension  $n$  and CR codimension  $d$ . For every  $p \in M$ , the Levi map of  $M$  is the (vector-valued) Hermitian form

$$\mathcal{L} : \mathcal{V}_p \times \mathcal{V}_p \rightarrow \mathbb{C}T_pM / \mathcal{V}_p \oplus \bar{\mathcal{V}}_p$$

defined by

$$\mathcal{L}_p(X_p, Y_p) = [X, \bar{Y}]_p \quad \text{mod } \mathcal{V}_p \oplus \bar{\mathcal{V}}_p. \tag{2.1}$$

In (2.1), the definition of the Levi form  $\mathcal{L}_p$  is independent of the choice of the vectors fields  $X$  and  $Y$  extending  $X_p$  and  $Y_p$  in a neighborhood of  $p$ .

We say that  $M$  is *Levi-nondegenerate* if for every  $p \in M$ ,  $\mathcal{L}_p(X_p, Y_p) = 0$  for all  $Y_p \in \mathcal{V}_p$  implies that  $X_p = 0$ . If  $M$  is of hypersurface type, then the Levi map is a Hermitian form and we say that  $M$  is *strongly pseudoconvex* if the Levi form  $\mathcal{L}_p$  is positive definite at every  $p$  (or negative definite at every  $p$ ).

If  $(M, \mathcal{V})$  and  $(M', \mathcal{V}')$  are abstract CR manifolds, and  $h : M \rightarrow M'$  is a map of class  $\mathcal{C}^1$ , then we say that  $h$  is CR provided that  $dh(\mathcal{V}) \subset \mathcal{V}'$ , or, equivalently, if  $h^*(T'M') \subset T'M$  (we again abuse notation by identifying bundles with sections here). If  $M' = \mathbb{C}^{N'}$ ,  $h = (h_1, \dots, h_{N'})$  is CR if and only if each  $h_j$  a CR function on  $M$  (for details, see e.g. [1]). We shall denote by  $\Gamma_p(M)$  the set of all germs at  $p$  of CR vector fields of  $M$ .

Now we come to a notion that will be important throughout the remainder of this paper.

**Definition 2.1.** Let  $M$  be an abstract CR manifold and  $p \in M$ . We say that  $M$  satisfies the *microlocal extension property* at  $p$ , if, for every neighborhood  $\Omega$  of  $p$ , there is a (nonempty) open convex cone  $\Gamma \subset T_p^0M$  such that for every continuous CR function  $u$  on  $\Omega$ , we have  $\text{WF}(u)|_p \subset \Gamma$ . We further say that  $M$  satisfies the microlocal extension property if it satisfies the microlocal extension property at every point of  $M$ .

We are going to recall the classical notion of wavefront set  $\text{WF}(u)$  in Definition 4.1. In Sec. 4, we will discuss the microlocal extension property in more detail, exhibit important instances of CR manifolds for which this property is satisfied, and relate it to *almost analytic extendability*. The reader should compare this notion with the work of Berhanu and Xiao [3, 4]. We also note that we require that

the (usual) wavefront set of a CR function/distribution in  $\Omega$  is *a priori* contained in a convex cone, which however is allowed to change as  $\Omega$  changes; this is a bit in contrast with the hypoanalytic wavefront set in the embedded setting (for this concept, we refer the reader to [23]).

Recall that for a subset  $X \subset \mathbb{C}^{N'}$ , and for every  $q \in X$ , we denote by  $\mathcal{I}_q(X) \subset \mathcal{C}^\infty(\mathbb{C}^{N'}, q)$  the ideal of all germs at  $q$  of  $\mathcal{C}^\infty$ -smooth functions  $\rho: (\mathbb{C}^{N'}, q) \rightarrow \mathbb{C}$  that vanish on  $X$  near  $q$ . For  $r \in \{1, \dots, N'\}$ , we define the *regular  $r$ -type* of  $X \subset \mathbb{C}^{N'}$  at  $q$  as follows:

$$\Delta_r(X, q) = \sup_{\substack{\alpha: \Delta^r \rightarrow \mathbb{C}^{N'} \\ \alpha(0)=q, \text{rk } \alpha'(0)=r}} \left( \inf_{\rho \in \mathcal{I}_q(X)} \nu_0(\rho \circ \alpha) \right) \in \mathbb{Z}_+ \cup \{\infty\}. \tag{2.2}$$

Here,  $\Delta^r = \{t = (t_1, \dots, t_r) \in \mathbb{C}^r : |t_j| < 1, j = 1, \dots, r\}$  and the supremum is taken over all holomorphic maps  $\alpha: \Delta^r \rightarrow \mathbb{C}^{N'}$  which are of full rank  $r$  at 0 and satisfy  $\alpha(0) = q$ . Note the inequalities

$$\Delta_{N'}(X, q) \leq \Delta_{N'-1}(X, q) \leq \dots \leq \Delta_1(X, q) \leq \Delta(X, q),$$

where  $\Delta(X, q)$  is the 1-type already defined in (1.2). We shall say that  $q$  is of  *$r$ -regular infinite-type* if  $\Delta_r(X, q) = \infty$ , and we denote the set of points in  $X$  which are of  $r$ -regular infinite-type by  $\mathcal{E}_X^r$ . We therefore have

$$\mathcal{E}_X^{N'} \subset \mathcal{E}_X^{N'-1} \subset \dots \subset \mathcal{E}_X^1 \subset \mathcal{E}_X.$$

A formal holomorphic subvariety  $\Gamma \subset \mathbb{C}^{N'}$  through a point  $p \in \mathbb{C}^{N'}$  is given by a (radical) ideal  $I_p(\Gamma) \subset \mathbb{C}[[Z' - p]]$ . We say that a formal holomorphic subvariety  $\Gamma \subset \mathbb{C}^{N'}$  through the point  $p$  is *formally contained* in  $X$  at  $p \in X$  if for every  $\mathbb{C}^{N'}$ -valued formal power series  $\varphi(t)$ ,  $t \in \mathbb{C}$ , with  $\varphi(0) = p$ , one has

$$\psi \circ \varphi(t) = 0, \quad \forall \psi \in I_p(\Gamma) \Rightarrow \nu_0(\varrho(\varphi(t), \overline{\varphi(t)})) = \infty, \quad \forall \varrho \in \mathcal{I}_p(X).$$

It follows from this definition that if there exists a nontrivial formal subvariety through  $p$  which is formally contained in  $X$ , then  $p$  is a D'Angelo infinite-type point. If there exists a formal (holomorphic) submanifold of dimension  $r$  through  $p$  which is formally contained in  $X$ , then  $p \in \mathcal{E}_X^r$ .

The next definition introduces the key geometrical concept to be used in our main theorem.

**Definition 2.2.** Let  $M$  be an abstract CR manifold and  $h: M \rightarrow \mathbb{C}^{N'}$  be a  $\mathcal{C}^1$ -smooth CR map. We say that  $(\Gamma_q)_{q \in M}$  is a CR family of  $r$ -dimensional formal (holomorphic) submanifolds through  $h(M)$  if for every  $p \in M$  we can find a neighborhood  $U$  of  $p$  such that there exists a map  $\psi: U \rightarrow (\mathbb{C}[[t_1, \dots, t_r]])^{N'}$  such that for  $\xi \in U$ , we can write a parametrization of  $\Gamma_\xi$  in the form

$$\psi(\xi): (\mathbb{C}_t^r, 0) \ni t \mapsto h(\xi) + \sum_{\alpha \in \mathbb{N}^r} \psi_\alpha(\xi) t^\alpha,$$

where each  $\psi_\alpha: U \rightarrow \mathbb{C}^{N'}$  is a  $\mathcal{C}^1$ -smooth CR map and  $\partial_t(\psi(\xi))(0)$  is of rank  $r$  for all  $\xi \in U$ .

For every  $p \in M$ , and every CR map  $h : M \rightarrow \mathbb{C}^{N'}$  of class  $\mathcal{C}^k$ , we define the following numerical invariant:

$$e_k(p) := \dim_{\mathbb{C}} \text{span}\{\bar{L}_1 \dots \bar{L}_j \rho_w(h(p), \overline{h(p)}) : \rho \in \mathcal{S}_{h(M)}(h(p)), \bar{L}_1, \dots, \bar{L}_j \in \Gamma_p(M), 0 \leq j \leq k\}. \tag{2.3}$$

The complex gradients

$$\rho_w(h(p), \overline{h(p)}) = \left( \frac{\partial \rho}{\partial w_1}(h(p), \overline{h(p)}), \dots, \frac{\partial \rho}{\partial w_{N'}}(h(p), \overline{h(p)}) \right),$$

which correspond to  $j = 0$  in (2.3), and their CR derivatives

$$\begin{aligned} & \bar{L}_1 \dots \bar{L}_j \rho_w(h(p), \overline{h(p)}) \\ &= \left( \bar{L}_1 \dots \bar{L}_j \frac{\partial \rho}{\partial w_1}(h(p), \overline{h(p)}), \dots, \bar{L}_1 \dots \bar{L}_j \frac{\partial \rho}{\partial w_{N'}}(h(p), \overline{h(p)}) \right), \end{aligned}$$

are considered as vectors in  $\mathbb{C}^{N'}$ . For  $k \geq 0$ , the function  $p \mapsto e_k(p)$  is integer valued and lower semicontinuous, and obviously  $e_k(p) \leq N'$  for every  $p \in M$ . We define

$$e_k := \max\{\ell \in \mathbb{Z}_+ : e_k(p) \geq \ell \text{ for } p \text{ on some dense subset of } M\}.$$

We may now state our most general result.

**Theorem 2.3.** *Let  $M$  be an abstract CR manifold with the microlocal extension property, and assume that  $k, \ell \in \mathbb{N}$  are given satisfying  $0 \leq k \leq \ell \leq N'$  and  $N' - \ell + k \geq 1$ . Let  $h : M \rightarrow \mathbb{C}^{N'}$  be a CR mapping of class  $\mathcal{C}^{N'-\ell+k}$ , and assume that  $e_k \geq \ell$ .*

*Then there exists a dense open subset  $\Omega$  of  $(\text{SingSupp } h)^\circ$  which can be decomposed into disjoint open sets,*

$$\Omega = \bigsqcup_{r=1}^{N'-\ell} \Omega_r,$$

*and for each  $r = 1, \dots, N' - \ell$ , a CR family  $(\Gamma_\xi^r)_{\xi \in \Omega_r}$  of  $r$ -dimensional formal submanifolds through  $h(\Omega_r)$  such that each  $\Gamma_\xi^r$  is formally contained in  $h(M)$ . In particular, it holds that  $h(\Omega_r) \subset \mathcal{E}_{h(M)}^r$  and hence  $h(\Omega) \subset \mathcal{E}_{h(M)}$ .*

We will see later (in view of Proposition 4.5) that Theorem 2.3 recovers [16, Theorem 2.2] when  $M$  is embedded.

By putting various geometric assumptions on  $M$  and  $h(M)$ , we may use Theorem 2.3 to get different types of (less technical) statements. We shall illustrate this by deriving several regularity results in different contexts.

For maps with no specific rank assumption, Theorem 2.3 provides the following.

**Corollary 2.4.** *Let  $M$  be an abstract CR manifold with the microlocal extension property,  $M' \subset \mathbb{C}^{N'}$  be a  $\mathcal{C}^\infty$ -smooth CR submanifold of CR dimension  $n' \geq 1$ . Then for every CR map  $h : M \rightarrow M'$  of class  $\mathcal{C}^{n'}$ , there is a dense open subset  $\Omega$*

of  $(\text{SingSupp } h)^\circ$  such that  $h(\Omega) \subset \mathcal{E}_{M'}$ . In particular, if  $M'$  is of D'Angelo finite type,  $h$  must be  $\mathcal{C}^\infty$ -smooth on a dense open subset of  $M$ .

For strictly noncharacteristic CR maps, we may assume less initial regularity than in Corollary 2.4 to boost higher regularity.

**Corollary 2.5.** *Let  $M$  be a Levi-nondegenerate abstract CR manifold with the microlocal extension property, of CR dimension  $n$  and CR codimension  $d$ , and  $M' \subset \mathbb{C}^{N'}$  be a  $\mathcal{C}^\infty$ -smooth CR submanifold with  $N' > N = n + d$ . Then for every strictly noncharacteristic CR map  $h: M \rightarrow M'$  of class  $\mathcal{C}^{N'-N+1}$ , there is a dense open subset  $\Omega$  of  $(\text{SingSupp } h)^\circ$  such that  $h(\Omega) \subset \mathcal{E}_{M'}$ . In particular, if  $M'$  is of D'Angelo finite type,  $h$  must be  $\mathcal{C}^\infty$ -smooth on a dense open subset of  $M$ .*

Observe that Corollary 2.5 can be seen as a generalization of Theorem 1.1 for CR manifolds of arbitrary CR codimension.

For CR immersions (that are not necessarily strictly noncharacteristic), Theorem 2.3 yields the following variant of Corollary 2.5.

**Corollary 2.6.** *Let  $M$  be an abstract CR manifold with the microlocal extension property,  $M' \subset \mathbb{C}^{N'}$  be a Levi-nondegenerate  $\mathcal{C}^\infty$ -smooth CR submanifold, of CR dimension  $n$  and  $n'$ , respectively. Then for every CR immersion  $h: M \rightarrow M'$  of class  $\mathcal{C}^{n'-n+1}$ , there is a dense open subset  $\Omega$  of  $(\text{SingSupp } h)^\circ$  such that  $h(\Omega) \subset \mathcal{E}_{M'}$ . In particular, if  $M'$  is of D'Angelo finite type,  $h$  must be  $\mathcal{C}^\infty$ -smooth on a dense open subset of  $M$ .*

In fact, it follows from the proof of Corollary 2.6 that the result also holds for CR maps whose differential is injective on  $T^{1,0}M$ . Hence, Corollary 2.6 recovers an earlier result by Berhanu–Xiao [3, Theorem 2.5] for strongly pseudoconvex real hypersurfaces  $M'$  as targets.

In the previous results,  $\mathcal{C}^\infty$  regularity of the maps follows automatically once the target manifold is of D'Angelo finite type. However, Theorem 2.3 can also be used to establish  $\mathcal{C}^\infty$  regularity results when the target manifold is everywhere of D'Angelo infinite-type. We will illustrate this by showing how the following other result due to Berhanu–Xiao [4] may also be derived from Theorem 2.3.

**Corollary 2.7.** *Let  $M$  be an abstract CR manifold of hypersurface type,  $M' \subset \mathbb{C}^{n'+1}$  be a  $\mathcal{C}^\infty$ -smooth (connected) real hypersurface, both Levi-nondegenerate. Denote by  $n$  the CR dimension of  $M$  and by  $\ell' > 0$  the signature of  $M'$ , with  $n' > n \geq 1$ . If  $\max(n' - \ell', \ell') \leq n$ , then every strictly noncharacteristic CR map  $h: M \rightarrow M'$ , of class  $\mathcal{C}^{n'-n+1}$ , is  $\mathcal{C}^\infty$ -smooth on some dense open subset of  $M$ .*

Recall here that the signature of  $M'$  is the minimum of the numbers of positive and negative eigenvalues of the Levi form at an arbitrary point of  $M'$ .

The reader can further exploit Theorem 2.3 to derive, in the spirit of [16], more applications for target manifolds, both of D'Angelo finite and infinite-type.

Let us discuss now the ingredients of the proof of the main result, Theorem 2.3, as well as the organization of the paper. To a given CR map  $h: M \rightarrow \mathbb{C}^{N'}$  (of a

certain *a priori* smoothness), we associate a disjoint union of open subsets of  $M$  (Sec. 5). Each open subset  $\omega$  in the obtained decomposition satisfies the following alternative: it is either contained in  $M \setminus (\text{SingSupp } h)$  (Proposition 5.5) or has the property that for every point  $q \in \omega$ , there is a formal holomorphic submanifold (of fixed positive dimension) through  $h(q)$  that is formally contained in  $h(M)$  (Proposition 5.6). Since the union of such open subsets happens to be dense in  $M$ , this roughly proves Theorem 2.3. The open subsets decomposition is constructed through the introduction of numerical invariants associated to rings of functions attached to the map  $h$ . This strategy is analogous to that carried out in [16] in the case where  $M$  is embedded. However, we should point out that the open subset decomposition in [16] uses heavily the minimality assumption on the embedded manifold  $M$  (and Tumanov's extension theorem) and therefore cannot be applied in the abstract case tackled in this paper. We instead proceed with a different construction of the rings and invariants attached to the map leading to the desired open subset decomposition. Our present construction, though still similar in spirit with that of [16], has the advantage to make no assumption on  $M$ , and hence, is more general than the one given in [16], even in the embedded case. Furthermore, in order to prove Proposition 5.5, we also need to establish a smooth version of the reflection principle of [16, Theorem 3.1] adapted to abstract CR structures. This is achieved in Theorem 4.8 and Corollary 4.10 where the *microlocal extension property* of  $M$  comes into play. In Sec. 4, we prove Theorem 4.8 and discuss in detail the microlocal extension property. Using [3, Theorem 2.9], we give instances of abstract as well as embeddable CR manifolds satisfying this condition, showing in particular that the results of this paper recover those of Berhanu–Xiao [3, 4] in the abstract case, and those of the authors [16] in the embedded case. We also relate the microlocal extension property to the notion of almost analytic extension, whose basic properties are recalled in Sec. 3. The proofs of Theorem 2.3 and its consequences are finalized in Sec. 6.

