IMRN International Mathematics Research Notices
2003, No. 4

Approximation and Convergence of Formal CR-Mappings

Francine Meylan, Nordine Mir, and Dmitri Zaitsev

1 Introduction and results

An important step in understanding the existence of analytic objects with certain properties consists of understanding the same problem at the level of formal power series. The latter problem can be reduced to a sequence of algebraic equations for the coefficients of the unknown power series and is often simpler than the original problem, where the power series are required to be convergent. It is therefore of interest to know whether such power series are automatically convergent or can possibly be replaced by other convergent power series satisfying the same properties. A celebrated result of this kind is Artin's approximation theorem [1] which states that a formal solution of a system of analytic equations can be replaced by a convergent solution of the same system that approximates the original solution at any prescribed order.

In this paper, we study convergence and approximation properties (in the spirit of [1]) of formal (holomorphic) mappings sending real-analytic submanifolds $M \subset \mathbb{C}^N$ and $M' \subset \mathbb{C}^{N'}$ into each other, $N, N' \geq 2$. In this situation, the above theorem of Artin cannot be applied directly. Moreover, without additional assumptions on the submanifolds, the analogous approximation statement is not even true. Indeed, in view of an example of Moser-Webster [23], there exist real-algebraic surfaces $M, M' \subset \mathbb{C}^2$ that are formally but not biholomorphically equivalent. However, our first main result shows that this phenomenon cannot happen if M is a minimal CR-submanifold (not necessarily algebraic) in $\mathbb{C}^N$ (see Section 2.1 for notation and definitions).

Theorem 1.1. Let $M \subset \mathbb{C}^N$ be a real-analytic minimal CR-submanifold and $M' \subset \mathbb{C}^{N'}$ a real-algebraic subset with $p \in M$ and $p' \in M'$. Then for any formal (holomorphic)

Received 27 May 2002. Revision received 30 July 2002.

mapping $f : (\mathbb{C}^N, p) \to (\mathbb{C}^{N'}, p')$ sending M into M' and any positive integer k, there exists a germ of a holomorphic map $f^k : (\mathbb{C}^N, p) \to (\mathbb{C}^{N'}, p')$ sending M into M', whose Taylor series at p agrees with f up to order k. □

Approximation results in the spirit of Theorem 1.1 have been recently obtained in [8, 9] in the important case when $N = N'$ and f is invertible. Note that under the assumptions of Theorem 1.1, there may exist nonconvergent maps f sending M into M'. For instance, it is easy to construct such maps in case M is not generic in $\mathbb{C}^N$. Also, if M' contains an irreducible complex-analytic subvariety E' of positive dimension through p', such maps f with $f(M) \subset E'$ (in the formal sense) always exist. Our next result shows that these are essentially the only exceptions. Denote by $\mathcal{E}'$ the set of all points of M' through which there exist irreducible complex-analytic subvarieties of M' of positive dimension. This set is always closed (see [12, 18]) but not real-analytic in general (see [20] for an example). In the following, we say that a formal (holomorphic) map $f : (\mathbb{C}^N, p) \to (\mathbb{C}^{N'}, p')$ sends M into $\mathcal{E}'$ if $\varphi(f(x(t))) \equiv 0$ holds for all germs of real-analytic maps $x : (\mathbb{R}_t^{\dim M}, 0) \to (M, p)$ and $\varphi : (M', p') \to (\mathbb{R}, 0)$ such that φ vanishes on $\mathcal{E}'$. We prove the following theorem.

Theorem 1.2. Let $M \subset \mathbb{C}^N$ be a minimal real-analytic generic submanifold and $M' \subset \mathbb{C}^{N'}$ a real-algebraic subset with $p \in M$ and $p' \in M'$. Then any formal (holomorphic) mapping $f : (\mathbb{C}^N, p) \to (\mathbb{C}^{N'}, p')$ sending M into M' is either convergent or sends M into $\mathcal{E}'$. □

As an immediate consequence we obtain the following characterization.

Corollary 1.3. Let $M \subset \mathbb{C}^N$ be a minimal real-analytic generic submanifold and $M' \subset \mathbb{C}^{N'}$ a real-algebraic subset with $p \in M$ and $p' \in M'$. Then all formal maps $f : (\mathbb{C}^N, p) \to (\mathbb{C}^{N'}, p')$ sending M into M' are convergent if and only if M' does not contain any irreducible complex-analytic subvariety of positive dimension through p'. □

In contrast to most previously known related results, Theorems 1.1, 1.2, and Corollary 1.3 do not contain any assumption on the map f. Indeed, Theorems 1.1 and 1.2 seem to be the first results of this kind and an analog of Corollary 1.3 appears only in the work of Baouendi, Ebenfelt, and Rothschild [6] for the case $M, M' \subset \mathbb{C}^N$ are real-analytic hypersurfaces containing no nontrivial complex subvarieties (see also Huang [14] for regularity results under the same assumptions). In fact they prove a more general result for M and M' of higher codimension assuming the map f to be finite and show (see the proof of [6, Proposition 7.1]) that the finiteness of f automatically holds (unless f is constant) in the mentioned case of hypersurfaces. However, in the setting of Corollary 1.3, the finiteness of a (nonconstant) map f may fail to hold even when $M, M' \subset \mathbb{C}^N$ are hypersurfaces, for example, for $M := S^3 \times \mathbb{C}$, $M' := S^5 \subset \mathbb{C}^3$, where $S^{2n-1} \subset \mathbb{C}^n$ is the unit sphere. Thus, even in this case, Theorems 1.1, 1.2, and Corollary 1.3 are new and do not follow from the same approach. It is worth mentioning that Corollary 1.3 is also new in the case of unit spheres $M = S^{2N-1}$ and $M' = S^{2N'-1}$ with $N' > N$. In conjunction with a rationality result due to Forstnerič [13], we obtain the following corollary.

Corollary 1.4. Any formal map sending the unit sphere $S^{2N-1} \subset \mathbb{C}^N$ into another unit sphere $S^{2N'-1} \subset \mathbb{C}^{N'}$ is a convergent rational map. □

Previous work in the direction of Theorem 1.2 is due to Chern and Moser [11] for real-analytic Levi-nondegenerate hypersurfaces. More recently, this result was extended in [3, 4, 6, 8, 9, 17, 21, 22] under weaker conditions on the submanifolds and mappings (see also [7, 15] for related references).

One of the main novelties of this paper, compared to previous related work, lies in the study of convergence properties of ratios of formal power series rather than of the series themselves. It is natural to call such a ratio convergent, if it is equivalent to a ratio of convergent power series. However, for our purposes, we need a refined version of convergence along a given submanifold that we define in Section 3.1 (see Definition 3.4). With this refined notion, we are able to conclude the convergence of a given ratio along a submanifold, provided its convergence is known to hold along a smaller submanifold and under suitable conditions on the ratio (see Lemmas 3.7 and 3.8).

Another novelty of our techniques consists of applying the mentioned convergence results of Section 3.1 and their consequences given in Section 3.2 to ratios defined on iterated complexifications of real-analytic submanifolds (in the sense of [25, 26]) rather than on single Segre sets (in the sense of [2]) associated to given fixed points. The choice of iterated complexifications is needed to guarantee the nonvanishing of the relevant ratios that may not hold when restricted to the Segre sets. These tools are then used to obtain the convergence of a certain type of ratios of formal power series that appear naturally in the proofs of Theorems 1.1 and 1.2. This is done in Theorem 4.1 that is, in turn, derived from Theorem 3.13 which is established in the more general context of a pair of submersions of a complex manifold.

After the necessary preparations in Sections 5 and 6, we state and prove Theorem 7.1 which is the main technical result of the paper and which implies, in particular, that the (formal) graph of f is contained in a real-analytic subset $Z_f \subset M \times M'$ satisfying a straightening property. If f is not convergent, the straightening property implies the existence of nontrivial complex-analytic subvarieties in M' and hence proves Theorem 1.2. To obtain Theorem 1.1, we use the additional property of the set Z_f (also given by Theorem 7.1), stating that Z_f also contains graphs of holomorphic maps approximating f up to any order. The fact that $Z_f \subset M \times M'$ then yields Theorem 1.1.

2 Notation and definitions

2.1 Formal mappings and CR-manifolds

A formal (holomorphic) mapping $f : (\mathbb{C}^N_Z, p) \to (\mathbb{C}^{N'}_{Z'}, p')$ is the data of N' formal power series $(f_1, \dots, f_{N'})$ in $Z - p$, with $f(p) = p'$. Let $M \subset \mathbb{C}^N$ and $M' \subset \mathbb{C}^{N'}$ be real-analytic submanifolds with $p \in M$ and $p' \in M'$, and let $\rho(Z, \overline{Z})$, $\rho'(Z', \overline{Z'})$ be real-analytic vector-valued defining functions for M near p and M' near p', respectively. Recall that a formal mapping f as above sends M into M' if there exists a matrix $a(Z, \overline{Z})$, with entries in $\mathbb{C}[[Z-p, \overline{Z} - \overline{p}]]$, such that the formal identity

$$\rho'\big(f(Z), \overline{f(Z)}\big) = a\big(Z, \overline{Z}\big) \cdot \rho\big(Z, \overline{Z}\big) \tag{2.1}$$

holds. Observe that (2.1) is independent of the choice of local real-analytic defining functions for M and M'. For M' merely a real-analytic subset in $\mathbb{C}^{N'}$, we also say that f sends M into M', and write $f(M) \subset M'$, if (2.1) holds for any real-analytic function ρ' (with some a depending on ρ'), defined in a neighborhood of p' in $\mathbb{C}^{N'}$, vanishing on M'. The notation $f(M) \subset M'$ is motivated by the fact that in case f is convergent, the above condition holds if and only if f is the Taylor series of a holomorphic map sending (M, p) into (M', p') in the sense of germs.

For a real-analytic CR-submanifold $M \subset \mathbb{C}^N$ (see, e.g., [5] for basic concepts related to CR-geometry), we write $T^c_p M$ for the complex tangent space of M at $p \in M$, that is, $T^c_p M := T_p M \cap i T_p M$. Recall that M is called *generic* if for any point $p \in M$, we have $T_p M + i T_p M = T_p \mathbb{C}^N$. Recall also that M is called *minimal* (in the sense of Tumanov [24]) at a point $p \in M$ if there is no real submanifold $S \subset M$ through p with $\dim S < \dim M$ and $T^c_q M \subset T_q S$ for all $q \in S$. It is well known that, if M is real-analytic, the minimality of M at p is equivalent to the finite type condition of Kohn [16] and Bloom and Graham [10].

2.2 Rings of formal power series

For a positive integer n, we write $\mathbb{C}[[t]]$ for the ring of formal power series (with complex coefficients) in the indeterminates $t = (t_1, \dots, t_n)$ and $\mathbb{C}\{t\}$ for the ring of convergent ones. If $t^0 \in \mathbb{C}^n$, $\mathbb{C}[[t - t^0]]$, and $\mathbb{C}\{t - t^0\}$ will denote the corresponding rings of series centered at t^0. For any formal power series $F(t)$, we denote by $\overline{F}(t)$ the formal power series obtained from $F(t)$ by taking complex conjugates of its coefficients.