### 3. Almost Analytic Extensions, Wedges and Boundary Values

In this section, we recall some standard facts about almost analytic extensions on wedges and boundary values, which will be useful when discussing the microlocal analytic extension property in Sec. 4. We also prove in this section (Proposition 3.5) a Hölder regularity result for  $\bar{\partial}$ -bounded extensions of Hölder continuous boundary values on some wedges, which will be used in the proof of the smooth reflection principle given in Theorem 4.8.

#### 3.1. Almost analytic extensions

We start the section by recalling that any smooth function possesses an *almost analytic extension*. This fact is commonly attributed to Nirenberg [21], and we also refer the reader to the paper of Dyn'kin [9].



**Proposition 3.1.** *Assume that  $u \in \mathcal{C}^\infty(\mathbb{R}_s^d)$ . There exists a function  $U \in \mathcal{C}^\infty(\mathbb{R}_s^d + i\mathbb{R}_t^d)$  such that for every compact set  $K \subset \mathbb{R}^d$ , every integer  $a \in \mathbb{N}$  and every multi-index  $\alpha \in \mathbb{N}^d$ , there exists a constant  $C = C(K, a, \alpha) > 0$  such that*

$$U(s, 0) = u(s), \quad \left| \frac{\partial^{|\alpha|}}{\partial s^\alpha} \left( \frac{\partial U}{\partial \bar{\sigma}_j} \right) (s, t) \right| \leq C \|t\|^a, \quad s \in K, \quad t \in \mathbb{R}^d, \quad j = 1, \dots, d.$$

In the above result and in what follows, we use the standard notation  $\frac{\partial}{\partial \bar{\sigma}} = \frac{1}{2}(\frac{\partial}{\partial s} + i\frac{\partial}{\partial t})$  for the CR operators on  $\mathbb{R}^{2d}$  associated to the complex coordinates  $\sigma = s + it$ .

Let  $D \subset \mathbb{C}_z^n \times \mathbb{R}_s^d$  be a fixed open subset. For a cone  $\Gamma \subset \mathbb{R}_t^d \setminus \{0\}$  (with vertex the origin) and  $r > 0$ , we denote by  $\Gamma_r = \{t \in \Gamma : \|t\| < r\}$ . Consider now an open cone  $\Gamma \subset \mathbb{R}^d$  and a set of the form

$$W_r := D + i\Gamma_r = \{(z, s + it) \in D + i\Gamma \subset \mathbb{C}^n \times \mathbb{C}^d : \|t\| < r\},$$

and define the set  $\mathfrak{B}(W_r)$  to consist of all functions  $U \in \mathcal{C}^1(W_r)$  which have the following properties:

- (a) For any compact set  $K \subset\subset D$  and any compactly contained subcone  $\Gamma' \subset\subset \Gamma$ , there is a constant  $C = C(K, \Gamma') > 0$  and a nonnegative integer  $k = k(K, \Gamma')$  such that

$$|U(z, \bar{z}, s, t)| \leq C \|t\|^{-k}, \quad (z, s) \in K, \quad t \in \Gamma'_r. \tag{3.1}$$

- (b) For  $j = 1, \dots, d$ ,  $\bar{\partial}_{\sigma_j} U$  is bounded on  $W_r$ , that is, for any  $K \subset\subset D$  and  $\Gamma' \subset\subset \Gamma$  as in (a), there is a constant  $C = C(K, \Gamma') > 0$  such that

$$\left| \frac{\partial U(z, \bar{z}, s, t)}{\partial \bar{\sigma}_j} \right| \leq C, \quad (z, s) \in K, \quad t \in \Gamma'_r, \quad j = 1, \dots, d. \tag{3.2}$$

As in [14], we introduce the families of functions that we are taking as “almost analytic extensions” as follows. We first define the set  $\mathfrak{A}(W_r)$  to consist of all functions  $U \in \mathcal{C}^\infty(W_r)$  with the property that for every  $\alpha, \beta \in \mathbb{N}^n$  and every  $\gamma \in \mathbb{N}^d$ ,

$$\frac{\partial^{|\alpha|+|\beta|+|\gamma|} U}{\partial z^\alpha \bar{z}^\beta s^\gamma} \in \mathfrak{B}(W_r).$$

In other words,  $U \in \mathfrak{A}(W_r)$  if for every compact subset  $K \subset D$ , subcone  $\Gamma' \subset \Gamma$  with  $\Gamma' \subset\subset \Gamma$ , every  $\alpha, \beta \in \mathbb{N}^n$ , every  $\gamma \in \mathbb{N}^d$ , there exists a  $k \in \mathbb{N}$ , and a constant  $C = C(K, \Gamma', \alpha, \beta, \gamma) > 0$  such that

$$\left| \frac{\partial^{|\alpha|+|\beta|+|\gamma|}}{\partial z^\alpha \bar{z}^\beta s^\gamma} U(z, \bar{z}, s, t) \right| \leq C \|t\|^{-k}, \quad (z, s) \in K, \quad t \in \Gamma'_r, \tag{3.3}$$

and such that

$$\left| \frac{\partial^{|\alpha|+|\beta|+|\gamma|}}{\partial z^\alpha \bar{z}^\beta s^\gamma} \frac{\partial}{\partial \bar{\sigma}_j} U(z, \bar{z}, s, t) \right| \leq C, \quad (z, s) \in K, \quad t \in \Gamma'_r, \quad j \in \{1, \dots, d\}. \tag{3.4}$$

If, furthermore,  $U \in \mathfrak{A}(W_r)$  has the property that, for every compact set  $K \subset D$ , every subcone  $\Gamma' \subset \Gamma$  with  $\Gamma' \subset\subset \Gamma$ , every  $\alpha, \beta \in \mathbb{N}^n$ , every  $\gamma \in \mathbb{N}^d$ , every  $a \in \mathbb{N}$ ,

there exists  $C_2 = C_2(K, a, \Gamma', \alpha, \beta, \gamma) > 0$  such that

$$\left| \frac{\partial^{|\alpha|+|\beta|+|\gamma|}}{\partial z^\alpha \bar{z}^\beta s^\gamma} \frac{\partial}{\partial \bar{\sigma}_j} U(z, \bar{z}, s, t) \right| \leq C_2 \|t\|^a, \quad (z, s) \in K, \quad t \in \Gamma'_r, \quad j \in \{1, \dots, d\}, \tag{3.5}$$

then we say that  $U \in \mathfrak{A}_\infty(W_r)$ .

**Definition 3.2.** We say that a distribution  $u$  defined on an open subset  $D \subset \mathbb{C}^n \times \mathbb{R}^d$  possesses a  $\bar{\partial}$ -bounded extension (respectively, a regular  $\bar{\partial}$ -bounded extension, respectively, an almost analytic extension) to

$$W_r = \{(z, s + it) : (z, s) \in D, t \in \Gamma, 0 < \|t\| < r\},$$

if there exists a function  $U \in \mathfrak{B}(W_r)$  (respectively,  $U \in \mathfrak{A}(W_r)$ , respectively,  $U \in \mathfrak{A}_\infty(W_r)$ ) with  $u = \text{bv}_{W_r} U$ . We also refer to such an  $U$  as a  $\bar{\partial}$ -bounded (respectively, a regular  $\bar{\partial}$ -bounded, respectively, an almost analytic) extension of  $u$  (to  $W_r$ ).

If a distribution  $u$  on  $D$  has one of the types of extensions introduced in Definition 3.2 to a function  $U \in \mathfrak{B}(W_r)$  (respectively,  $U \in \mathfrak{A}(W_r)$ , respectively,  $U \in \mathfrak{A}_\infty(W_r)$ ), for every  $0 < r' \leq r$ , we can also write  $u = \text{bv}_{W_{r'}} \tilde{U}$ , where  $\tilde{U}(z, s, t) = \chi(\|t\|)U(z, s, t)$  with  $\chi \in \mathcal{C}^\infty(\{\|t\| < r'\})$  satisfying  $\chi(\|t\|) = 1$  for  $2\|t\| < r'$ . We shall consequently drop the index  $r$  from consideration when appropriate.

Observe also that  $u$  possesses one of the extensions introduced in Definition 3.2 to  $D+i\Gamma$  if and only if  $\bar{u}$  possesses the same type of extension to  $D-i\Gamma = D+i(-\Gamma)$ .

**Remark 3.3.** We observe that  $\mathfrak{A}_\infty(W_r)$  is a subalgebra of  $\mathcal{C}^\infty(W_r)$  and therefore, distributions  $u_1, u_2$  which have almost analytic extensions  $U_1$  and  $U_2$ , respectively, can be multiplied by setting  $u_1 u_2 = \text{bv}(U_1 U_2)$ .

**Remark 3.4.** The preceding remark also shows that for a vector field  $X$  on  $D$ , whose coefficients are boundary value distributions of functions in  $\mathfrak{A}_\infty(W)$ , thus extending to a vector field  $X_+$  on  $W$ , for  $u = \text{bv}_W U$  with  $U \in \mathfrak{A}_\infty(W)$  the distribution  $Xu$  is defined, and  $Xu = \text{bv}_W(X_+U)$ . In particular, derivatives with respect to smooth vector fields of functions/distributions on  $D$  which extend almost analytically to  $W$  also extend almost analytically to  $W$ .

### 3.2. A priori regularity for $\bar{\partial}$ -bounded extensions

Our goal in this section is to prove a Hölder regularity result for extensions of Hölder continuous functions which are  $\bar{\partial}$ -bounded and whose derivative is of slow growth. We are following in our approach the paper of Coupet [5]. However, we need a slightly more general result than what is stated in [5], which we could not locate in the literature. We therefore include the details of the proof.

We first recall that a continuous function  $f : \Omega \rightarrow \mathbb{C}$  is Hölder continuous on a set  $\Omega \subset \mathbb{R}^d$  with Hölder exponent  $\alpha \in (0, 1]$  if there exists a constant  $C > 0$  such that

$$|f(x) - f(y)| \leq C \|x - y\|^\alpha.$$

The space of all Hölder continuous functions with Hölder exponent  $\alpha$  is denoted by  $\mathcal{C}^{0,\alpha}(\Omega)$ . If  $\Omega$  is compact, it becomes a Banach space if endowed with the norm

$$\|f\|_{0,\alpha} = \|f\|_\infty + \|f\|_\alpha,$$

where

$$\|f\|_\infty = \max_{x \in \Omega} |f(x)|, \quad \|f\|_\alpha = \max_{x \neq y \in \Omega} \frac{|f(x) - f(y)|}{\|x - y\|^\alpha}.$$

In what follows, we write  $\mathbb{C}_\sigma^d = \mathbb{R}_s^d + i\mathbb{R}_t^d$  and  $W = \mathbb{R}^d + i\Gamma$ , where  $\Gamma$  is an open convex cone linearly equivalent to  $\mathbb{R}_+^d := \{(t_1, \dots, t_d) \in \mathbb{R}^d : t_j > 0, \forall j\}$ .

**Proposition 3.5.** *Assume that  $h \in \mathcal{C}^1(W) \cap \mathcal{C}^0(\overline{W})$  has compact support in  $\overline{W}$  and satisfies on  $W$*

$$|(\bar{\partial}_{\sigma_j} h)(s, t)| \leq C, \quad |h_{s_j}(s, t)| \leq \frac{C}{\|t\|^k}, \quad j = 1, \dots, d,$$

for some constant  $C > 0$  and some integer  $k \geq 1$ . If  $h|_{t=0} \in \mathcal{C}^{0,\alpha}(\mathbb{R}^d)$  for some  $0 < \alpha < 1$  and  $\beta := \frac{\alpha}{\alpha+k}$ , then there exists a universal constant  $A > 0$ , depending only on  $\alpha$  and the support of  $h$  such that

$$h \in \mathcal{C}^{0,\beta}(W), \quad \text{with } \|h\|_\beta \leq A(C + \|h|_{t=0}\|_{0,\alpha}).$$

**Proof.** As in the aforementioned paper of Coupet (which treats the case  $k = 1$ ), we divide the proof in several steps. First note that without loss of generality, we may assume that  $\Gamma = \mathbb{R}_+^d$ . In each of the following steps, for  $\sigma = s + it$  and  $\sigma' = s' + it'$  with  $t, t' \in \Gamma$ , we estimate  $|h(\sigma) - h(\sigma')|$  satisfying different restrictions in each step.

**Step 1.**  $s = s', t = \lambda t', \lambda > 0$ . We consider the map  $\varphi : H_+ = \{\zeta \in \mathbb{C} : \text{Im } \zeta > 0\} \rightarrow W$  given by

$$\varphi(\zeta) = s + \zeta \frac{t}{\|t\|}.$$

Note that  $\varphi(i\|t\|) = \sigma$ ,  $\varphi(i\lambda\|t\|) = \sigma'$ . The function  $u = h \circ \varphi$  is defined on  $H_+$ , continuous up to  $\mathbb{R}$ , and  $u|_{\mathbb{R}} \in \mathcal{C}^{0,\alpha}$ . Furthermore,  $\frac{\partial u}{\partial \zeta}$  is bounded (by  $C$ ) on  $H_+$ , and  $C^1(H_+)$ . We can therefore apply the generalized Cauchy formula to write the function  $u$  as

$$u(\zeta) = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{u(x)}{x - \zeta} dx + \frac{1}{2\pi i} \int_{H_+} \frac{\partial u}{\partial \bar{z}}(z, \bar{z}) \frac{1}{z - \zeta} dz \wedge d\bar{z} = f_1(\zeta) + f_2(\zeta).$$

The first integral can be estimated via

$$|f_1(\zeta) - f_1(\zeta')| \leq C_1 \|u\|_{\mathbb{R}} \|0, \alpha\| |\zeta - \zeta'|^\alpha \leq C_1 \|h|_{t=0}\|_{0, \alpha} |\zeta - \zeta'|^\alpha,$$

with a universal constant  $C_1$ , by e.g. Muskhelishvili's book [20, Chap. 2, Paragraphs 19 and 22].

We estimate the second integral as follows. Let us assume that  $\text{supp } u \subset D_R := \{\zeta \in \mathbb{C} : |\zeta| < R\}$ . First note that for any  $r > 2$ , we have that

$$\int_{D_r} \frac{1}{|\xi(1 - \xi)|} dV(\xi) \leq \tilde{C}_2 \ln(r).$$

$$\begin{aligned} |f_2(\zeta) - f_2(\zeta')| &= \frac{1}{2\pi} \left| \int_{H_+ \cap D_R} \frac{\partial u}{\partial \bar{z}}(z) \frac{\zeta - \zeta'}{(z - \zeta)(z - \zeta')} dz \wedge d\bar{z} \right| \\ &\leq \frac{C}{\pi} \int_{H_+ \cap D_R} \left| \frac{\zeta - \zeta'}{(z - \zeta)(z - \zeta')} \right| dV(z) \\ \xi &= \frac{z - \zeta}{\zeta' - \zeta}, \quad dV(\xi) = \frac{dV(z)}{|\zeta - \zeta'|^2} \\ &= \frac{C}{\pi} |\zeta - \zeta'| \int_{D_{2R/|\zeta - \zeta'|}} \frac{1}{|\xi(1 - \xi)|} dV(\xi) \\ &\leq \tilde{C}_2 \ln(2R/|\zeta - \zeta'|) C |\zeta - \zeta'| \\ &\leq C_2 C |\zeta - \zeta'|^\alpha, \end{aligned}$$

where  $C_2$  depends on the support of  $u$  and  $\alpha$ , but not on  $u$ .