An ideal $I \subset \mathbb{C}[[t]]$ is called a *manifold ideal* if it has a set of generators with linearly independent differentials (at 0). If $I \subset \mathbb{C}[[t]]$ is a manifold ideal, then any set of generators with linearly independent differentials has the same number of elements that we call the *codimension* of I. In general, we say that a manifold ideal I defines a *formal submanifold* $\mathcal{S} \subset \mathbb{C}^l$ and write $I = I(\mathcal{S})$. Note that if $I \subset \mathbb{C}\{t\}$, then I defines a (germ of a) complex submanifold $\mathcal{S} \subset \mathbb{C}^n$ through the origin in the usual sense. Given a formal submanifold $\mathcal{S} \subset \mathbb{C}^n$ of codimension d, a (local) parametrization of $\mathcal{S}$ is a formal map $j : (\mathbb{C}^{n-d}, 0) \to (\mathbb{C}^n, 0)$ of rank $n - d$ (at 0) such that $V \circ j = 0$ for all $V \in I(\mathcal{S})$. If $\mathcal{S}, \mathcal{S}' \subset \mathbb{C}^n$ are two formal submanifolds, we write $\mathcal{S} \subset \mathcal{S}'$ to mean that $I(\mathcal{S}') \subset I(\mathcal{S})$. For a formal map $h : (\mathbb{C}^n_t, 0) \to (\mathbb{C}^r_T, 0)$, we define its graph $\Gamma_h \subset \mathbb{C}^n \times \mathbb{C}^r$ as the formal submanifold given by $I(\Gamma_h)$, where $I(\Gamma_h) \subset \mathbb{C}[[t, T]]$ is the ideal generated by $T_1 - h_1(t), \dots, T_r - h_r(t)$.

For a formal power series $F(t) \in \mathbb{C}[[t]]$ and a formal submanifold $\mathcal{S} \subset \mathbb{C}^l$, we write $F|_{\mathcal{S}} \equiv 0$ (or sometimes also $F(t) \equiv 0$ for $t \in \mathcal{S}$) to mean that $F(t) \in I(\mathcal{S})$. If k is a nonnegative integer, we also write $F(t) = O(k)$ for $t \in \mathcal{S}$ to mean that for one (and hence for any) parametrization $j = j(t)$ of $\mathcal{S}$, $(F \circ j)(t)$ vanishes up to order k at the origin. We also say that another power series $G(t)$ agrees with $F(t)$ up to order k (at the origin) if $F(t) - G(t) = O(k)$.

A convenient criterion for the convergence of a formal power series is given by the following well-known result (see, e.g., [6, 21] for a proof).

Proposition 2.1. Any formal power series which satisfies a nontrivial polynomial identity with convergent coefficients is convergent. □

It will be also convenient to consider formal power series defined on an abstract complex manifold (of finite dimension) $\mathcal{X}$ centered at a point $x_0 \in \mathcal{X}$ without referring to specific coordinates. In each coordinate chart such a power series is given by a usual formal power series that transforms in the obvious way under biholomorphic coordinate changes. Given such a series H, we write $H(x_0)$ for the value at x_0 that is always defined. It is easy to see that the set of all formal power series on a complex manifold centered at x_0 forms a (local) commutative ring that is an integral domain. The notion of convergent power series extends to power series on abstract complex manifolds in the obvious way.

In a similar way, we may consider formal holomorphic vector fields on abstract complex manifolds and apply them to formal power series. If F and G are such formal power series on $\mathcal{X}$ centered at x_0, we write $\mathcal{L}(F/G) \equiv 0$ if and only if $F\mathcal{L}G - G\mathcal{L}F \equiv 0$ (as formal power series on $\mathcal{X}$).

Completely analogously, we may define formal power series mappings between complex manifolds and their compositions.

3 Meromorphic extension of ratios of formal power series

The ultimate goal of this section is to establish a meromorphic extension property for ratios of formal power series (see Theorem 3.13).

3.1 Convergence of ratios of formal power series

Throughout Section 3, for any formal power series $F = F(t) \in \mathbb{C}[[t]]$ in $t = (t_1, \ldots, t_n)$ and any nonnegative integer k, we denote by $j^k F$ or by $j^k_t F$ the formal power series mapping corresponding to the collection of all partial derivatives of F up to order k. We will use the first notation when there is no risk of confusion and the second one when other indeterminates appear. For $F(t), G(t) \in \mathbb{C}[[t]]$, we write $(F : G)$ for a pair of two formal power series thinking of it as a ratio, where we allow both series to be zero.

Definition 3.1. Let $(F_1 : G_1)$ and let $(F_2 : G_2)$ be ratios of formal power series in $t = (t_1, \ldots, t_n)$, and $S \subset \mathbb{C}^n$ be a (germ of a) complex submanifold through $0 \in \mathbb{C}^n$. We say that the ratios $(F_1 : G_1)$ and $(F_2 : G_2)$ are k-*similar along* S if $(j^k(F_1G_2 - F_2G_1))|_S \equiv 0$.

The defined relation of similarity for formal power series is obviously symmetric but not transitive, for example, any ratio is k-similar to $(0 : 0)$ along any complex submanifold S and for any nonnegative integer k. However, we have the following weaker property.

Lemma 3.2. Let $(F_1 : G_1)$, $(F_2 : G_2)$, and $(F_3 : G_3)$ be ratios of formal power series in $t = (t_1, \ldots, t_n)$, $S \subset \mathbb{C}^n$ a complex submanifold through the origin and k a nonnegative integer. Suppose that both ratios $(F_1 : G_1)$ and $(F_3 : G_3)$ are k-similar to $(F_2 : G_2)$ along S. Then, if there exists $l \leq k$ such that $(j^l(F_2, G_2))|_S \not\equiv 0$, then $(F_1 : G_1)$ and $(F_3 : G_3)$ are $(k - l)$-similar along S. □

Proof. Without loss of generality, we may assume that $(j^l F_2)|_S \not\equiv 0$. By the assumptions, we have $(j^k(F_1G_2 - F_2G_1))|_S \equiv 0$ and $(j^k(F_3G_2 - F_2G_3))|_S \equiv 0$. Multiplying the first identity by F_3, the second by F_1, and subtracting from each other, we obtain

$$\left(j^k\left(F_2\left(F_1G_3 - F_3G_1\right)\right)\right)\Big|_S \equiv 0. \tag{3.1}$$

Since $(j^l F_2)|_S \not\equiv 0$, the last identity is only possible if $(j^{k-l}(F_1G_3 - F_3G_1))|_S \equiv 0$ as required. ■

We will actually use the following refined version of Lemma 3.2 whose proof is completely analogous. In what follows, for some splitting of indeterminates $t = (t^1, t^2, t^3) \in \mathbb{C}^{n_1} \times \mathbb{C}^{n_2} \times \mathbb{C}^{n_3}$ and for any formal power series $F(t) \in \mathbb{C}[[t]]$, we write $j^k_{t^i} F$ for the collection of all partial derivatives up to order k of F with respect to t^i, $i = 1, 2, 3$.

Lemma 3.3. Let $(F_1 : G_1)$, $(F_2 : G_2)$, and $(F_3 : G_3)$ be ratios of formal power series in $t = (t^1, t^2, t^3) \in \mathbb{C}^{n_1} \times \mathbb{C}^{n_2} \times \mathbb{C}^{n_3}$ and set $S := \mathbb{C}^{n_1} \times \{(0,0)\} \subset \mathbb{C}^{n_1} \times \mathbb{C}^{n_2} \times \mathbb{C}^{n_3}$. Suppose that there exist integers $l \geq 0$ and $k_2, k_3 \geq l$ such that

$$\left(j^l(F_2, G_2)\right)\big|_S \not\equiv 0, \qquad \left(j_{t^2}^{k_2} j_{t^3}^{k_3}(F_1 G_2 - F_2 G_1)\right)\big|_S \equiv 0,$$
$$\left(j_{t^2}^{k_2} j_{t^3}^{k_3}(F_3 G_2 - F_2 G_3)\right)\big|_S \equiv 0.$$

Then $(j_{t^2}^{k_2-l} j_{t^3}^{k_3-l}(F_1 G_3 - F_3 G_1))|_S \equiv 0$. □

Clearly, given a complex submanifold $S \subset \mathbb{C}^n$ through the origin, any fixed ratios $(F_1 : G_1)$ and $(F_2 : G_2)$ are k-similar along S for any k if and only if $F_1 G_2 - F_2 G_1 \equiv 0$, that is, if they are equivalent as ratios. We now define a notion of convergence along S for any ratio of formal power series.

Definition 3.4. Let $S \subset \mathbb{C}^n$ be a complex submanifold through the origin and $F(t), G(t) \in \mathbb{C}[[t]]$, $t = (t_1, \ldots, t_n)$. The ratio $(F : G)$ is said to be *convergent along* S if there exist a nonnegative integer l and, for any nonnegative integer k, convergent power series $F_k(t), G_k(t) \in \mathbb{C}\{t\}$, such that the ratio $(F_k : G_k)$ is k-similar to $(F : G)$ along S and $(j^l(F_k, G_k))|_S \not\equiv 0$.

The uniformity of the choice of the integer l is a crucial requirement in Definition 3.4 (see, e.g., the proof of Lemma 3.8). This notion of convergence for ratios of formal power series has the following elementary properties.

Lemma 3.5. For $F(t), G(t) \in \mathbb{C}[[t]]$, $t = (t_1, \ldots, t_n)$, the following properties hold:

(i) $(F : G)$ is always convergent along $S = \{0\}$;

(ii) if F and G are convergent, then $(F : G)$ is convergent along any submanifold $S \subset \mathbb{C}^n$ (through 0);

(iii) if $(F : G)$ is equivalent to a nontrivial ratio that is convergent along a submanifold S, then $(F : G)$ is also convergent along S;

(iv) if $(F : G)$ is convergent along $S = \mathbb{C}^n$, then it is equivalent to a nontrivial ratio of convergent power series. □

Proof. Properties (i), (ii), and (iv) are easy to derive from Definition 3.4. Property (iii) is a consequence of Lemma 3.2. ■

An elementary useful property of ratios of formal power series is given by the following lemma.

Lemma 3.6. Let $(F : G)$ be a ratio of formal power series in $t = (t^1, t^2) \in \mathbb{C}^{n_1} \times \mathbb{C}^{n_2}$ with $G \not\equiv 0$, and such that $(\partial/\partial t^2)(F/G) \equiv 0$. Then there exists $\widetilde{F}(t^1), \widetilde{G}(t^1) \in \mathbb{C}[[t^1]]$ with $\widetilde{G} \not\equiv 0$ such that $(F : G)$ is equivalent to $(\widetilde{F} : \widetilde{G})$. □

Proof. From the assumption it is easy to obtain, by differentiation, the identity

$$\big(\partial_{t^2}^{\nu} F\big)\big(\partial_{t^2}^{\alpha} G\big) - \big(\partial_{t^2}^{\nu} G\big)\big(\partial_{t^2}^{\alpha} F\big) \equiv 0, \tag{3.2}$$

for all multi-indices $\alpha, \nu \in \mathbb{N}^{n_2}$. Since $G \not\equiv 0$, there exists $\alpha \in \mathbb{N}^{n_2}$ such that $(\partial_{t^2}^{\alpha} G)|_{t^2=0} \not\equiv 0$. Define $\widetilde{F}(t^1) := \partial_{t^2}^{\alpha} F(t^1, 0) \in \mathbb{C}[[t^1]]$ and $\widetilde{G}(t^1) := \partial_{t^2}^{\alpha} G(t^1, 0) \in \mathbb{C}[[t^1]]$. Then, by putting $t^2 = 0$ in (3.2), we obtain that $(\widetilde{F}\partial_{t^2}^{\nu} G - \widetilde{G}\partial_{t^2}^{\nu} F)|_{t^2=0} \equiv 0$ for any multi-index $\nu \in \mathbb{N}^{n_2}$. From this, it follows that the ratios $(F : G)$ and $(\widetilde{F} : \widetilde{G})$ are equivalent, which completes the proof of the lemma since by construction $\widetilde{G} \not\equiv 0$. ■

The following lemma will be used in Section 3.2 to pass from smaller sets of convergence to larger ones.