We combine the preceding estimates to obtain

$$\begin{aligned} |h(\sigma) - h(\sigma')| &= |u(i \|t\|) - u(i\lambda \|t\|)| \\ &\leq (C_1 \|h|_{t=0}\|_{0, \alpha} + C_2 C) \|t\| - \lambda \|t\|^\alpha \\ &\leq C_3 (\|h|_{t=0}\|_{0, \alpha} + C) \|\sigma - \sigma'\|^\alpha. \end{aligned} \tag{3.6}$$

**Step 2.**  $t = t'$ . First assume that  $\|s - s'\|^\frac{1}{\alpha+k} \leq \|t\|$ . Then, we use the mean value theorem to estimate

$$|h(\sigma) - h(\sigma')| \leq \frac{C}{\|t\|^k} \|s - s'\| \leq C \|s - s'\|^\frac{\alpha}{\alpha+k}. \tag{3.7}$$

If on the other hand,  $\|s - s'\|^\frac{1}{\alpha+k} > \|t\|$ , we set

$$\lambda = \frac{\|s - s'\|^\frac{1}{\alpha+k}}{\|t\|} > 1.$$

By Step 1, (3.6), we can estimate

$$\begin{aligned} |h(s + it) - h(s + i\lambda t)| &\leq C_3 (\|h|_{t=0}\|_{0, \alpha} + C) \|t\|^\alpha (\lambda - 1)^\alpha \\ &\leq C_3 (\|h|_{t=0}\|_{0, \alpha} + C) \|s - s'\|^\frac{\alpha}{\alpha+k}, \end{aligned}$$

as well as

$$|h(s' + it) - h(s' + i\lambda t)| \leq C_3(\|h|_{t=0}\|_{0,\alpha} + C) \|s - s'\|^{\frac{\alpha}{\alpha+k}}.$$

Therefore, using these estimates and (3.7), we have

$$\begin{aligned} |h(s + it) - h(s' + it)| &\leq |h(s + it) - h(s + i\lambda t)| + |h(s + i\lambda t) - h(s' + i\lambda t)| \\ &\quad + |h(s' + i\lambda t) - h(s' + it)| \\ &\leq 2C_3(\|h|_{t=0}\|_{0,\alpha} + C) \|s - s'\|^{\frac{\alpha}{\alpha+k}} + C \|s - s'\|^{\frac{\alpha}{\alpha+k}} \\ &\leq C_4(\|h|_{t=0}\|_{0,\alpha} + C) \|s - s'\|^{\frac{\alpha}{\alpha+k}}. \end{aligned}$$

Without loss of generality, we assume that  $C_4 > 1$ , to obtain now for all  $s, s'$  that

$$|u(s + it) - u(s' + it)| \leq C_4(\|h|_{t=0}\|_{0,\alpha} + C) \|s - s'\|^{\frac{\alpha}{\alpha+k}}. \tag{3.8}$$

**Step 3.**  $s = s'$ . As in [5], we define the points

$$P_1 = s + it, \quad P_2 = s + it^2, \dots, P_n = s + it^n, \quad P_{n+1} = s + it',$$

where

$$t^2 = \begin{pmatrix} t'_1 \\ t_2 \\ \vdots \\ t_n \end{pmatrix}, \quad t^3 = \begin{pmatrix} t'_1 \\ t'_2 \\ \vdots \\ t_n \end{pmatrix}, \dots, t^{n+1} = \begin{pmatrix} t'_1 \\ t'_2 \\ \vdots \\ t'_n \end{pmatrix},$$

and the functions  $h_j(\tilde{s} + i\tilde{t}) := h(s + iP_j + \tilde{s} + i\tilde{t})$ . Now note that for every  $j$ , by Step 2, the function  $h_j|_{\tilde{t}=0}$  is Höder regular of order  $\frac{\alpha}{\alpha+k}$ , with

$$\|h_j|_{\tilde{t}=0}\|_{0,\frac{\alpha}{\alpha+k}} \leq C_4(\|h|_{t=0}\|_{0,\alpha} + C).$$

We can apply Step 1 (note that  $h_j$  is also  $\bar{\partial}$ -bounded, with the same constant  $C$ ), with  $\alpha$  replaced by  $\frac{\alpha}{\alpha+k}$ , to obtain (if  $t'_j < t_j$  we can replace  $h_j$  by  $h_{j+1}$ )

$$\begin{aligned} |h(s + iP_j) - h(s + iP_{j+1})| &= |h_j(0) - h_j((t'_j - t_j)e_j)| \\ &\leq C_5(\|h|_{t=0}\|_{0,\alpha} + C) |t'_j - t_j|^{\frac{\alpha}{\alpha+k}}, \end{aligned}$$

so that

$$|h(s + it) - h(s + it')| \leq C_6(\|h|_{t=0}\|_{0,\alpha} + C) \|t - t'\|^{\frac{\alpha}{\alpha+k}}.$$

**Step 4.** We can now estimate

$$\begin{aligned} |h(\sigma) - h(\sigma')| &\leq |h(s + it) - h(s + it')| + |h(s + it') - h(s' + it')| \\ &\leq A(\|h|_{t=0}\|_{0,\alpha} + C) \|\sigma - \sigma'\|^{\frac{\alpha}{\alpha+k}}, \end{aligned}$$

which finishes the proof. □

#### 4. Abstract CR Manifolds and the Microlocal Extension Property

Our goal in this section is to discuss in detail the *microlocal extension property* introduced in Sec. 2 for abstract CR manifolds. We will furthermore show that CR mappings on such CR manifolds satisfying certain “regular” systems of smooth equations are actually smooth. Even though in spirit this follows our recent paper [16], the abstract case, as already indicated, poses specific problems, which need special treatment that we address in this section.

##### 4.1. The microlocal extension property

Let  $M$  be an abstract CR manifold of CR dimension  $n$  and of real dimension  $2n + d$ . In what follows, we set  $N = n + d$ . We first recall the definition of the wavefront set of a function  $u : M \rightarrow \mathbb{R}$  (since we shall only deal with smooth wavefront sets here, we drop it from the notation).

**Definition 4.1.** Let  $u$  be a distribution on  $M$ . A point  $(p_0, \xi_0) \in T^*M$ , where  $T^*_{p_0}M \ni \xi_0 \neq 0$ , is *not* in the wavefront set  $\text{WF}(u) \subset T^*M \setminus \{0\}$  (here  $0$  denotes the image of the zero section) if there exists a coordinate neighborhood  $U$  of  $p_0$ , an open (nonempty) convex cone  $\Gamma \subset T^*_{p_0}M$  containing  $\xi_0$  and a function  $\chi$ , compactly supported in  $U$  with  $\chi(p_0) \neq 0$ , such that the Fourier transform of  $\chi u$  decays rapidly, uniformly for  $\xi \in \Gamma$ , i.e.

$$\forall N \in \mathbb{N}, \quad \exists C_N > 0 : |\mathfrak{F} \chi u(\xi)| = \left| \int_{\mathbb{R}^{n+d}} (\chi u)(x) e^{-ix\xi} dx \right| \leq C_N (1 + |\xi|)^{-N}, \quad \xi \in \Gamma.$$

We note that the preceding definition does not depend on the (bundle) coordinates used for defining the Fourier transform, and that  $\text{WF}(u)$  is a closed subset of  $T^*M \setminus \{0\}$  (meaning that the image of the zero section is removed). We also note that  $(x, \xi) \in \text{WF}(u)$  if and only if  $(x, -\xi) \in \text{WF}(\bar{u})$ .

The (closed) set of points where  $u$  is *not smooth* (i.e. the set of points  $p$  for which there does not exist a neighborhood  $U$  on which  $u$  is smooth) is called the *singular support* of  $u$  and coincides with the projection of the wavefront set of  $u$  to  $M$ :

$$\text{SingSupp } u = \{x \in M : \exists \xi \in T^*_x M, \xi \neq 0, (x, \xi) \in \text{WF}(u)\}.$$

The notions we introduced above for functions generalize in a straightforward manner to maps: if  $h = (h_1, \dots, h_{N'})$  is a  $\mathbb{C}^{N'}$ -valued map (with components being either distributions or functions), then we define

$$\text{WF}(h) = \bigcup_{j=1}^{N'} \text{WF}(h_j) \quad \text{and} \quad \text{SingSupp } h = \bigcup_{j=1}^{N'} \text{SingSupp } h_j.$$

Let  $P$  be a (classical) pseudodifferential operator with principal symbol  $\mathcal{P}$  (we do not go into too much detail about the fact that we are working on a manifold

instead of an open subset of  $\mathbb{R}^{n+d}$ , as all of the notions we introduce are really local).

**Definition 4.2.** A point  $(x, \xi) \in T^*M$ , where  $\xi \neq 0$ , does not belong to the characteristic set  $\text{Char}(P) \subset T^*M \setminus \{0\}$  of  $P$  if  $\mathcal{P}(x, \xi) \neq 0$ .

For all of the (pseudodifferential) operators  $P$  which we will consider in this paper, it will hold that

$$\text{WF}(Pu) \subset \text{WF}(u), \tag{4.1}$$

for any distribution  $u$  on  $M$ . We also have the elliptic regularity theorem (see e.g. [12]), which states that

$$\text{WF}(u) \subset \text{Char}(P) \cup \text{WF}(Pu). \tag{4.2}$$

A direct consequence of the elliptic regularity theorem is that for any CR function (or distribution)  $u$  on  $M$ , we have  $\text{WF}(u) \subset T^0M$ , where  $T^0M$  is the characteristic bundle of  $M$ . Indeed, the CR vector fields, considered as pseudodifferential operators, have the property that the intersection of their characteristic sets coincides with  $T^0M$  (see [2]). We will therefore, when speaking about wavefront sets of CR distributions, only consider them as subsets of  $T^0M$ .

We now introduce the notion of *standard coordinate patch* which will be useful for us in order to characterize the microlocal extension property of an abstract CR manifold.

We say that an open subset  $\mathcal{D} \subset M$  is a *standard coordinate patch* if there exists an open subset  $D \subset \mathbb{C}^n \times \mathbb{R}^d$ , a  $\mathcal{C}^\infty$ -smooth diffeomorphism  $\Phi : D \rightarrow \mathcal{D}$  with the property that, for every  $z_0 \in \pi_1(D)$ , where  $\pi_1 : D \rightarrow \mathbb{C}^n$  is the canonical projection, the submanifold  $N_{z_0} := \Phi(\{z = z_0\})$  of  $\mathcal{D}$  is totally real and transverse to the complex tangent directions in  $M$ . On a standard coordinate patch  $\mathcal{D}$ , for  $p \in \mathcal{D}$ , we have  $T_p^0M \cong T_p^*N_{\pi_1(p)}$  (induced by the restriction of evaluation of the forms) and we require that this yields a well-defined identification  $T^0\mathcal{D} \cong D \times \mathbb{R}^d$ . We shall refer to any such a choice of local coordinates as *standard coordinates*. It is quite simple to see that for every  $p \in M$ , there exists a standard coordinate patch  $\mathcal{D} \subset M$  containing  $p$ : Starting with any smooth chart  $\Psi : \tilde{\mathcal{D}} \rightarrow \mathbb{R}^{2n+d}$  centered at  $p$ . Then  $d\Psi(T_p^cM) \subset T_0\mathbb{R}^{2n+d} = \mathbb{R}^{2n+d}$  is a  $2n$ -dimensional subspace, and we can choose smooth coordinates  $(x, y, s)$  in  $\mathbb{R}^{2n+d}$  such that  $d\Psi(T_p^cM) = \{s = 0\}$ . We claim that for  $(x, y) = z \in \mathbb{C}^n$  close by the origin and for a small  $\varepsilon > 0$ , the map  $\gamma_{x+iy} : (-\varepsilon, \varepsilon) \ni s \mapsto \Psi^{-1}(x, y, s)$  parametrizes a smooth submanifold  $N_z$  of  $M$  which is actually totally real. Indeed,  $d\Psi^{-1}(\{s = 0\} \oplus \{z = 0\}) = T_p^cM \oplus T_pN_0$ , and this direct sum decomposition necessarily stays stable for small perturbations of  $(x, y, s)$ , i.e.

$$d\Psi^{-1}(\{z = z_0\} \oplus d\Psi(T_{\Psi^{-1}(x_0, y_0, s_0)}^cM)) = T_qN_{z_0} \oplus T_q^cM, \\ \text{where } q = \Psi^{-1}(x_0, y_0, s_0).$$

Now set  $D$  to be a neighborhood of the origin for which the above equation holds,  $\Phi = \Psi^{-1}|_D$ , and  $\mathcal{D} = \Phi(D)$ .

Therefore, any abstract CR manifold can be covered by standard coordinate patches. Furthermore, any distribution  $u$  defined in some standard coordinate patch  $\mathcal{D}$  will be identified as a distribution over  $D \subset \mathbb{R}^{2n} \times \mathbb{R}^d$  (through the associated diffeomorphism  $\Phi : D \rightarrow \mathcal{D}$ ).

The dual cone of a cone  $\Gamma \subset \mathbb{R}^d$  is defined by

$$\Gamma^\circ = \{t' \in \mathbb{R}^d \setminus \{0\} : \langle t', t \rangle \geq 0, \forall t \in \Gamma\}.$$

We also recall that  $(\Gamma^\circ)^\circ$  (if nonempty) is the smallest closed (in  $\mathbb{R}^d \setminus \{0\}$ ) convex cone containing  $\Gamma$ .

The basic result about almost analytic extensions and the wavefront set we are going to use is the following well-known result (see e.g. [18] or the proof of [2, Theorem 5.3.7]).

**Theorem 4.3.** *Let  $M$  be an abstract CR manifold of CR dimension  $n$  and CR codimension  $d$ . Let  $u$  be a CR distribution defined on a standard coordinate patch  $\mathcal{D} \subset M$ ,  $\Gamma \subset \mathbb{R}^d$  an open, convex cone and set  $W = D + i\Gamma$ . If there exists  $U \in \mathcal{A}_\infty(W)$  with  $\text{bv}_W U = u$ , then  $\text{WF}(u) \subset D \times \Gamma^\circ$ . On the other hand, if  $\text{WF}(u) \subset D \times \Gamma^\circ$ , then for any open subset  $D' \subset\subset D$  and every cone  $\Gamma'$  with  $\overline{\Gamma'} \subset \Gamma$ ,  $u|_{D'}$  has an almost analytic extension to  $W' = D' + i\Gamma'$ .*

Using Theorem 4.3, we reach the following characterization of the microlocal extension property for abstract CR manifolds.

**Proposition 4.4.** *Let  $M$  be an abstract CR manifold of CR dimension  $n$  and CR codimension  $d$ . Then  $M$  has the microlocal extension property at a point  $p \in M$  if and only if for every neighborhood  $\Omega$  of  $p$ , there exists a standard coordinate patch  $\mathcal{D} \subset \Omega$  and a nonempty open convex cone  $\Gamma \subset \mathbb{R}^d$  such that every continuous CR function  $u : \Omega \rightarrow \mathbb{C}$  admits an almost analytic extension to  $D + i\Gamma$ .*

**Proof.** Suppose that  $M$  has the microlocal extension property at  $p$ . Fix a neighborhood  $\Omega$  of  $p$  and let  $\mathcal{D} \subset \Omega$  be a standard coordinate patch centered at  $p$ . By assumption there exists a nonempty open convex cone  $\Gamma \subset T_p^0 M$  such that any continuous CR function  $u : \Omega \rightarrow \mathbb{C}$  satisfies  $\text{WF}(u)|_p \subset \Gamma$ . Since by assumption  $\Gamma$  is open, and by definition  $\text{WF}(u)$  is closed, shrinking  $\mathcal{D}$  if necessary, we may assume that  $\text{WF}(u|_D) \subset D \times \Gamma$ . Furthermore, since  $\Gamma$  is convex, we may find an acute convex cone  $\tilde{\Gamma} \subset \mathbb{R}^d$  such that  $\Gamma \subset \tilde{\Gamma}^\circ$ . Applying Theorem 4.3 and shrinking  $D$  and  $\tilde{\Gamma}$ , we get that any such function  $u$  extends almost analytically to  $D + i\tilde{\Gamma}$ .