Lemma 3.7. Let $F(t), G(t) \in \mathbb{C}[[t]]$ be formal power series in $t = (t^1, t^2, t^3) \in \mathbb{C}^{n_1} \times \mathbb{C}^{n_2} \times \mathbb{C}^{n_3}$ that depend only on (t^1, t^3). Then, if the ratio $(F : G)$ is convergent along $\mathbb{C}^{n_1} \times \{(0,0)\}$, it is also convergent along $\mathbb{C}^{n_1} \times \mathbb{C}^{n_2} \times \{0\}$. □

Proof. We set $S := \mathbb{C}^{n_1} \times \{(0,0)\} \subset \mathbb{C}^{n_1} \times \mathbb{C}^{n_2} \times \mathbb{C}^{n_3}$ and $\widetilde{S} := \mathbb{C}^{n_1} \times \mathbb{C}^{n_2} \times \{0\} \subset \mathbb{C}^{n_1} \times \mathbb{C}^{n_2} \times \mathbb{C}^{n_3}$. By the assumptions and Definition 3.4, there exist a nonnegative integer l and, for any nonnegative integer k, convergent power series $F_k(t), G_k(t) \in \mathbb{C}\{t\}$ such that

$$\big(j_t^k\big(F(t^1, t^3)G_k(t) - G(t^1, t^3)F_k(t)\big)\big)\big|_S \equiv 0, \tag{3.3}$$

and $(j^l(F_k, G_k))|_S \not\equiv 0$. We fix $k \geq l$. Choose $\beta_0 \in N^{n_2}$ with $|\beta_0| \leq l$ such that

$$\Big(j_{(t^1,t^3)}^l\big(\big(\partial_{t^2}^{\beta_0} F_k\big)(t^1, 0, t^3), \big(\partial_{t^2}^{\beta_0} G_k\big)(t^1, 0, t^3)\big)\Big)\Big|_{t^3=0} \not\equiv 0. \tag{3.4}$$

Define $\widetilde{F}_k(t) := \partial_{t^2}^{\beta_0} F_k(t^1, 0, t^3)$ and $\widetilde{G}_k(t) := \partial_{t^2}^{\beta_0} G_k(t^1, 0, t^3)$. By the construction, we have $(j^l(\widetilde{F}_k, \widetilde{G}_k))|_{\widetilde{S}} \not\equiv 0$ and it is also easy to see from (3.3) that $(F : G)$ is $(k-l)$-similar to $(\widetilde{F}_k : \widetilde{G}_k)$ along $\widetilde{S}$. This finishes the proof of Lemma 3.7. ■

The next less obvious lemma will be also used for the same purpose.

Lemma 3.8. Consider formal power series $F(t), G(t) \in \mathbb{C}[[t]]$ in $t = (t^1, t^2, t^3) \in \mathbb{C}^{n_1} \times \mathbb{C}^{n_2} \times \mathbb{C}^{n_3}$ of the form

$$F(t) = \varphi\big(Y(t^1, t^3), t^2\big), \qquad G(t) = \psi\big(Y(t^1, t^3), t^2\big), \tag{3.5}$$

where $Y(t^1, t^3) \in (\mathbb{C}[[t^1, t^3]])^r$ for some integer $r \geq 1$ and φ and ψ are convergent power series in $\mathbb{C}^r \times \mathbb{C}^{n_2}$ centered at $(Y(0), 0)$. Then the conclusion of Lemma 3.7 also holds. □

Proof. The statement obviously holds if F and G are both zero, hence we may assume that $(F, G) \not\equiv 0$. Then there exists a nonnegative integer d such that $(j^d(F, G))|_S \not\equiv 0$, where $S := \mathbb{C}^{n_1} \times \{(0, 0)\} \subset \mathbb{C}^{n_1} \times \mathbb{C}^{n_2} \times \mathbb{C}^{n_3}$. Since $(F : G)$ is assumed to be convergent along S, there exist a nonnegative integer l and ratios $(F_s : G_s)$, $s = 0, 1, \ldots$, of convergent power series such that $(F : G)$ is s-similar to $(F_s : G_s)$ and

$$\left(j^l(F_s, G_s)\right)\big|_S \not\equiv 0 \tag{3.6}$$

for all s. Then, for any $k \geq l$ and $s \geq k + l$, we have

$$\left(j_{t^2}^{s-k} j_{t^3}^{k}\left(FG_s - F_sG\right)\right)\big|_S \equiv 0. \tag{3.7}$$

In view of (3.5) we may rewrite (3.7) in the form

$$R_{s,k}\left(\left(j_{t^3}^{k} Y\right)(t^1, 0), t^1\right) \equiv 0, \tag{3.8}$$

where $R_{s,k}$ is a convergent power series in the corresponding variables. We view (3.8) as a system of analytic equations $R_{s,k}(y, t^1) = 0$ for $k \geq l$ fixed, $s \geq k + l$ arbitrary, and $y(t^1) := (j_{t^3}^{k} Y)(t^1, 0)$ as a formal solution of the system. By applying Artin's approximation theorem [1], for any positive integer κ, there exists a convergent solution $y^\kappa(t^1)$ agreeing up to order κ (at $0 \in \mathbb{C}^{n_1}$) with $y(t^1)$ (and depending also on k) and satisfying $R_{s,k}(y^\kappa(t^1), t^1) \equiv 0$ for all $s \geq k + 1$. It is easy to see that there exists a convergent power series $Y^\kappa(t^1, t^3)$ (e.g., a polynomial in t^3) satisfying $(j_{t^3}^{k} Y^\kappa)|_{t^3=0} \equiv y^\kappa(t^1)$. Hence the power series $\widetilde{F}_k^\kappa(t) := \varphi(Y^\kappa(t^1, t^3), t^2)$ and $\widetilde{G}_k^\kappa(t) := \psi(Y^\kappa(t^1, t^3), t^2)$ are convergent and agree with $F(t)$ and $G(t)$, respectively, up to order κ, therefore by choosing κ sufficiently large (depending on k), we may assume that $(j^d(\widetilde{F}_k^\kappa, \widetilde{G}_k^\kappa))|_S \not\equiv 0$. In what follows, we fix such a choice of κ. By our construction, (3.8) is satisfied with Y replaced by Y^κ and thus (3.7) is satisfied with (F, G) replaced by $(\widetilde{F}_k^\kappa, \widetilde{G}_k^\kappa)$, that is,