Conversely, let  $\Omega$  be a neighbourhood of  $p$ . Then there exists a standard coordinate patch  $\mathcal{D} \subset \Omega$  centered at  $p$  and an open convex cone  $\Gamma \subset \mathbb{R}^d$  such that every continuous CR function  $u$  on  $\Omega$  extends almost analytically to  $D + i\Gamma$ . Hence, Theorem 4.3 implies that  $\text{WF}(u) \subset D \times \Gamma^\circ$ . Since  $\Gamma^\circ$  is a closed acute cone, we may find an open convex cone  $\hat{\Gamma}$  such that  $\Gamma^\circ \subset \hat{\Gamma}$  and hence  $\text{WF}(u)|_p \subset \hat{\Gamma}$ . The proof is complete. □



### 4.2. CR manifolds satisfying the microlocal extension property

We want to discuss here some classes of CR manifolds for which the microlocal extension property does hold. Though this is not the focus of this present paper, let us first look at the case of embedded CR manifolds. Recall that an embedded CR submanifold  $M \subset \mathbb{C}^N$  is said to be minimal at a point  $p \in M$  if there is no proper CR submanifold  $\Sigma \subset M$  through  $p$  with the same CR dimension as that of  $M$  (see e.g. [1, 24]). We have the following:

**Proposition 4.5.** *Let  $M \subset \mathbb{C}^N$  be a generic  $\mathcal{C}^\infty$ -smooth CR submanifold in  $\mathbb{C}^N$  and  $p \in M$ . If  $M$  is minimal at  $p$ , then  $M$  satisfies the microlocal extension property at  $p$ .*

**Proof.** In order to prove the proposition, we use the characterization given by Proposition 4.4 to show that the almost analytic extension property holds for any continuous CR function defined in any neighborhood of  $p$ . To this end, we just note that by Tumanov’s theorem [24], for any neighborhood  $\Omega$  of  $p$  in  $M$ , there exists a wedge of edge  $M$  at  $p$  in  $\mathbb{C}^N$  to which all continuous CR functions on  $\Omega$  extend holomorphically. The desired almost analytic extension property for all CR functions on  $\Omega$  then follows by using standard coordinates attached to generic submanifolds in complex space such as in [16, Proposition 3.2]. The details are left to the reader. □

To the authors’ knowledge, there is no known characterization of the microlocal extension property for abstract CR manifolds. However, the following general sufficient condition for abstract CR manifolds with nondegenerate Levi form (analogous to Lewy’s extension theorem [17]) is due to Berhanu and Xiao [3, Theorem 2.9], and is also, to the best of our knowledge, currently the only such result.

**Theorem 4.6.** *Let  $M$  be an abstract CR manifold. If the Levi form of  $M$  at a point  $p \in M$  has the property that it possesses a nonzero eigenvalue in the direction of every characteristic form  $\eta \in T_p^0 M$ , then  $M$  has the microlocal extension property at  $p$ . In particular, if  $M$  is of hypersurface type, then  $M$  has the microlocal extension property at every point where the Levi-form has a nonzero eigenvalue.*

### 4.3. Smooth edge-of-the-wedge theory

Let  $\mathcal{D}$  be a standard coordinate patch in an abstract CR manifold  $M$ . If  $u: \mathcal{D} \rightarrow \mathbb{C}$  is a CR function that admits an almost analytic extension  $U \in \mathfrak{A}_\infty(W)$ , with  $W = D + i\Gamma$  for some cone  $\Gamma \subset \mathbb{R}^d$ , and if furthermore  $u \in \mathcal{C}^k(\mathcal{D})$ , then, by a result of Rosay [22],  $U$  is actually  $k$  times continuously differentiable up to the edge  $D$ , i.e.  $U \in \mathcal{C}^k(D + i(\Gamma' \cup \{0\}))$ , for any cone  $\Gamma' \subset \Gamma$ . Proposition 3.5 proved Sec. 3 can be seen as a variant of this result, for Hölder continuous CR functions.

Another known smoothness result, which we now recall, states that it is enough to check smoothness in *enough independent directions* (by the edge of the wedge

theorem). The version we will need for the purpose of this paper can be stated as follows.

**Theorem 4.7.** *Let  $M$  be an abstract CR manifold of CR dimension  $n$  and CR codimension  $d$ ,  $u$  be a CR distribution on  $M$  and  $p \in M$ . Then  $p \notin \text{SingSupp}(u)$  if and only if there is a standard coordinate neighborhood  $\mathcal{D}$  of  $p$  in  $M$  and an open convex cone  $\Gamma \subset \mathbb{R}^d$ , such that  $u|_{\mathcal{D}}$  admits an almost analytic extension to both  $W_+ = D + i\Gamma$  and  $W_- = D - i\Gamma$ .*

The proof of this theorem is rather simple now: Replacing  $M$  by a suitable neighborhood of  $p$ , we apply Theorem 4.3 to see that  $\text{WF}(u)|_p \subset (\Gamma^o \cap (-\Gamma)^o) \setminus \{0\} = \emptyset$ , and hence  $u$  is smooth near  $p$ .

#### 4.4. A smooth reflection principle on abstract CR manifolds

We now turn to an important *a priori* regularity result for CR maps, which, for embedded CR manifolds, is contained in [16].

**Theorem 4.8.** *Let  $M$  be an abstract CR manifold,  $h: M \rightarrow \mathbb{C}^{N'}$  and  $g: M \rightarrow \mathbb{C}^k$  be, respectively, a CR  $\mathcal{C}^1$ -smooth and a  $\mathcal{C}^0$  map. Let  $\mathcal{D} \subset M$  be a standard coordinate patch,  $p \in \mathcal{D}$ ,  $\Gamma \subset \mathbb{R}^d$  an open convex cone, and assume that (all components of)  $h$  and  $g$  extend almost analytically to  $D + i\Gamma$ . Let  $r$  be a  $\mathcal{C}^\infty$ -smooth,  $\mathbb{C}^{N'}$ -valued map, defined in a neighbourhood of  $(p, h(p), \overline{g(p)}) \in M \times \mathbb{C}^{N'} \times \mathbb{C}^k$ , holomorphic in its last variable, satisfying for  $q \in M$  near  $p$  the following properties:*

$$r(q, h(q), \overline{h(q)}, \overline{g(q)}) = 0;$$

$$\det r_w(p, h(p), \overline{h(p)}, \overline{g(p)}) \neq 0.$$

Then  $p \notin \text{SingSupp } h$ .

Let us stress that even though  $h$  is assumed to be CR,  $g$  need not be CR in Theorem 4.8. We are going to exploit this in a more specific result that follows from Theorem 4.8. In order to state it, we introduce the following notion (that will appear later in this paper).

**Definition 4.9.** Let  $M$  be an abstract CR manifold,  $r \in \mathbb{N}$ , and let  $f: \Omega \rightarrow \mathbb{C}$  be a  $\mathcal{C}^r$ -smooth function on some open subset  $\Omega \subset M$ . We say that  $f$  is  $\mathcal{C}^r$ -admissible if there exists an integer  $\ell \geq 0$ , a  $\mathcal{C}^{r+\ell}$ -smooth CR function  $F: \Omega \rightarrow \mathbb{C}$ , and  $\mathcal{C}^\infty$ -smooth  $(1, 0)$  vector fields  $X_1, \dots, X_\ell$  defined on  $\Omega$  such that  $f = X_1 \dots X_\ell F$ .

This notion of admissible functions extends obviously to  $\mathbb{C}^n$ -valued maps by requiring that each component be admissible.

**Corollary 4.10.** *Let  $M$  be an abstract CR manifold,  $p \in M$ , and assume that  $M$  has the microlocal extension property at  $p$ . Let  $h: M \rightarrow \mathbb{C}^{N'}$  and  $g: M \rightarrow \mathbb{C}^k$  be, respectively, a  $\mathcal{C}^1$  CR map and a  $\mathcal{C}^0$ -admissible map. Let  $r$  be a  $\mathcal{C}^\infty$ -smooth,*

$\mathbb{C}^{N'}$ -valued map, defined in a neighbourhood of  $(p, h(p), \overline{g(p)}) \in M \times \mathbb{C}^{N'} \times \mathbb{C}^k$ , holomorphic in its last variable, satisfying for  $q \in M$  near  $p$  the following properties:

$$r(q, h(q), \overline{h(q)}, \overline{g(q)}) = 0;$$

$$\det r_w(p, h(p), \overline{h(p)}, \overline{g(p)}) \neq 0.$$

Then  $p \notin \text{SingSupp } h$ .

**Proof of Corollary 4.10.** From the microlocal extension property at  $p$ , we know that there exists a coordinate patch  $\mathcal{D} \subset M$  containing  $p$  and an open convex cone  $\Gamma \subset \mathbb{R}^d$  such that all continuous CR functions on  $M$  extend almost analytically to  $D+i\Gamma$ . The corollary then follows from Theorem 4.8 by noticing that any admissible function is a higher order derivative of a CR function along  $(1, 0)$  smooth vector fields, and therefore, also extends almost analytically to  $D + i\Gamma$ , as a consequence of Remark 3.4. □

**Proof of Theorem 4.8.** Let  $(z, \bar{z}, s)$  be standard coordinates defined on  $D \subset \mathbb{R}^{2n} \times \mathbb{R}^d$ , in which  $p$  may be assumed to be the origin. We may also assume that  $h(0) = 0, g(0) = 0$ . Since the conclusion of the theorem is local, we shall do a number of steps each of which requires us to possibly shrink  $D$ ; we shall do so without explicitly mentioning it, and will not rename  $D$ . We will apply the same policy when denoting constants, that may change from one line to the other. The proof given here is an adaptation of the proof of [16, Theorem 3.1], that includes the appropriate changes needed to deal with abstract CR manifolds.

First, we consider  $\mathbb{R}_s^d$  and  $\mathbb{C}^{N'}$  as totally real subspaces of  $\mathbb{C}_\sigma^d$  and  $\mathbb{C}_{Z, \zeta}^{2N'}$  by  $\sigma = s + it, Z = \bar{\zeta} = w$ , respectively, i.e.  $\text{Re } w = \text{Re}(\frac{Z+\zeta}{2}), \text{Im } w = \text{Re}(\frac{Z-\zeta}{2i})$ . Proposition 3.1 yields, for appropriate open subsets  $\Omega \subset \mathbb{C}^{N'}$  and  $O \subset \mathbb{C}^k$  containing 0, an extension  $R(z, \bar{z}, s, t, Z, \bar{Z}, \zeta, \bar{\zeta}, \Lambda)$  of  $r$  to  $D \times \mathbb{R}^d \times \Omega \times \bar{\Omega} \times O$ , i.e.

$$R(z, \bar{z}, s, 0, w, \bar{w}, \bar{w}, w, \Lambda) = r(z, \bar{z}, s, w, \bar{w}, \Lambda, \bar{\Lambda}),$$

and such that for every  $m, \ell \in \mathbb{N}$ , there exists a  $C = C(m, \ell) > 0$  such that, denoting

$$P = \frac{\partial^{|\alpha|+|\beta|+|\gamma|+|\delta|+|\epsilon|+|\delta'|+|\epsilon'|}}{\partial z^\alpha \bar{z}^\beta s^\gamma Z^\delta \bar{Z}^\epsilon \zeta^{\delta'} \bar{\zeta}^{\epsilon'}}, \quad \text{with } |\alpha| + |\beta| + |\gamma| + |\delta| + |\epsilon| + |\delta'| + |\epsilon'| \leq m,$$

for every partial derivative of the form, it holds that

$$\|PR_\sigma\| + \|PR_{\bar{z}}\| + \|PR_{\bar{\zeta}}\| \leq C (\|t\| + \|Z - \bar{\zeta}\|)^\ell, \tag{4.3}$$

shrinking  $D, \Omega$  and  $O$  if necessary. The Jacobian of the map  $R$  with respect to  $Z$  (considered as a map  $\mathbb{R}^{2N'}$  to  $\mathbb{R}^{2N'}$ ) at the origin is computed to be

$$R_{(Z, Z)} = \begin{pmatrix} R_Z & R_{\bar{Z}} \\ \bar{R}_Z & \bar{R}_{\bar{Z}} \end{pmatrix} = \begin{pmatrix} r_w & 0 \\ 0 & \bar{r}_w \end{pmatrix},$$

which has a nonzero determinant by assumption. We can therefore apply the implicit function theorem to obtain a map  $\Phi(z, \bar{z}, s, t, \zeta, \bar{\zeta}, \Lambda)$ , holomorphic in its

last component, such that  $R(z, \bar{z}, s, t, Z, \bar{Z}, \zeta, \bar{\zeta}, \Lambda, \bar{\Lambda}) = 0$  if and only if  $Z = \Phi(z, \bar{z}, s, t, \zeta, \bar{\zeta}, \Lambda)$ . In particular, one has for  $(z, s) \in D$

$$h(z, \bar{z}, s) = \Phi(z, \bar{z}, s, 0, \overline{h(z, \bar{z}, s)}, h(z, \bar{z}, s), \overline{g(z, \bar{z}, s)}). \tag{4.4}$$

We now claim that, for  $j = 1, \dots, d$  and  $k = 1, \dots, N'$ , the  $\Phi_{\bar{\sigma}_j}$  and  $\Phi_{\bar{\zeta}_k}$  are controlled in the following way.

**Claim 1.** For every compact  $K \subset D$ , every open convex cone  $\Gamma' \subset\subset \Gamma$ , and for every  $\ell, m \in \mathbb{N}$ , there exists a constant  $C > 0$  such that for  $(z, s) \in K, \zeta \in \bar{\Omega}, \Lambda \in O$  and  $t \in \Gamma'$  with  $\|t\|$  sufficiently small,

$$\begin{aligned} & \sum_{|\alpha|+|\beta|+|\gamma|+|\delta|+|\epsilon| \leq m} \left\| \frac{\partial^{|\alpha|+|\beta|+|\gamma|+|\delta|+|\epsilon|}}{\partial z^\alpha \bar{z}^\beta s^\gamma \zeta^\delta \bar{\zeta}^\epsilon} \Phi_{\bar{\sigma}} \right\| + \left\| \frac{\partial^{|\alpha|+|\beta|+|\gamma|+|\delta|+|\epsilon|}}{\partial z^\alpha \bar{z}^\beta s^\gamma \zeta^\delta \bar{\zeta}^\epsilon} \Phi_{\bar{\zeta}} \right\| \\ & \leq C(\|t\| + \|\Phi(z, \bar{z}, s, t, \zeta, \bar{\zeta}, \Lambda) - \bar{\zeta}\|)^\ell. \end{aligned}$$

This can be seen in the following way: First, we have that

$$\Phi_{\bar{\sigma}} = -R_Z^{-1}(R_{\bar{\sigma}} + R_{\bar{Z}}\bar{\Phi}_{\bar{\sigma}}), \quad \Phi_{\bar{\zeta}} = -R_Z^{-1}(R_{\bar{\zeta}} + R_{\bar{Z}}\bar{\Phi}_{\bar{\zeta}}).$$

Then any partial derivative (in  $(z, \bar{z}, s, \zeta, \bar{\zeta})$ ) acting on this equation gives (by the chain rule) rise to expressions which can be controlled in the claimed way using (4.3).

Now, by assumption,  $h$  extends almost analytically to  $D + i\Gamma$ . Let us denote an almost analytic extension of  $h$  by  $h_+(z, \bar{z}, s, t)$ , defined for  $(z, s) \in D$  and  $t \in \Gamma$ , and similarly for  $g$ . In the remainder of the proof, we will show that

$$h_-(z, \bar{z}, s, t) := \Phi(z, \bar{z}, s, t, \overline{h_+(z, \bar{z}, s, -t)}, h_+(z, \bar{z}, s, -t), \overline{g_+(z, \bar{z}, s, -t)}) \tag{4.5}$$

is an almost analytic extension of  $h$  to  $D - i\Gamma$  (here  $\|t\|$  is small). An application of Theorem 4.7 then yields that  $h$  is smooth near the origin.

Let us first check that  $h_-$  is of slow growth. Let  $\alpha, \beta \in \mathbb{N}^n$  and  $\gamma \in \mathbb{N}^d, K \subset D$  compact, and  $\Gamma' \subset\subset -\Gamma$  be given. By assumption,  $h_+$  and  $g_+$  are of slow growth in  $D + i\Gamma$ . In particular, there exist constants  $C = C(\alpha, \beta, \gamma) > 0$  and  $k = k(\alpha, \beta, \gamma) \in \mathbb{N}$ , such that for  $(z, s) \in K$  and  $t \in -\Gamma'$

$$\begin{aligned} & \sum_{\substack{\alpha', \beta' \in \mathbb{N}^n, \gamma' \in \mathbb{N}^d \\ |\alpha'|+|\beta'|+|\gamma'| \leq |\alpha|+|\beta|+|\gamma|}} \left\| \frac{\partial^{|\alpha'|+|\beta'|+|\gamma'|} h_+(z, \bar{z}, s, -t)}{\partial z^{\alpha'} \bar{z}^{\beta'} s^{\gamma'}} \right\| \\ & + \left\| \frac{\partial^{|\alpha'|+|\beta'|+|\gamma'|} g_+(z, \bar{z}, s, -t)}{\partial z^{\alpha'} \bar{z}^{\beta'} s^{\gamma'}} \right\| \leq C \|t\|^{-k}. \end{aligned} \tag{4.6}$$

We can also assume that  $D, \Omega$  and  $O$  have been chosen so small that all of the derivatives of  $R$  of order at most  $|\alpha| + |\beta| + |\gamma|$  stay bounded, say by  $\Delta > 0$ . The

chain rule, (4.6) and (4.5) then imply that there exists a combinatorial factor  $\tilde{\Delta}$  such that

$$\left\| \frac{\partial^{|\alpha|+|\beta|+|\gamma|} h_-(z, \bar{z}, s, t)}{\partial z^\alpha \bar{z}^\beta s^\gamma} \right\| \leq \tilde{\Delta} \Delta C \|t\|^{-k}, \tag{4.7}$$

for  $(z, s) \in K$  and  $t \in -\Gamma'$  with  $\|t\|$  sufficiently small; i.e.  $h_-$  is of slow growth as claimed.

Before we turn to the growth behaviour of the derivatives  $\frac{\partial h_-}{\partial \bar{\sigma}_j}$ ,  $j = 1, \dots, d$ , i.e.

$$\frac{\partial h_-}{\partial \bar{\sigma}_j} = \Phi_{\bar{\sigma}_j} + \Phi_\zeta \overline{\frac{\partial h_+}{\partial \bar{\sigma}_j}} + \Phi_\zeta \frac{\partial h_+}{\partial \sigma_j} + \Phi_\Lambda \overline{\frac{\partial g_+}{\partial \bar{\sigma}_j}}, \tag{4.8}$$

we need some preparation. Fix  $K$  and  $\Gamma'$  as before. We will first establish the following.

**Claim 2.** There exist  $\beta > 0$  and  $C > 0$  such that

$$\begin{aligned} \|h_+(z, \bar{z}, s, -t) - h_-(z, \bar{z}, s, t)\| &\leq C \|t\|^\beta, \\ (z, \bar{z}, s) \in K, \quad t \in \Gamma', \quad \|t\| &\text{ small.} \end{aligned}$$

In order to establish Claim 2, note that by Rosay’s result already mentioned in the beginning of Sec. 4.3, we have that  $h_+$  is actually  $\mathcal{C}^1(D \times (\Gamma'' \cup \{0\}))$  for any cone  $\Gamma'' \subset\subset \Gamma$ , i.e. it is  $\mathcal{C}^1$  up to the edge. Since  $g_+$  is an almost holomorphic extension of  $g$ , we thus see that (4.8), (4.5) and (4.4) imply that  $h_-(z, \bar{z}, s, t)$  is a  $\bar{\partial}$ -bounded extension to  $D - i\Gamma''$  of the function  $h_+|_{t=0} \in \mathcal{C}^1(D)$ . Choose a cutoff function  $\chi$  such that for an  $\epsilon > 0$  we have  $\text{supp } \chi \subset D \times (-2\epsilon, 2\epsilon)^d$ ,  $\chi|_{K \times (-\epsilon, \epsilon)^d} = 1$ , and which is almost holomorphic (in the  $\sigma_j$ ’s).

Since  $\Gamma' \subset\subset \Gamma$  is a compact subcone, there exists a finite number of closed convex cones  $\Gamma_1, \dots, \Gamma_e$ , each of which is linearly equivalent to  $\mathbb{R}_+^d$ , and an open convex cone  $\Gamma'' \subset\subset \Gamma$  such that

$$\Gamma' \subset \bigcup_{j=1}^e \Gamma_j \subset \Gamma'' \subset\subset \Gamma.$$

For  $z_0 \in \mathbb{R}^{2n}$  sufficiently close to 0, considering  $h_{z_0, j} = (\chi h_-)|_{\{z_0\} \times \mathbb{R}^d \times -\Gamma_j}$ , we see that  $h_{z_0, j}$  satisfies the conditions of Proposition 3.5, since  $h$  is  $\mathcal{C}^1$  over  $D$  and (4.7) and (4.6) hold. So that for some  $\beta > 0$  (independent of  $z_0$  and  $j$ ) we have that

$$h_{z_0, j} \in \mathcal{C}^{0, \beta}(-\Gamma_j), \text{ with } \|h_{z_0, j}\|_{0, \beta} \leq C.$$

*A priori*  $C$  depends on  $z_0$  and  $j$ , but the concrete form of the estimates in Proposition 3.5, and the fact mentioned above that  $h_+ \in \mathcal{C}^1(D \times (\Gamma'' \cup \{0\}))$ , ensures that  $C$  is actually independent of  $z_0$  and  $j$ . We thus conclude that for  $z \in K$  and  $t \in -\Gamma'$ ,  $\|t\|$  small enough, we have that

$$\|h_+(z, \bar{z}, s, -t) - h_-(z, \bar{z}, s, t)\| \leq C \|t\|^\beta.$$

Now, similarly to [16], Claims 1 and 2 can be used to show in particular that we can control the partial derivatives of  $\Phi_{\bar{\sigma}}$  and  $\Phi_{\bar{\zeta}}$  in the following way: For every  $K, \Gamma'$

as above, and  $m, \ell \in \mathbb{N}$ , there exists a  $C > 0$  such that if  $|\alpha| + |\gamma| + |\gamma| + |\delta| + |\epsilon| \leq m$ , then for  $(z, s) \in K, t \in \Gamma'$  with  $\|t\|$  small enough, we have

$$\begin{aligned} & \left\| \frac{\partial^{|\alpha|+|\beta|+|\gamma|+|\delta|+|\epsilon|}}{\partial z^\alpha \bar{z}^\beta s^\gamma \zeta^\delta \bar{\zeta}^\epsilon} \Phi_{\bar{\sigma}}(z, \bar{z}, s, t, \overline{h_+(z, \bar{z}, s, -t)}, h_+(z, \bar{z}, s, -t), \overline{g_+(z, \bar{z}, s, -t)}) \right\| \\ & \leq C \|t\|^\ell, \\ & \left\| \frac{\partial^{|\alpha|+|\beta|+|\gamma|+|\delta|+|\epsilon|}}{\partial z^\alpha \bar{z}^\beta s^\gamma \zeta^\delta \bar{\zeta}^\epsilon} \Phi_{\bar{\zeta}}(z, \bar{z}, s, t, \overline{h_+(z, \bar{z}, s, -t)}, h_+(z, \bar{z}, s, -t), \overline{g_+(z, \bar{z}, s, -t)}) \right\| \\ & \leq C \|t\|^\ell. \end{aligned} \tag{4.9}$$

We can now complete the proof of the theorem largely similar to [16]: If we apply a partial derivative of the form

$$P = \frac{\partial^{|\alpha|+|\beta|+|\gamma|}}{\partial z^\alpha \bar{z}^\beta s^\gamma}$$

to (4.8) yields an expression which we can control because it is a sum of products each of which contains only factors which are of slow growth towards  $t = 0$ , and at least one factor which vanishes to infinite order in  $t$  at  $t = 0$ ; this is obvious for terms containing derivatives with respect to  $\bar{\sigma}$  of  $\overline{h_+}$  and  $\overline{g_+}$ , and for the other terms we can use (4.9). The details are analogous to those of [16] and are therefore left to the reader.  $\square$

## 5. Invariants of CR Maps and Associated Open Subsets Decomposition

In this section, we carry out the construction of the open subsets decomposition associated to any given CR map mentioned in Sec. 2. To this end, we at first introduce a number of rings of functions attached to the map as well as study numerical invariants related to these rings. We follow the lines of thought of the construction done for embedded CR manifolds in [16], but the present construction for abstract CR manifolds needs a number of substantial changes that we explain. In fact, even in the embedded case, our construction in this paper improves the one carried out in [16] (as the required minimality assumption in a number of properties in [16, Secs. 4 and 5] is no longer necessary with the present new construction). We indicate here the main differences and will drop proofs and refer to [16] where they are very similar to the embedded case.

### 5.1. Function rings attached to a CR map and numerical invariants

Let  $(M, \mathcal{V})$  be an abstract CR manifold of CR dimension  $n$  and CR codimension  $d$ . We introduce here a sequence of local numerical invariants attached to a (germ of a) CR map at  $p \in M$ . In what follows, if  $X$  is a real manifold,  $x_0 \in X$  and

$\ell \in \mathbb{Z}_+ \cup \{\infty\}$ , we denote by  $\mathcal{C}^\ell(X, x_0)$  the ring of germs of  $\mathcal{C}^\ell$ -smooth functions at  $x_0$  and by  $\mathcal{C}^\ell(X)$  the ring of  $\mathcal{C}^\ell$ -functions over  $X$ .

For  $p \in M$  and  $k \in \mathbb{Z}_+ \cup \{\infty\}$ , we denote by  $\mathcal{C}_{CR}^k(M, p)$  the ring of germs of  $\mathcal{C}^k$ -smooth CR functions at  $p$ . Analogously to Definition 4.9, we say that a germ at  $p$  of a  $\mathcal{C}^m$ -smooth function  $g$ ,  $m \in \mathbb{N}$ , is *admissible* if there exists an integer  $\ell \in \mathbb{N}$ , a germ at  $p$  of  $\mathcal{C}^{\ell+m}$ -smooth CR function  $G$  and  $\mathcal{C}^\infty$ -smooth  $(1, 0)$  vector fields defined near  $p$ ,  $X_1, \dots, X_\ell$ , such that  $g = X_1, \dots, X_\ell G$  (as germs at  $p$ ). Note that germs at  $p$  of CR functions are obviously admissible. This notion extends in the obvious way to  $\mathbb{C}^n$ -valued mappings by requiring that every component be admissible.

Let  $h: M \rightarrow \mathbb{C}_w^{N'}$  be a  $\mathcal{C}^1$  CR map. Even though the notation for the rings and invariants associated to  $h$  is the same as we used in [16], let us stress that the rings and invariants introduced here are different from those introduced in the above-mentioned paper.

**Definition 5.1.** Let  $M$  and  $h$  be as above,  $p \in M$ , and  $\mu, j \in \mathbb{N}$  with  $0 \leq j \leq \mu$ .

- (a) We denote by  $\mathcal{A}_p^{j, \mu}$  the set of all pairs  $(g, r)$ , where  $g = (g_1, \dots, g_k) \in (\mathcal{C}^{\mu-j}(M, p))^k$  for some integer  $k$ , each  $g_\nu$  being admissible, and where  $r(q, w, \bar{w}, \Lambda) \in \mathcal{C}^\infty(M \times \mathbb{C}^{N'} \times \mathbb{C}^k, (p, h(p), \overline{g(p)}))$  is holomorphic in  $\Lambda$ , and

$$r(q, h(q), \overline{h(q)}, \overline{g(q)}) = 0 \quad \text{for } q \in M \text{ near } p.$$

- (b) If  $h$  is  $\mathcal{C}^{\mu-j}$ -smooth, we denote by  $\mathcal{F}_p^{j, \mu}$  the subring of  $\mathcal{C}^{\mu-j}(M, p)$  consisting of those functions  $\psi$  that may written in the form

$$\psi(q) = r(q, h(q), \overline{h(q)}, \overline{g(q)})$$

for  $q \in M$  near  $p$ , where  $g = (g_1, \dots, g_k) \in (\mathcal{C}^{\mu-j}(M, p))^k$  for some integer  $k$ , each  $g_\nu$  being admissible, and  $r(q, w, \bar{w}, \Lambda) \in \mathcal{C}^\infty(M \times \mathbb{C}^{N'} \times \mathbb{C}^k, (p, h(p), \overline{g(p)}))$  is holomorphic in  $\Lambda$ .

- (c) If  $h$  is  $\mathcal{C}^{\mu-j}$ -smooth, for  $(g, r) \in \mathcal{A}_p^{j, \mu}$ , we define  $r_w \in (\mathcal{F}_p^{j, \mu})^{N'}$  by

$$\begin{aligned} r_w &:= r_w(q, h(q), \overline{h(q)}, \overline{g(q)}) \\ &= (r_{w_1}(q, h(q), \overline{h(q)}, \overline{g(q)}), \dots, r_{w_{N'}}(q, h(q), \overline{h(q)}, \overline{g(q)})) \end{aligned}$$

for  $q \in M$  near  $p$ .

Observe that if  $h$  is  $\mathcal{C}^{\mu-j}$ -smooth, then for any  $\psi \in \mathcal{F}_p^{j, \mu}$  there is a neighborhood of  $p$  in  $M$  such that for any  $q$  in that neighborhood, (the germ at  $q$  of)  $\psi \in \mathcal{F}_q^{j, \mu}$ .

For every  $p \in M$ , we define the following vector subspace of  $\mathbb{C}^{N'}$ :

$$\mathcal{D}_j^\mu(p) = \{r_w(p, h(p), \overline{h(p)}, \overline{g(p)}) : (g, r) \in \mathcal{A}_p^{j, \mu}\}.$$

We define, for  $p \in M$  and any integer  $0 \leq j \leq \mu$ :

$$\mathcal{S}_j^\mu(p) := \dim_{\mathbb{C}} \mathcal{D}_j^\mu(p). \tag{5.1}$$

For every  $p \in M$  and each  $\mu \in \mathbb{N}$ , we clearly have

$$\mathcal{D}_0^\mu(p) \subset \mathcal{D}_1^\mu(p) \subset \dots \subset \mathcal{D}_\mu^\mu(p),$$

and therefore

$$\mathcal{S}_0^\mu(p) \leq \mathcal{S}_1^\mu(p) \leq \dots \leq \mathcal{S}_\mu^\mu(p).$$

Even though for  $0 \leq j \leq \mu$ ,  $\mathcal{S}_j^\mu(p)$  was defined using specific coordinates in  $\mathbb{C}^{N'}$ , the reader may easily check that  $\mathcal{S}_j^\mu(p)$  is actually independent of the specific choice of (local) holomorphic coordinates in  $\mathbb{C}^{N'}$  near  $h(p)$ .

For  $p \in M$ , we set

$$\mathbb{V}_p^{j,\mu} := (\mathcal{D}_j^\mu(p))^\perp = \{V \in \mathbb{C}^{N'} : V \cdot r_w(p, h(p), \overline{h(p)}, \overline{g(p)}) = 0, \forall (g, r) \in \mathcal{A}_p^{j,\mu}\}. \tag{5.2}$$

Since, as mentioned above,  $\mathcal{D}_j^\mu(p)$  is increasing in  $j$ , we have that

$$\mathbb{V}_p^{\mu,\mu} \subset \mathbb{V}_p^{\mu-1,\mu} \subset \dots \subset \mathbb{V}_p^{0,\mu} \quad \text{and} \quad \dim \mathbb{V}_p^{j,\mu} = N' - \mathcal{S}_j^\mu(p).$$

In Remark 5.2, we define a certain type of ‘‘holomorphic’’ derivatives for any element of  $\mathcal{F}_p^{j,\mu}$ , that we will frequently use in the sequel.

**Remark 5.2.** Assume that  $h$  is  $\mathcal{C}^{\mu-j}$ -smooth and let  $p \in M$ , with  $\mu, j \in \mathbb{N}$  satisfying  $0 \leq j \leq \mu$ .

- (i) If  $\psi \in \mathcal{F}_p^{j,\mu}$  can be written in two different ways as

$$\psi(q) = r^1(q, h(q), \overline{h(q)}, \overline{g(q)}) = r^2(q, h(q), \overline{h(q)}, \overline{g(q)})$$

for  $q \in M$  near  $p$ , where  $g \in (\mathcal{C}^{\mu-j}(M, p))^k$ , for some integer  $k$ , is admissible, then  $r = r_1 - r_2$  satisfies  $(g, r) \in \mathcal{A}_p^{j,\mu}$ . In particular, for every  $V \in \mathbb{V}_p^{j,\mu}$ , since  $V \cdot r_w(p, h(p), \overline{h(p)}, \overline{g(p)}) = 0$  we have

$$V \cdot r_w^1(p, h(p), \overline{h(p)}, \overline{g(p)}) = V \cdot r_w^2(p, h(p), \overline{h(p)}, \overline{g(p)}). \tag{5.3}$$

It follows that for every  $V \in \mathbb{V}_p^{j,\mu}$ , we may define

$$V \cdot \psi_w(p) := V \cdot r_w^1(p, h(p), \overline{h(p)}, \overline{g(p)}) \tag{5.4}$$

since the right-hand side of (5.4) is independent of a particular choice of representative for  $\psi$  by (5.3).