$$\left(j_{t^2}^{s-k} j_{t^3}^{k}\left(\widetilde{F}_k^\kappa G_s - F_s \widetilde{G}_k^\kappa\right)\right)\big|_S \equiv 0. \tag{3.9}$$

In view of Lemma 3.3, (3.7), (3.9), and (3.6) imply

$$\left(j_{t^2}^{s-k-l} j_{t^3}^{k-l}\left(F\widetilde{G}_k^\kappa - \widetilde{F}_k^\kappa G\right)\right)\big|_S \equiv 0. \tag{3.10}$$

Since s can be taken arbitrarily large, (3.10) implies that $(F:G)$ and $(\widetilde{F}^\kappa_k:\widetilde{G}^\kappa_k)$ are $(k-l)$-similar along $\widetilde{S}:=\mathbb{C}^{n_1}\times\mathbb{C}^{n_2}\times\{0\}$. Since $(j^d(\widetilde{F}^\kappa_k,\widetilde{G}^\kappa_k))|_S\not\equiv 0$ implies $(j^d(\widetilde{F}^\kappa_k,\widetilde{G}^\kappa_k))|_{\widetilde{S}}\not\equiv 0$, the ratio $(F:G)$ is convergent along $\widetilde{S}$ (in the sense of Definition 3.4) and the proof is complete. ∎

For the proof of Theorem 3.13, we will also need the following lemma.

Lemma 3.9. Let η be a holomorphic map from a neighborhood of 0 in $\mathbb{C}^r$ into a neighborhood of 0 in $\mathbb{C}^n$, with $\eta(0)=0$, and $\alpha(t),\beta(t)\in\mathbb{C}[[t]]$, $t=(t_1,\dots,t_n)$. Suppose that there exists a (germ of a) complex submanifold $S\subset\mathbb{C}^r$ through 0 such that $\eta|_S:S\to\mathbb{C}^n$ has maximal rank n at points of the intersection $S\cap\eta^{-1}(\{0\})$ that are arbitrarily close to $0\in\mathbb{C}^r$. Suppose also that the ratio $(\alpha\circ\eta:\beta\circ\eta)$ is convergent along S (in the sense of Definition 3.4). Then $(\alpha:\beta)$ is equivalent to a nontrivial ratio of convergent power series. □

Proof. Without loss of generality, S is connected. By Definition 3.4, there exist a nonnegative integer l and, for any positive integer k, convergent power series $A_k(z),B_k(z)\in\mathbb{C}\{z\}$, $z=(z_1,\dots,z_r)$, such that

$$\left(j_z^k\left(B_k(\alpha\circ\eta)-A_k(\beta\circ\eta)\right)\right)\big|_S\equiv 0 \tag{3.11}$$

and $(j_z^l(A_k,B_k))|_S\not\equiv 0$. We may assume that A_k,B_k are convergent in a polydisc neighborhood Δ_k of $0\in\mathbb{C}^r$. Choose $\nu_0\in\mathbb{N}^r$, $|\nu_0|\le l$, of minimal length such that

$$\left(\partial_z^{\nu_0}A_l,\partial_z^{\nu_0}B_l\right)\big|_S\not\equiv 0. \tag{3.12}$$

Then, since $(\partial_z^{\nu}A_l,\partial_z^{\nu}B_l)|_S\equiv 0$ for $|\nu|<|\nu_0|$, (3.11) with $k=l$ implies

$$\left(\left(\partial_z^{\nu_0}B_l\right)(\alpha\circ\eta)-\left(\partial_z^{\nu_0}A_l\right)(\beta\circ\eta)\right)\big|_S\equiv 0. \tag{3.13}$$

By assumption on $\eta|_S$, we may choose a point $s_0\in S\cap\Delta_l$ arbitrarily close to 0 with $s_0\in\Delta_l$, such that $\eta|_S$ has rank n at s_0 and $\eta(s_0)=0$. By the rank theorem, we may choose a right inverse of η, $\theta:\Omega\to S$, holomorphic in some neighborhood Ω of $0\in\mathbb{C}^n$ with $\theta(0)=s_0$. Since $(\eta\circ\theta)(t)\equiv t$, we obtain from (3.13) that $((\partial_z^{\nu_0}B_l)\circ\theta)(t)(\alpha(t))-((\partial_z^{\nu_0}A_l)\circ\theta)(t)(\beta(t))\equiv 0$. To complete the proof of the lemma, it remains to observe that, in view of (3.12), θ can be chosen so that $(((\partial_z^{\nu_0}A_l)\circ\theta)(t),((\partial_z^{\nu_0}B_l)\circ\theta)(t))\not\equiv 0$. ∎

3.2 Applications to pullbacks of ratios of formal power series

The notion of convergence of a ratio of formal power series along a submanifold introduced in Definition 3.4 extends in an obvious way to formal power series defined on a complex manifold $\mathcal{X}$. We also say that two ratios $(F_1 : G_1)$, $(F_2 : G_2)$ of formal power series on $\mathcal{X}$ are equivalent if $F_1G_2 - F_2G_1$ vanishes identically as a formal power series on $\mathcal{X}$.