- (ii) For any polynomial  $P(t, \bar{t}) = \sum_{\alpha, \beta} P^{\alpha, \beta} t^\alpha \bar{t}^\beta \in \mathcal{F}_p^{j,\mu}[t, \bar{t}]$ ,  $t \in \mathbb{C}^r$ , and any  $V \in \mathbb{V}_p^{j,\mu}$ , we define

$$V \cdot P_w(t, \bar{t}) := \sum_{\alpha, \beta} (V \cdot P_w^{\alpha, \beta}) t^\alpha \bar{t}^\beta,$$

which is well defined (at  $p$ ) by (i).



- (iii) If  $\psi \in \mathcal{F}_p^{j,\mu}$  is defined over a neighborhood  $U$  of  $p$ , and  $V : U \rightarrow \mathbb{C}^{N'}$  is a map satisfying  $V(q) \in \mathbb{V}_q^{j,\mu}$  for every  $q \in U$ , then  $V \cdot \psi_w$  is well defined over all of  $U$ ; the same holds for polynomials as in (ii).

The next result is a version of [16, Lemma 4.5] suitable for abstract CR manifolds. We note that the proof of [16, Lemma 4.5] uses in an essential way the fact that the CR manifold was embedded as well as Tumanov’s extension theorem for CR functions on generic minimal submanifolds in complex space. Such techniques are not available in the abstract setting. Instead, we will present a proof that relies on the definition of the new rings introduced in this paper.

**Lemma 5.3.** *Let  $M$  be an abstract CR manifold,  $\mu, j$  be integers satisfying  $0 \leq j < \mu$ , and let  $h : M \rightarrow \mathbb{C}^{N'}$  be a CR map of class  $\mathcal{C}^{\mu-j}$ . Let  $p \in M$  and  $\bar{K}$  be a  $\mathcal{C}^\infty$ -smooth CR vector field on  $M$  (near  $p$ ).*

- (i) *For  $\psi \in \mathcal{F}_p^{j,\mu}$  we have that  $\bar{K}\psi \in \mathcal{F}_p^{j+1,\mu}$ , and there exists a neighborhood  $U_p$  of  $p$  such that for every  $q \in U_p$ , (the germ at  $q$  of)  $\bar{K}\psi$  belongs to  $\mathcal{F}_q^{j+1,\mu}$ . Furthermore, if  $V : U_p \rightarrow \mathbb{C}^{N'}$  is a CR map of class  $\mathcal{C}^1$  which satisfies  $V(q) \in \mathbb{V}_q^{j+1,\mu}$  for  $q \in U_p$ , then  $V \cdot (\bar{K}\psi)_w$  is defined over  $U_p$  and we have*

$$V \cdot (\bar{K}\psi)_w = \bar{K}(V \cdot \psi_w).$$

- (ii) *Let  $(g, r) \in \mathcal{A}_p^{j,\mu}$ . Then there exists  $(\hat{g}, \hat{r}) \in \mathcal{A}_p^{j+1,\mu}$  (depending on  $\bar{K}$ ) such that  $\bar{K}r_w = \hat{r}_w$ .*

**Proof.** Let  $\psi \in \mathcal{F}_p^{j,\mu}$ . By definition there exist  $g = (g_1, \dots, g_k) \in (\mathcal{C}^{\mu-j}(M, p))^k$  for some integer  $k$ , such that each  $g_\nu$  is admissible, and  $r \in \mathcal{C}^\infty(M \times \mathbb{C}^{N'} \times \mathbb{C}^k, (p, h(p), \overline{g(p)}))$ , holomorphic in its last argument (denoted by  $\Lambda$  in what follows) such that, for  $q \in U_p \subset M$  in some neighborhood of  $p$ ,

$$\psi(q) = r(q, h(q), \overline{h(q)}, \overline{g(q)}).$$

Therefore, for  $q \in U_p$ , since  $h$  is CR, the chain rule implies that we may write

$$\bar{K}\psi = \tilde{\psi} + r_{\bar{w}} \cdot \bar{K}\bar{h} + r_\Lambda \cdot \bar{K}\bar{g},$$

for some  $\tilde{\psi} \in \mathcal{F}_p^{j,\mu}$ . Now we can certainly write  $\bar{K}\bar{h}$  and  $\bar{K}\bar{g}$  in the form  $\overline{K\bar{h}}$  and  $\overline{K\bar{g}}$  for some smooth  $(1, 0)$  complex vector field  $K$  on  $U_p$ . Hence, we have

$$\bar{K}\psi = \tilde{\psi} + r_{\bar{w}} \cdot \overline{K\bar{h}} + r_\Lambda \cdot \overline{K\bar{g}}. \tag{5.5}$$

Hence for  $q \in U_p$ , we can write

$$(\bar{K}\psi)(q) = \tilde{r}(q, h(q), \overline{h(q)}, \overline{g(q)}),$$

where  $\tilde{g} = (g, Kg, Kh) \in (\mathcal{C}^{\mu-j-1}(M, p))^{2k+N'}$  is admissible; since  $\tilde{r}$  is clearly holomorphic in its last argument, we see that  $\bar{K}\psi \in \mathcal{F}_p^{j+1,\mu}$  as claimed. Finally, as observed after Definition 5.1, for every  $q \in U_p$ , the germ at  $q$  of  $\bar{K}\psi$  belongs to  $\mathcal{F}_q^{j+1,\mu}$ .

Next, suppose that we are given a neighborhood  $U_p$  of  $p$  in  $M$  as before, and  $U_p \ni q \mapsto V(q) \in \mathbb{V}_q^{j+1,\mu}$  is CR of class  $\mathcal{C}^1$ . Then, we have on  $U_p$

$$V \cdot (\bar{K}\psi)_w = V \cdot \bar{K}(\psi_w) = \bar{K}(V \cdot \psi_w),$$

since  $V$  is CR. This completes the proof of part (i) of the lemma; part (ii) follows by very similar arguments.  $\square$

### 5.2. Open subset decomposition and its properties

Let  $M$  be as above and  $h : M \rightarrow \mathbb{C}^{N'}$  be a  $\mathcal{C}^1$ -smooth CR map. For  $\mu, j$  integers satisfying  $0 \leq j \leq \mu \leq N'$ , since the functions  $M \ni p \mapsto \mathcal{S}_j^\mu(p)$  are all lower semicontinuous and integer valued, the set

$$\widetilde{M} = \bigcup_{\mu=0}^{N'} \bigcup_{j=0}^{\mu} \{p \in M : \exists U_p, \mathcal{S}_j^\mu(p) = \mathcal{S}_j^\mu(q), \forall q \in U_p, 0 \leq j \leq \mu\}$$

is open and dense in  $M$ . We denote by  $M_h^\infty = M \setminus \text{SingSupp}(h)$  the open subset of  $M$  consisting of those points  $p \in M$  such that  $h$  is  $\mathcal{C}^\infty$ -smooth in a neighborhood of  $p$ , and similarly  $\widetilde{M}_h^\infty = \widetilde{M} \setminus \text{SingSupp}(h)$ .

For  $k, \ell, \nu \in \mathbb{N}$  with  $0 \leq k \leq \ell \leq N'$ ,  $k \leq \nu \leq N' - \ell + k - 1$ , we define

$$\Omega_{k,\nu}^\ell = \{p \in \widetilde{M} : \ell \leq \mathcal{S}_k^{N'-\ell+k}(p) < \dots < \mathcal{S}_\nu^{N'-\ell+k}(p) = \mathcal{S}_{\nu+1}^{N'-\ell+k}(p)\}, \quad (5.6)$$

and for  $\nu = N' - \ell + k$ , we set

$$\Omega_{k,N'-\ell+k}^\ell = \{p \in \widetilde{M} : \ell \leq \mathcal{S}_k^{N'-\ell+k}(p) < \dots < \mathcal{S}_{N'-\ell+k}^{N'-\ell+k}(p) = N'\}. \quad (5.7)$$

We further decompose each of the sets  $\Omega_{k,\nu}^\ell$  defined for  $k, \ell, \nu \in \mathbb{N}$  with  $k \leq \nu \leq N' - \ell + k$  by either (5.6) or (5.7) into

$$\widehat{\Omega}_{k,\nu}^{\ell,m} := \{p \in \Omega_{k,\nu}^\ell : \mathcal{S}_\nu^{N'-\ell+k}(p) = m\}, \quad \ell \leq m \leq N', \quad (5.8)$$

so that we have

$$\bigsqcup_{m=\ell}^{N'} \widehat{\Omega}_{k,\nu}^{\ell,m} = \Omega_{k,\nu}^\ell. \quad (5.9)$$

Note that each  $\widehat{\Omega}_{k,\nu}^{\ell,m}$  is open in  $\Omega_{k,\nu}^\ell$ , in  $\widetilde{M}$ , and thus also open in  $M$ . Let us finally note that (5.7) implies that

$$\widehat{\Omega}_{k,N'-\ell+k}^{\ell,m} = \emptyset, \quad \text{for } m < N'. \quad (5.10)$$

With the same proof as in [16, Proposition 6.4], we see that.

**Proposition 5.4.** *Let  $M$  be an abstract CR manifold and  $h : M \rightarrow \mathbb{C}^{N'}$  a  $\mathcal{C}^1$ -smooth CR map, and  $\ell, k \in \mathbb{N}$  such that  $0 \leq k \leq \ell \leq N'$ . If the open subset  $M_\ell^k := \{q \in M : \mathcal{S}_k^{N'-\ell+k}(q) \geq \ell\}$  is dense in  $M$ , then the open set  $\bigcup_{\nu=k}^{N'-\ell+k} \Omega_{k,\nu}^\ell$  is dense in  $M$ , where  $\Omega_{k,\nu}^\ell$  are defined by (5.6) and (5.7).*

We will relate the open subsets  $\widehat{\Omega}_{k,\nu}^{\ell,m}$  defined in (5.7) to smoothness properties of the map  $h$  (using Theorem 4.8) and to the CR geometry of  $h(M)$  in the next two results.

**Proposition 5.5.** *Let  $M$  be an abstract CR manifold,  $p \in M$  and assume that  $M$  has the microlocal extension property at  $p$ . Let  $h: M \rightarrow \mathbb{C}^{N'}$  be a CR map of class  $\mathcal{C}^1$ . If there exist  $j \leq \mu \leq N'$  such that  $\mathcal{S}_j^\mu(p) = N'$ , then  $p \in M_h^\infty$ . In particular it holds that*

$$\bigcup_{\nu=k}^{N'-\ell+k} \widehat{\Omega}_{k,\nu}^{\ell,N'} \subset \widetilde{M}_h^\infty, \quad \text{where } 0 \leq k \leq \ell \leq N' \quad \text{with } k \leq \nu \leq N' - \ell + k.$$

**Proof.** If there are  $j, \mu$  such that  $\mathcal{S}_j^\mu(p) = N'$ , then we can find  $(g, r^1), \dots, (g, r^{N'}) \in \mathcal{A}_p^{j,\mu}$  such that for  $q \in M$  near  $p$ ,

$$r^j(q, h(q), \overline{h(q)}, \overline{g(q)}) = 0, \quad j = 1, \dots, N',$$

and such that the Jacobian  $r_w$  of the map  $r = (r^1, \dots, r^{N'})$  is invertible at  $p$ . Hence we may apply Corollary 4.10 to conclude that  $h$  is  $\mathcal{C}^\infty$ -smooth in a neighborhood of  $p$ . The remainder of the proposition follows in a straightforward way. The proof is complete.  $\square$

Once we have dealt with what is happening for all points belonging to each of the open subsets  $\widehat{\Omega}_{k,\nu}^{\ell,N'}$ , we now turn to describing how the existence of points in one of the sets  $\widehat{\Omega}_{k,\nu}^{\ell,m}$  for  $m < N'$  impacts the CR geometry of the set  $h(M)$ . This is explained in the following result.

**Proposition 5.6.** *Let  $M$  be an abstract CR manifold and let  $h: M \rightarrow \mathbb{C}^{N'}$  be a CR map of class  $\mathcal{C}^1$ . Let  $k, \ell, m, \nu \in \mathbb{N}$  with  $k \leq \nu \leq N' - \ell + k - 1$  and  $0 \leq k \leq \ell \leq m < N'$ . If  $h$  is of class  $\mathcal{C}^{N'-\ell+k-\nu}$  on  $\widehat{\Omega}_{k,\nu}^{\ell,m}$ , then there exists a CR family of  $(N' - m)$ -dimensional formal holomorphic submanifolds  $(\Gamma_q)_{q \in \widehat{\Omega}_{k,\nu}^{\ell,m}}$ , depending in a  $\mathcal{C}^{N'-\ell+k-\nu}$  fashion on  $q$ , such that for every  $q \in \widehat{\Omega}_{k,\nu}^{\ell,m}$ ,  $\Gamma_q$  is formally contained in  $h(M)$  at  $h(q)$ .*

The proof of Proposition 5.6 follows along the lines of the arguments from [16]; we highlight in what follows the main steps of the proof adapted to the abstract case studied in this paper. Throughout the rest of this section, we fix  $k, \ell, m, \nu \in \mathbb{N}$  as in Proposition 5.6.

For  $p \in \widehat{\Omega}_{k,\nu}^{\ell,m}$ , we have by definition  $\dim \mathbb{V}_p^{\nu, N'-\ell+k} = N' - m$ . One of the main properties which our construction needs is that locally one can find a basis of CR vector fields which span  $\mathbb{V}_q^{\nu, N'-\ell+k}$  for  $q$  close to  $p$ . Before we formulate the proposition, let us recall the following result [16, Lemma 5.4], which also holds in the abstract setting.

**Lemma 5.7.** *Let  $M$  be an abstract CR manifold of CR dimension  $n$ ,  $p \in M$ , and  $\mathcal{R}_p$  be a subring of  $\mathcal{C}^\tau(M, p)$ , for some  $\tau \in \mathbb{Z}_+$ , satisfying the following*

condition: for every  $\psi \in \mathcal{R}_p$ , if  $\psi(p) \neq 0$  then  $1/\psi \in \mathcal{R}_p$ . Let  $N' \geq 1$ ,  $1 \leq \delta < N'$ , and  $A^1, \dots, A^\delta$  be germs of  $p$  of  $\mathbb{C}^{N'}$ -valued mappings with components in  $\mathcal{R}_p$ . Assume that

- (i) The rank of the  $N' \times \delta$  matrix  $A := (A^1, \dots, A^\delta)$  is equal to  $\delta$  at  $p$ .
- (ii) For any smooth CR vector field  $\bar{L}$  of  $M$  near  $p$ , the rank of the  $N' \times 2\delta$  matrix  $(A, \bar{L}A)$  is constantly equal to  $\delta$  in a neighborhood of  $p$ .

Then there exist  $N' - \delta$  germs at  $p$  of  $\mathbb{C}^{N'}$ -valued mappings, with components in  $\mathcal{R}_p \cap \mathcal{C}_{CR}^\tau(M, p)$ , denoted by  $V^1, \dots, V^{N'-\delta}$  such that for  $1 \leq j \leq N' - \delta$  and  $1 \leq \gamma \leq \delta$ , we have

$$V^j \cdot A^\gamma := \sum_{i=1}^{N'} V_i^j A_i^\gamma = 0 \quad \text{in } \mathcal{R}_p, \tag{5.11}$$

and such that  $V^1(p), \dots, V^{N'-\delta}(p)$  are linearly independent.

Proposition 5.8 is obtained from Lemma 5.7 in exactly the same way as [16, Proposition 5.3] and [16, Lemma 5.5] are obtained from [16, Lemma 5.4].