Let $\mathcal{Y}$ be another complex manifold and $\nu : \mathcal{Y} \to \mathcal{X}$ a holomorphic map defined in a neighborhood of a reference point $y_0 \in \mathcal{Y}$ with $x_0 := \nu(y_0)$. Consider the pullback under ν of a ratio $(F : G)$ of formal power series on $\mathcal{X}$ (centered at x_0) and assume that it is convergent along a submanifold $S \subset \mathcal{Y}$ through y_0. Under certain assumptions on the map ν and on the formal power series we show in this section that $\mathcal{Y}$ can be embedded into a larger manifold $\widetilde{\mathcal{Y}}$ and ν holomorphically extended to $\widetilde{\nu} : \widetilde{\mathcal{Y}} \to \mathcal{X}$ such that the pullback of $(F : G)$ under $\widetilde{\nu}$ is convergent along a larger submanifold $\widetilde{S} \subset \widetilde{\mathcal{Y}}$. The precise statement is the following proposition.

Proposition 3.10. Let $\mathcal{X}$ and $\mathcal{Y}$ be complex manifolds and $\nu : \mathcal{Y} \to \mathcal{X}$ a holomorphic submersion with $y_0 \in \mathcal{Y}$. Let $S \subset \mathcal{Y}$ be a complex submanifold through y_0 and $(F : G)$ a ratio of formal power series on $\mathcal{X}$, centered at $x_0 := \nu(y_0)$, whose pullback under ν is convergent along S. Let $\eta : \mathcal{X} \to \mathcal{C}$ be a holomorphic submersion onto a complex manifold $\mathcal{C}$. Define

$$\begin{aligned} \widetilde{\mathcal{Y}} &:= \big\{(y, x) \in \mathcal{Y} \times \mathcal{X} : \eta(\nu(y)) = \eta(x)\big\}, \\ \widetilde{S} &:= \big\{(y, x) \in \widetilde{\mathcal{Y}} : y \in S\big\}, \\ \widetilde{\nu} &: \widetilde{\mathcal{Y}} \ni (y, x) \longmapsto x \in \mathcal{X}. \end{aligned} \tag{3.14}$$

Assume that one of the following conditions hold:

(i) the ratio $(F : G)$ is equivalent to a nontrivial ratio $(\alpha \circ \eta : \beta \circ \eta)$, where α and β are formal power series on $\mathcal{C}$ centered at $\eta(x_0)$;

(ii) the ratio $(F : G)$ is equivalent to a nontrivial ratio of the form $(\Phi(Y(\eta(x)), x) : \Psi(Y(\eta(x)), x))$, where Y is a $\mathbb{C}^r$-valued formal power series on $\mathcal{C}$ centered at $\eta(x_0)$ and Φ, Ψ are convergent power series centered at $(Y(x_0), x_0) \in \mathbb{C}^r \times \mathcal{X}$.

Then the pullback of $(F : G)$ under $\widetilde{\nu}$ is convergent along $\widetilde{S}$. □

Remark 3.11. The conclusion of Proposition 3.10 obviously holds in the case $\dim \mathcal{X} = \dim \mathcal{C}$ (without assuming neither (i) nor (ii)), and therefore, we may assume, in what follows, that $\dim \mathcal{X} > \dim \mathcal{C}$ which implies $\dim \widetilde{\mathcal{Y}} > \dim \mathcal{Y}$.

In order to reduce Proposition 3.10 to an application of Lemmas 3.7 and 3.8, we need the following lemma.

Lemma 3.12. In the setting of Proposition 3.10, define

$$\widehat{S} := \big\{\big(y, \nu(y)\big) : y \in S\big\} \subset \widetilde{\mathcal{Y}}. \tag{3.15}$$

Then the pullback of $(F : G)$ under $\widetilde{\nu}$ is convergent along the complex submanifold $\widehat{S}$. □

The idea of the proof lies in the fact that the derivatives of the pullbacks under $\widetilde{\nu}$ can be expressed through derivatives of the pullbacks under ν of the same power series. For this property to hold, it is essential to assume that ν is submersive.

Proof. The manifold $\mathcal{Y}$ can be seen as embedded into $\widetilde{\mathcal{Y}}$ via the map $\vartheta : \mathcal{Y} \ni y \mapsto (y, \nu(y)) \in \widetilde{\mathcal{Y}}$. Therefore, by considering $\vartheta(\mathcal{Y})$, we may also think of $\mathcal{Y}$ as a submanifold in $\widetilde{\mathcal{Y}}$. Since ν is a submersion, after possibly shrinking $\mathcal{Y}$ near y_0 and $\mathcal{X}$ near x_0, we may choose for every $y \in \mathcal{Y}$ a holomorphic right inverse of ν, $\nu_y^{-1} : \mathcal{X} \to \mathcal{Y}$, such that $\nu_y^{-1}(\nu(y)) = y$. Such a choice can be made by the rank theorem so that the map $\mathcal{Y} \times \mathcal{X} \ni (y, x) \mapsto \nu_y^{-1}(x) \in \mathcal{Y}$ is holomorphic.