**Proposition 5.8.** *Let  $M$  be an abstract CR manifold and let  $h: M \rightarrow \mathbb{C}^{N'}$  be a CR map of class  $\mathcal{C}^1$ . Let  $k, \ell, m, \nu \in \mathbb{N}$  with  $k \leq \nu \leq N' - \ell + k - 1$  and  $0 \leq k \leq \ell \leq m < N'$  and assume that  $h$  is of class  $\mathcal{C}^{N'-\ell+k-\nu}$  on  $\widehat{\Omega}_{k,\nu}^{\ell,m}$ . Then for every  $p \in \widehat{\Omega}_{k,\nu}^{\ell,m}$ , there exist a neighborhood  $U_p \subset \widehat{\Omega}_{k,\nu}^{\ell,m}$  of  $p$  and CR maps  $V^j: U_p \rightarrow \mathbb{C}^{N'}$  of class  $\mathcal{C}^{N'-\ell+k-\nu}$ ,  $j = 1, \dots, N' - m$ , whose components belong to  $\mathcal{F}_p^{\nu, N'-\ell+k}$ , such that  $\{V^1(q), \dots, V^{N'-\ell}(q)\}$  forms a basis of  $\mathbb{V}_q^{\nu, N'-\ell+k}$  for every  $q \in U_p$ . Furthermore, for every  $q \in U_p$ , we have  $\mathbb{V}_q^{\nu, N'-\ell+k} = \mathbb{V}_q^{\nu+1, N'-\ell+k}$  and for every  $(g, r) \in \mathcal{A}_q^{\nu+1, N'-\ell+k}$  defined on a neighborhood  $\tilde{U}_q \subset U_p$  of  $q$ , it holds that*

$$V^j(\tilde{q}) \cdot r_w(\tilde{q}, h(\tilde{q}), \overline{h(\tilde{q})}, g(\tilde{q})) = 0, \quad \tilde{q} \in \tilde{U}_q, \quad j = 1, \dots, N' - m.$$

Once we have established all the useful properties of the spaces  $\mathbb{V}_p^{\nu, N'-\ell+k}$ , we can now construct in the next proposition the formal holomorphic submanifolds that appear in the statement of Proposition 5.6. Note that the proposition does not make any assumption on  $M$ ; in particular no microlocal extension property on  $M$  is required.

**Proposition 5.9.** *Under the assumptions of Proposition 5.6, for every  $p \in \widehat{\Omega}_{k,\nu}^{\ell,m}$ , let  $\mathbf{V} = (V^1, \dots, V^{N'-m})$  and  $U_p$  be the open set from Proposition 5.8. For  $t = (t_1, \dots, t_{N'-m}) \in \mathbb{C}^{N'-m}$ , we set  $t \cdot \mathbf{V} := \sum_{i=1}^{N'-m} t_i V^i$ . For  $d \in \mathbb{Z}_+$ , define a family of homogeneous polynomial maps of degree  $d$  in  $(\mathcal{F}_p^{\nu, N'-\ell+k}[t])^{N'}$  inductively by setting*

$$D^1(t) := t \cdot \mathbf{V}, \quad D^{d+1}(t) := \frac{1}{d+1} (t \cdot \mathbf{V}) \cdot D_w^d(t), \quad d \geq 1. \tag{5.12}$$

Further set  $D(t) := \sum_{d=1}^{\infty} D^d(t) \in (\mathcal{F}_p^{\nu, N'-\ell+k}[[t]])^{N'}$  and write  $D(t) = \sum_{\alpha \in \mathbb{N}^{N'-m}} d_{\alpha} t^{\alpha}$ . Then the following hold:

- (a) For each  $\alpha \in \mathbb{N}^{N'-m}$ ,  $d_{\alpha}$  is a well-defined CR map on  $U_p$  and of class  $\mathcal{C}^{N'-\ell+k-\nu}$ .
- (b) For every  $q \in U_p$ ,  $t \mapsto D(q; t) := h(q) + \sum_{\alpha \in \mathbb{N}^{N'-m}} d_{\alpha}(q) t^{\alpha}$  defines an  $(N' - m)$ -dimensional formal holomorphic submanifold through  $h(q)$ , which we denote by  $\Gamma_q$ .
- (c) For every  $q \in U_p$ ,  $\Gamma_q$  is formally contained in  $h(M)$  at  $h(q)$ .

The proof of Proposition 5.9 is carried out exactly the same way as the proof of [16, Proposition 5.5] by using the following properties:

- (i)  $\{V^1(q), \dots, V^{N'-\ell}(q)\}$  forms a basis of  $\mathbb{V}_q^{\nu, N'-\ell+k}$  for every  $q \in U_p$ ,
- (ii) each  $V^i(q)$  belongs to  $\mathbb{V}_q^{\nu, N'-\ell+k}$ , i.e. that for every  $(g, r) \in \mathcal{A}_q^{\nu, N'-\ell+k}$ ,
 
$$V^i \cdot r_w(q) = 0, \text{ for } q \in U_p, \quad i = 1, \dots, N' - m,$$
- (iii) for each  $i = 1, \dots, N' - m$ , the conjugate  $\overline{V^i}$  also belongs to  $\mathcal{F}_q^{\nu, N'-\ell+k}$  for  $q \in U_p$  (since  $V^i$  is CR and therefore admissible),
- (iv) an appropriate use of the chain rule (e.g. [16, Lemma 5.7]).

**Proof of Proposition 5.6.** A direct application of Proposition 5.9 yields the (local) existence of  $\Gamma_q$ , for every  $q \in \widehat{\Omega}_{k, \nu}^{\ell, m}$ , with the required property. We just have to check that the construction from Proposition 5.9 actually yields the same formal submanifolds on overlaps of neighborhoods  $U_p \cap U_{\tilde{p}} \subset \widehat{\Omega}_{k, \nu}^{\ell, m}$  with  $p, \tilde{p} \in \widehat{\Omega}_{k, \nu}^{\ell, m}$  (the following argument also yields that the  $\Gamma_q$  are independent of the chosen basis  $\mathcal{V}$  in Proposition 5.9).

On such an intersection, by construction of  $U_p$  and  $U_{\tilde{p}}$ , if we denote the local bases used in Proposition 5.9 by  $\mathbf{V} = (V^1, \dots, V^{N'-m})$  and  $\tilde{\mathbf{V}} = (\tilde{V}^1, \dots, \tilde{V}^{N'-m})$ , respectively, there exist an invertible matrix  $A = (A_s^r)_{r,s=1}^{N'-m}$  whose components belong to  $\mathcal{C}_{CR}^{N'-\ell+k-\nu}(U_p \cap U_{\tilde{p}})$  (and  $\mathcal{F}_q^{\nu, N'-\ell+k}$  for every  $q \in U_p \cap U_{\tilde{p}}$ ) such that

$$\tilde{V}^r = \sum_{s=1}^{N'-m} A_s^r V^s.$$

Denote by  $D(q; t)$  the parametrization obtained from Proposition 5.9 for the formal submanifolds  $\Gamma_q$ 's, associated to  $\mathbf{V}$ , and  $\tilde{D}(q; \tilde{t})$  the one corresponding to the  $\tilde{\Gamma}_q$ 's associated to  $\tilde{\mathbf{V}}$ , for  $q \in U_p \cap U_{\tilde{p}}$ . The reader may check that, using the chain rule, one may easily construct, for every  $q \in U_p \cap U_{\tilde{p}}$ , a formal (holomorphic) invertible map  $\Phi_q: (\mathbb{C}^{N'-m}, 0) \rightarrow (\mathbb{C}^{N'-m}, 0)$ , whose coefficients depend  $\mathcal{C}^{N'-\ell+k-\nu}$  on  $q$ , such that

$$\Phi_q(t) = A(q)t + \dots, \quad \tilde{D}(q; \Phi_q(t)) = D(q; t).$$

Hence for  $q \in U_p \cap U_{\tilde{p}}$ , the formal submanifolds  $\Gamma_q$  and  $\tilde{\Gamma}_q$  coincide. The proof of the proposition is complete. □

## 6. Proofs of Theorem 2.3 and its Consequences

### 6.1. Proofs of Theorems 1.1 and 2.3, Corollaries 2.4, 2.6 and 2.7

**Proof of Theorem 2.3.** First of all, since  $h$  is of class  $\mathcal{C}^{N'-k+\ell}$ , it is not difficult to see that for every  $p \in M$ , we have  $\mathcal{S}_k^{N'-\ell+k}(p) \geq e_k(p)$ , where  $e_k(p)$  is given by (2.3). Hence  $\mathcal{S}_k^{N'-\ell+k}(p) \geq e_k \geq \ell$  for  $p$  on some dense open subset of  $M$ . This means that the open subset  $M_\ell^k$  given in Proposition 5.4 is dense in  $M$  and we therefore conclude from that proposition that  $\bigcup_{\nu=k}^{N'-\ell+k} \Omega_{k,\nu}^\ell$  is dense too in  $M$ , where each  $\Omega_{k,\nu}^\ell$  is given by (5.6) and where the union is a disjoint union. We therefore get that the disjoint union

$$\bigsqcup_{\nu=k}^{N'-\ell+k} \bigsqcup_{m=\ell}^{N'} \widehat{\Omega}_{k,\nu}^{\ell,m}$$

is dense in  $M$ . By Proposition 5.5, we have  $\bigcup_{\nu=k}^{N'-\ell+k} \widehat{\Omega}_{k,\nu}^{\ell,N'} \subset \widetilde{M}_h^\infty$  and hence, using in addition (5.10),

$$\bigsqcup_{\nu=k}^{N'-\ell+k-1} \bigsqcup_{m=\ell}^{N'-1} \widehat{\Omega}_{k,\nu}^{\ell,m}$$

is a dense open subset of  $(\text{SingSupp } h)^\circ$ . Applying Proposition 5.6, we reach the conclusion of the theorem.  $\square$

**Proof of Corollary 2.4.** As in [16, Lemma 6.1], it is easy to show that  $e_0 \geq N' - n'$ . Applying Theorem 2.3 with  $k = 0$  and  $\ell = N' - n'$ , we get a dense open subset  $\Omega$  of  $(\text{SingSupp } h)^\circ$  satisfying  $h(\Omega) \subset \mathcal{E}_{M'}$ . The result is proved.  $\square$

**Proof of Corollary 2.6.** As in [16, Lemma 6.2], one can show that the Levi-nondegeneracy assumption on  $M'$  and the fact that  $dh$  is injective on  $T^{1,0}M$  imply that  $e_1 \geq N' - n' + n$ . Now applying Theorem 2.3 with  $k = 1$  and  $\ell = N' - n' + n$ , we reach the desired result.  $\square$

For the proof of Corollary 2.7, we need the following version of [16, Proposition 7.1] in the abstract case for which we briefly sketch its proof.

**Proposition 6.1.** *Let  $M$  be an abstract strongly pseudoconvex CR manifold of hypersurface type, of CR dimension  $n$ , and  $M' \subset \mathbb{C}^{n'+1}$  be (connected)  $\mathcal{C}^\infty$ -smooth Levi-nondegenerate of signature  $\ell'$ ,  $n' > n \geq 1$ . Assume that there exists a point  $p \in M$  and a germ at  $p$  of strictly noncharacteristic CR map  $h: (M, p) \rightarrow M'$  of class  $\mathcal{C}^2$  satisfying the following: there exists a neighborhood  $V \subset M$  of  $p$ , and for every  $\xi \in V$ , a smooth complex curve  $\Upsilon_\xi$  in  $\mathbb{C}^{n'+1}$  containing  $h(\xi)$ , depending in a  $\mathcal{C}^1$  manner on  $\xi \in V$ , such that the order of contact of  $\Upsilon_\xi$  with  $M'$  at  $h(\xi)$  is greater or equal to 3. Then necessarily  $n < n' - \ell' < n'$ .*

**Sketch of proof of Proposition 6.1.** The proof of [16, Proposition 7.1] is obtained by adapting the arguments of [19, Proposition 3.1]. We claim that the

same arguments may be used in the present situation (of an abstract strongly pseudoconvex CR manifold  $M$  of hypersurface type). Indeed, using the fact that there exists an integrable strongly pseudoconvex CR structure  $\hat{\mathcal{V}}$  on  $M$  near  $p$  whose CR bundle agrees with  $\mathcal{V}$  to infinite order at  $p$  (see [23, Theorem IV.1.3]), one obtains, analogously to what is done in [4], first order normalization conditions for the map  $h$  at  $p$ . Once these normalizations are obtained, the proof of [19, Proposition 3.1] can be carried out in the same way (with obvious adjustments) to prove Proposition 6.1 (see also [15, Lemma 6.7] for similar arguments). We leave the details to the reader.  $\square$

**Proof of Corollary 2.7.** By [3, Theorem 2.9],  $M$  satisfies the microlocal extension property (at every point). Furthermore, without loss of generality, we may assume that  $M$  is connected. If there is point on  $M$  whose Levi form has one positive and one negative eigenvalue, then the same holds at every point on  $M$ , and by [3, Theorem 2.9] and Theorem 4.7,  $h$  is smooth all over  $M$ . Hence, we may assume that  $M$  is strongly pseudoconvex. Now using again the fact that there exists an integrable strongly pseudoconvex CR structure  $\hat{\mathcal{V}}$  on  $M$  near  $p$  whose CR bundle agrees with  $\mathcal{V}$  to infinite order at  $p$ , one can show that, since  $h$  is strictly non-characteristic,  $h$  must be CR immersive too (see [4]). Therefore, as in the proof of Corollary 2.6, we have that  $e_1 \geq n + 1$ . Applying Theorem 2.3 with  $k = 1$  and  $\ell = n + 1$ , we get that if  $(\text{SingSupp } h)^\circ$  were not empty, there would exist a nonempty open subset  $V$  of  $M$  and a family of formal holomorphic curves  $\Gamma_q$ , for  $q \in V$ , depending on a  $\mathcal{C}^1$ -smooth and CR manner on  $q$ , such that  $h(q) \in \Gamma_q$  and  $\Gamma_q$  is formally contained in  $M'$  (at  $h(q)$ ). From such a family, we easily get another family  $\Upsilon_q$ , for  $q \in V$ , of smooth complex curves passing through  $h(q)$ , depending in a  $\mathcal{C}^1$  manner on  $q$ , that are tangent to order  $\geq 3$  to  $M'$  at  $h(q)$ . It follows from Proposition 6.1 that  $n < n' - \ell'$ , a contradiction. The proof is complete.  $\square$

### 6.2. Proof of Corollary 2.5

We shall prove Corollary 2.5 by establishing the following more general result, which contains Corollary 2.5 as a special case for  $k = 1$ . First let us recall that an abstract CR manifold  $M$  is called  $k$ -finitely nondegenerate at a point  $p \in M$ ,  $\sigma \in \mathbb{Z}_+$ , if the Lie derivatives

$$\mathcal{L}_{\bar{K}_1} \cdots \mathcal{L}_{\bar{K}_j} \vartheta(p), \quad j \leq k, \quad \vartheta \in \Gamma(M, T^0 M), \quad \bar{K}_\nu \in \Gamma(M, \mathcal{V})$$

span  $T'_p M$ , and it is called  $k$ -finitely nondegenerate if it is  $k$ -finitely nondegenerate at each point. Furthermore,  $M$  is 1-finitely nondegenerate if and only if it is Levi-nondegenerate (see [1] for more on this).

**Theorem 6.2.** *Let  $M$  be a  $k$ -finitely nondegenerate abstract CR manifold with the microlocal extension property, of CR dimension  $n$  and CR codimension  $d$ , and  $M' \subset \mathbb{C}^{N'}$  be a  $\mathcal{C}^\infty$ -smooth CR submanifold with  $N' > N = n + d$ . Then for every*

strictly noncharacteristic CR map  $h : M \rightarrow M'$  of class  $\mathcal{C}^{N'-N+k}$ , there is a dense open subset  $\Omega$  of  $(\text{SingSupp } h)^\circ$  such that  $h(\Omega) \subset \mathcal{E}_{M'}$ . In particular, if  $M'$  is of D'Angelo finite type,  $h$  is  $\mathcal{C}^\infty$ -smooth on a dense open subset of  $M$ .

For the proof of Theorem 6.2, we need to prove the following lemma, which can be viewed as a version of [16, Lemma 6.3].

**Lemma 6.3.** *Let  $M$  be an abstract CR manifold and  $M' \subset \mathbb{C}^{N'}$  be a  $\mathcal{C}^\infty$ -smooth CR submanifold. If  $M$  is  $k$ -finitely nondegenerate and if  $h : M \rightarrow M'$  is a strictly noncharacteristic CR map of class  $\mathcal{C}^{k+1}$ , then  $e_k \geq n + d$  where  $n = \dim_{\text{CR}} M$  and  $d = \text{codim}_{\text{CR}} M$ .*

In order to prove Lemma 6.3, we need to recall a number of facts and establish a basic smooth-to-formal transition. We are going to write

$$e_\ell^M(p) := \dim_{\mathbb{C}} \{ \mathcal{L}_{\bar{K}_1} \cdots \mathcal{L}_{\bar{K}_k} \vartheta(p), \quad k \leq \ell, \quad \vartheta \in \Gamma(M, T^0 M), \quad \bar{K}_j \in \Gamma_p(M, \mathcal{V}) \},$$

and similarly for  $M'$ .

We recall that a formal generic submanifold  $\hat{M} \subset \mathbb{C}_z^N$  of codimension  $d$  (through the origin) is given by a formal manifold ideal  $\hat{I} = \hat{I}(\hat{M}) \subset \mathbb{C}[[z, \bar{z}]]$ , that is,  $\hat{I} = (\rho^1, \dots, \rho^d)$  and the generators  $\rho^1, \dots, \rho^d \in \mathbb{C}[[z, \bar{z}]]$  can be chosen to satisfy the following conditions:

- (1) Each of the  $\rho^j$  is real:  $\rho^j(z, \bar{z}) = \bar{\rho}^j(\bar{z}, z)$ , for  $j = 1, \dots, d$ .
- (2) The complex gradients of the  $\rho^j$  are linearly independent at 0:

$$\text{rk} \begin{pmatrix} \frac{\partial \rho^1}{\partial z_1}(0) & \cdots & \frac{\partial \rho^1}{\partial z_n}(0) \\ \vdots & & \vdots \\ \frac{\partial \rho^d}{\partial z_1}(0) & \cdots & \frac{\partial \rho^d}{\partial z_n}(0) \end{pmatrix} = d.$$

A formal generic submanifold has a formal coordinate ring

$$\mathbb{C}[[\hat{M}]] = \mathbb{C}[[z, \bar{z}]] / \hat{I}(\hat{M}),$$

which is isomorphic to a power series ring in  $2n + d$  variables. We are denoting the image of a formal power series  $A(z, \bar{z})$  in  $\mathbb{C}[[\hat{M}]]$  as  $A(z, \bar{z})|_{\hat{M}}$ .

Assume that  $\hat{M} \subset \mathbb{C}_z^N$  and  $\hat{M}' \subset \mathbb{C}_w^{N'}$  are formal generic submanifolds. A formal power series map of the form  $\mathcal{H}(z, \bar{z}) = (H(z), \bar{H}(\bar{z}))$  is said to be a formal holomorphic map from  $\hat{M}$  to  $\hat{M}'$  if  $\mathcal{H}^* \hat{I}(\hat{M}') \subset \hat{I}(\hat{M})$ , i.e. if  $\rho'(H(z), \bar{H}(\bar{z})) \in \hat{I}(\hat{M})$  for every  $\rho' \in \hat{I}(\hat{M}')$ . If we denote the maximal ideal of  $\mathbb{C}[[\hat{M}]]$  by  $\mathfrak{m}(\hat{M})$ , then we say that a formal power series map of the form above is a  $k$ -approximate formal map from  $M$  to  $M'$  if  $\rho'(H(z), \bar{H}(\bar{z}))|_M \in \mathfrak{m}(\hat{M})^{k+1}$  for every  $\rho' \in \hat{I}(\hat{M}')$ .



We also need to introduce the notion of a formal abstract CR structure before we will show that this is actually the same type of concept as that of a formal generic submanifold. For this, we let  $x = (x_1, \dots, x_m)$  be (real) formal unknowns, which we think of as coordinate functions of a formal manifold, i.e. we are identifying its coordinate ring with  $\mathbb{R}[[x]]$ . The ring  $\mathbb{C}[[x]]$  has a natural involution given by

$$a(x) = \sum_{\alpha} a_{\alpha} x^{\alpha} \mapsto \bar{a}(x) = \sum_{\alpha} \bar{a}_{\alpha} x^{\alpha},$$

which extends in a natural way to free modules over  $\mathbb{C}[[x]]$ . For any subset  $A \subset \mathbb{C}[[x]]^m$  of such a free module we denote by  $\bar{A}$  the set of conjugates of elements of  $A$ . If  $A \subset \mathbb{C}[[x]]^m$  is a submodule, so is  $\bar{A}$ . We define the Lie brackets of elements of  $\mathbb{C}[[x]]^m$  by identifying

$$a(x) = \begin{pmatrix} a_1(x) \\ \vdots \\ a_m(x) \end{pmatrix} \quad \text{with} \quad \sum_{j=1}^m a_j(x) \frac{\partial}{\partial x_j},$$

and also use this to define the action of  $a \in A$  on a formal function  $f$ , i.e.

$$a \cdot f = \sum_{j=1}^m a_j(x) \frac{\partial f}{\partial x_j}(x).$$

We identify the elements of

$$A^{\perp} = \left\{ (b^1(x), \dots, b^m(x)) : b \cdot a = \sum_j a_j(x) b^j(x) = 0 \forall a \in A \right\}$$

with forms, i.e.

$$(b^1(x), \dots, b^m(x)) \text{ is identified with } \sum_j b^j(x) dx_j.$$

The exterior differential of a formal power series  $f$  is defined in the usual way; note that a function  $f$  is a solution to the differential equations  $a \cdot f = 0$  for  $a \in A$  if and only if  $df \in A^{\perp}$ .

**Definition 6.4.** We say that a submodule  $A$ , of constant rank  $n$ , of  $\mathbb{C}[[x]]^m$  defines a formal abstract CR structure if

- (1)  $A \cap \bar{A} = \{0\}$ ; and
- (2)  $[A, A] \subset A$ .

Let  $A \subset \mathbb{C}[[x]]^m$  and  $B \subset \mathbb{C}[[y]]^{m'}$  be formal abstract CR structures. We say that a formal power series map  $\hat{h}(x) = (\hat{h}^1(x), \dots, \hat{h}^{m'}(x)) \in \mathbb{R}[[x]]^{m'}$  is CR if  $h^* B^{\perp} \subset A^{\perp}$ , where the pullback operation is defined by

$$\hat{h}^* \left( \sum_j b'_j(y) dy_j \right) = \sum_j b'_j(h(x)) d\hat{h}^j(x).$$

Two formal abstract CR structures are said to be *formally equivalent* if there exists an invertible CR map taking one into the other.

An example of a formal abstract CR structure comes from a formal generic submanifold  $\hat{M} \subset \mathbb{C}^N$  of codimension  $d$  as discussed before: If we define the  $(0, 1)$ -vector fields tangent to  $\hat{M}$  to be the formal vector fields of the form

$$\bar{L} = \sum_{j=1}^N \bar{L}_j(z, \bar{z}) \frac{\partial}{\partial \bar{z}_j}, \quad \text{satisfying } \bar{L}\rho(z, \bar{z}) = 0 \quad \forall \rho \in \hat{I}(\hat{M}),$$

then each such  $\bar{L}$  defines a formal vector field acting on  $\mathbb{C}[[z, \bar{z}]]/\hat{I}(\hat{M})$ ; we call the module of all these vector fields  $\mathcal{D}^{(0,1)}(\hat{M})$ . Now note that because of the assumption on the rank of  $\rho_w$ ,  $\mathcal{D}^{(0,1)}(\hat{M})$  is a module of (constant) rank  $n = N - d$  over the ring  $\mathbb{C}[[z, \bar{z}]]/\hat{I}(\hat{M})$ , which as observed above is the coordinate ring of a formal real manifold of dimension  $m = 2n + d$ , and thus isomorphic to some power series ring  $\mathbb{C}[[x_1, \dots, x_{2n+d}]]$ ; a convenient coordinate choice is  $(z, \bar{z}')$ , where  $\bar{z}'$  consists of  $n$  of the  $\bar{z}$ . In this case,  $\mathcal{D}^{(0,1)}(\hat{M})$  is generated by the  $dz_j$ .

Note that a formal CR map between these formal abstract CR structures is the same as a formal holomorphic map defined above. On the one hand, if  $\mathcal{H}(z, \bar{z}) = (H(z), \bar{H}(\bar{z}))$  is a formal holomorphic map, then for any  $j = 1, \dots, N'$ , we have that  $H^*dw_j = \sum_k \frac{\partial H_j}{\partial z_k}(z) dz_k$ . If on the other hand we have a map  $h = (H(z, \bar{z}), \bar{H}(\bar{z}, z))$  which takes the  $dw_j$  to linear combinations of the  $dz_k$ , then necessarily  $\bar{L}H_k(z, \bar{z}) = 0$  for every formal CR vector field  $\bar{L}$  and for every  $k = 1, \dots, N'$ . Because  $M$  is generic, this is only possible if every  $H_k$  is independent of  $\bar{z}$ .

An easy application of the Frobenius theorem yields that a formal abstract CR structure  $A$  is integrable in the sense that  $A^\perp$  is spanned by linearly independent differentials of solutions  $Z(x)$  of the equations  $\bar{L}Z = 0$ , where  $\bar{L} \in A$ , just as in the case of a real-analytic CR structure: We recall the corresponding statement and proof.

**Lemma 6.5.** *Let  $A \subset \mathbb{C}[[x_1, \dots, x_m]]^m$  be an abstract formal CR structure, of rank  $n$  and codimension  $d = m - 2n$ . Then there exists a formal generic manifold  $\hat{M} \subset \mathbb{C}^N$ , with  $N = n + d$ , such that the formal CR structure of  $\hat{M}$  is formally equivalent to  $A$ .*

**Proof.** Let  $x = (x_1, \dots, x_m)$  be the real coordinates in the theorem, and choose a basis  $\bar{L}_1, \dots, \bar{L}_n$  of  $A$ ,

$$\bar{L}_j = \sum_{k=1}^m \bar{L}_j^k(x) \frac{\partial}{\partial x_k}.$$

Since  $[\bar{L}_j, \bar{L}_\ell] = \sum_k a_{j,\ell}^k \bar{L}_k$  for some formal power series  $a_{j,\ell}^k$ , we can use the Frobenius theorem in the formal category for the family of vector fields  $\hat{L}_1, \dots, \hat{L}_n$  and

obtain that there exist  $N = n + d$  formal integrals  $\hat{z}_1(x), \dots, \hat{z}_N(x) \in \mathbb{C}[[x]]$ , with linearly independent differentials (at 0). The image of  $\mathbb{R}^{2n+d}$  (as a formal manifold) under the formal map  $x \mapsto \hat{z}(x) = (\hat{z}_1(x), \dots, \hat{z}_N(x))$  is a formal generic manifold of  $\mathbb{C}^N$  possessing the desired properties.  $\square$

Yet another way to obtain a formal abstract CR structure is from an abstract CR structure  $(M, \mathcal{V})$  at a point  $p \in M$ . In order to do that, let us denote for a germ of a smooth function  $a \in \mathcal{C}^\infty(M, p)$  by  $\hat{a}_p$  the formal function associated to  $a$ , i.e. the image of  $a$  in the quotient of  $\mathcal{C}^\infty(M, p)$  under the equivalence relation that  $a$  and  $b$  agree to infinite order at  $p$ . This ring can, in any set of smooth coordinates vanishing at  $p$ , be identified with the formal Taylor series at  $p$  via  $\hat{a}_p(x) = \sum_{\alpha} \frac{1}{\alpha!} \frac{\partial^{|\alpha|} a}{\partial x^\alpha}(0) x^\alpha$ , and the construction can be adapted to also define formal maps associated to a smooth map in the same way. We now define  $A$  to consist of all of the formal vector fields coming from local sections of  $\mathcal{V}$ , i.e.

$$A = \left\{ \sum_k \hat{L}^k \frac{\partial}{\partial x_k} : \bar{L} = \sum_k \bar{L}^k(x) \frac{\partial}{\partial x_k} \in \Gamma(M, \mathcal{V}) \right\}.$$

We will denote the formal CR structure gotten in this way by  $\hat{M}_p$ .

The formal map associated to a smooth CR map  $h : (M, \mathcal{V}) \rightarrow (M', \mathcal{V}')$  between abstract CR structures gives rise to a formal CR map  $\hat{h}_p : \hat{M}_p \rightarrow \hat{M}'_{h(p)}$ . If  $h$  is a  $\mathcal{C}^k$ -smooth CR map, then we analogously obtain a  $k$ -approximate formal CR map  $\hat{h}_p^{[k]} : \hat{M}_p \rightarrow \hat{M}'_{h(p)}$ .

We are now ready to give the proof of Lemma 6.3.

**Proof of Lemma 6.3.** Assume that  $h$  is strictly noncharacteristic at  $p$ . We are going to show that the inequality  $e_s^M(p) \leq e_s(p)$  holds for every  $s \leq k$ . From this, the lemma follows. Since  $h$  is strictly noncharacteristic, there exist functions  $\rho^1, \dots, \rho^d \in \mathcal{S}_{h(p)}(M')$ , satisfying  $\partial\rho^1 \wedge \dots \wedge \partial\rho^d(p) \neq 0$ , such that  $h^*i\partial\rho^1, \dots, h^*i\partial\rho^d$  span  $T^0M$  near  $p$ . We replace  $M'$  by the (larger) smooth manifold near  $h(p)$  defined by  $\rho^1 = \dots = \rho^d = 0$ .

Now, we compute the first  $k$  Lie derivatives of the  $h^*i\partial\rho^1, \dots, h^*i\partial\rho^d$  on the associated formal manifold  $\hat{M}_p$ ; we are allowed to do this because the map  $h$  is  $\mathcal{C}^{k+1}$  and therefore  $\hat{h} = \hat{h}_p^{[k+1]}$  is a  $(k + 1)$ -approximate formal CR map from  $\hat{M}_p$  to  $\hat{M}'_{h(p)}$ .

It follows that, with the  $\hat{z}_j$  denoting the formal integrals of  $\hat{M}_p$  as before, that

$$\begin{aligned} & \mathcal{L}_{\bar{K}_1} \dots \mathcal{L}_{\bar{K}_\ell} h^*i\partial\rho^j(p) \\ &= \mathcal{L}_{\bar{K}_1} \dots \mathcal{L}_{\bar{K}_k} \sum_{k=1}^{N'} (\varrho_{w_k}^j \circ h) dh_k \Big|_p \end{aligned}$$

$$\begin{aligned}
 &= \left( \hat{K}_1 \dots \hat{K}_\ell \sum_{k=1}^{N'} \sum_{\nu=1}^N \left( \hat{\varrho}_{w_k}^j \left( \hat{h}(\hat{z}), \overline{\hat{h}(\hat{z})} \right) \right) \frac{\partial \hat{h}_k}{\partial \hat{z}_\nu} d\hat{z}_\nu \right) \Big|_p \\
 &= \left( \hat{K}_1 \dots \hat{K}_\ell \hat{\varrho}_w^j \left( \hat{h}(\hat{z}), \overline{\hat{h}(\hat{z})} \right) \right) \Big|_p \begin{pmatrix} \frac{\partial \hat{h}_1}{\partial \hat{z}_1}(p) & \dots & \frac{\partial \hat{h}_1}{\partial \hat{z}_N}(p) \\ \vdots & & \vdots \\ \frac{\partial \hat{h}_{N'}}{\partial \hat{z}_1}(p) & \dots & \frac{\partial \hat{h}_{N'}}{\partial \hat{z}_N}(p) \end{pmatrix}.
 \end{aligned}$$

In the last line, we have identified forms with row vectors as before, and used the fact that the  $\hat{h}^k$  are actually formal holomorphic functions. By definition, the space spanned by the collection of all derivatives appearing on the left-hand side for  $\ell \leq s$  of this equation has dimension  $e_s^M(p)$ , and therefore the space spanned by vectors of the form

$$\left( \hat{K}_1 \dots \hat{K}_\ell \hat{\varrho}_w^j \left( \hat{h}(\hat{z}), \overline{\hat{h}(\hat{z})} \right) \right) \Big|_p = \left( \bar{K}_1 \dots \bar{K}_\ell (\varrho_w^j \circ h) \right) \Big|_p$$

also has at least that dimension as claimed. □

**Proof of Corollary 2.5.** We apply Lemma 6.3 and Theorem 2.3 with  $k = 1$  and  $\ell = N$  to conclude the existence of the open subset  $\Omega$  with the required properties. □

**Proof of Theorem 1.1.** We first note that since  $M$  is an abstract strongly pseudoconvex CR manifold of hypersurface type, it is everywhere Levi-nondegenerate and therefore satisfies the microlocal extension property by Theorem 4.6. We may therefore apply Corollary 2.5 with  $N = n + 1$  and  $N' = n' + 1$ , to conclude that for every strictly noncharacteristic CR map  $h : M \rightarrow M'$  of class  $\mathcal{C}^{n'-n+1}$  there exists a dense open subset  $\Omega$  of  $(\text{SingSupp } h)^\circ$  such that  $h(\Omega) \subset \mathcal{E}_{M'}$ . But since  $\mathcal{E}_{M'}$  is closed (see [6]), we even have  $h((\text{SingSupp } h)^\circ) \subset \mathcal{E}_{M'}$ . The proof is complete. □

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