Choose open neighborhoods $\Omega_1 \subset \mathbb{C}^{\dim \mathcal{Y}}$ and $\Omega_2 \subset \mathbb{C}^{\dim \widetilde{\mathcal{Y}} - \dim \mathcal{Y}}$ of the origin and local holomorphic coordinates $(z, w) = (z(y, x), w(y, x)) \in \Omega_1 \times \Omega_2$ on $\widetilde{\mathcal{Y}}$ vanishing at $(y_0, x_0) \in \widetilde{\mathcal{Y}}$ such that $\vartheta(\mathcal{Y})$ is given by $\{(y, x) \in \widetilde{\mathcal{Y}} : w = 0\}$. (Hence $z|_{\mathcal{Y}} : \mathcal{Y} \ni y \mapsto z(y, \nu(y)) \in \Omega_1$ is a system of holomorphic coordinates for $\mathcal{Y}$.) In what follows, as is customary, we identify S and $z(S)$. Since $(F : G)$ is convergent along S, for any nonnegative integer k, there exist convergent power series f_k, g_k in $\mathbb{C}\{z\}$ such that

$$\left(j_z^k R_k\right)\big|_S \equiv 0, \quad R_k(z) := (F \circ \nu)(z) g_k(z) - (G \circ \nu)(z) f_k(z), \tag{3.16}$$

and $(j_z^l(f_k, g_k))|_S \not\equiv 0$ for some nonnegative integer l independent of k. In what follows we fix k and may assume, without loss of generality, that f_k, g_k are holomorphic in Ω_1. We will define convergent power series $\widetilde{f}_k, \widetilde{g}_k \in \mathbb{C}\{z, w\}$ whose restrictions to $\{w = 0\}$ are f_k, g_k. For this, we set, for $z \in \Omega_1$, $\nu_z^{-1} := \nu_y^{-1}$ where $y \in \mathcal{Y}$ is uniquely determined by the relation $z = z(y, \nu(y))$. Define holomorphic functions on $\Omega_1 \times \Omega_2$ by setting $\widetilde{f}_k(z, w) := (f_k \circ \nu_z^{-1} \circ \widetilde{\nu})(z, w)$ and $\widetilde{g}_k(z, w) := (g_k \circ \nu_z^{-1} \circ \widetilde{\nu})(z, w)$. We also set $\widetilde{R}_k(z, w) := (F \circ \widetilde{\nu})(z, w)\widetilde{g}_k(z, w) - (G \circ \widetilde{\nu})(z, w)\widetilde{f}_k(z, w)$. Since ν_z^{-1} is a right convergent inverse for ν, it follows from the above construction that $\widetilde{R}_k(z, w) = (R_k \circ \nu_z^{-1} \circ \widetilde{\nu})(z, w)$. Therefore, by the chain rule, the power series mapping $j_{(z,w)}^k \widetilde{R}_k$ is a linear combination (with holomorphic coefficients in (z, w)) of the components of $(j_z^k R_k) \circ (\nu_z^{-1} \circ \widetilde{\nu})$. By restricting to $z \in S$ and $w = 0$, we obtain, in view of (3.16) and the fact that $\nu_z^{-1}(\widetilde{\nu}(z, 0)) = z$,

$$\left(j_{(z,w)}^k \left((F \circ \widetilde{\nu})\widetilde{g}_k - \left(G \circ \widetilde{\nu}\right)\widetilde{f}_k\right)\right)\big|_{z \in S,\, w=0} \equiv 0. \tag{3.17}$$

We therefore conclude that $(\widetilde{f}_k : \widetilde{g}_k)$ is k-similar to $(F \circ \widetilde{\nu} : G \circ \widetilde{\nu})$ along $\widehat{S}$ since the submanifold $\widehat{S}$ is given by $\{(z, 0) : z \in S\}$ in the (z, w)-coordinates. Since (f_k, g_k) is the restriction of $(\widetilde{f}_k, \widetilde{g}_k)$ to $\{w = 0\}$ by construction, we have $(j_{(z,w)}^l(\widetilde{f}_k, \widetilde{g}_k))|_{\widehat{S}} \not\equiv 0$. This shows that $(F \circ \widetilde{\nu} : G \circ \widetilde{\nu})$ is convergent along $\widehat{S}$ and hence completes the proof of the lemma. ■

Proof of Proposition 3.10. The statement obviously holds when F and G are both zero, so we may assume that the ratio $(F : G)$ is nontrivial. Choose local holomorphic coordinates

$Z = Z(y) \in \mathbb{C}^{\dim \mathcal{Y}}$ for $\mathcal{Y}$, vanishing at y_0, such that S is given in these coordinates by $\{Z = (Z^1, Z^2) \in \mathbb{C}^{n_1} \times \mathbb{C}^{n_2} : Z^2 = 0\}$, with $\dim \mathcal{Y} = n_1 + n_2$. By the construction of $\widetilde{\mathcal{Y}}$, we may choose holomorphic coordinates $\widetilde{Z}$ for $\widetilde{\mathcal{Y}}$ near (y_0, x_0) of the form $\widetilde{Z} = \widetilde{Z}(y, x) = (Z(y), Z^3(y, x)) \in \mathbb{C}^{\dim \mathcal{Y}} \times \mathbb{C}^{n_3}$, where Z is as above, $n_3 = \dim \widetilde{\mathcal{Y}} - \dim \mathcal{Y}$ and such that $\vartheta(\mathcal{Y})$ is given by $\{Z^3 = 0\}$. Note that the submanifolds $\widehat{S}$ and $\widetilde{S}$ are given in the $\widetilde{Z}$-coordinates by $\{Z^2 = Z^3 = 0\}$ and $\{Z^2 = 0\}$, respectively, and $\eta \circ \widetilde{\nu}$ is independent of Z^3 (again by the construction of $\widetilde{\mathcal{Y}}$).

To prove the conclusion assuming (i), we first note that since ν is a submersion and $\vartheta(\mathcal{Y}) \subset \widetilde{\mathcal{Y}}$, it follows that $\widetilde{\nu}$ is a submersion too. Therefore, the nontrivial ratio $(F \circ \widetilde{\nu} : G \circ \widetilde{\nu})$ is equivalent to the nontrivial ratio $(\alpha \circ \eta \circ \widetilde{\nu} : \beta \circ \eta \circ \widetilde{\nu})$, and this latter is convergent along $\widehat{S}$ by Lemma 3.5(iii) and Lemma 3.12. To complete the proof of (i), it is enough to prove that $(\alpha \circ \eta \circ \widetilde{\nu} : \beta \circ \eta \circ \widetilde{\nu})$ is convergent along $\widetilde{S}$ (again by Lemma 3.5(iii)). Using the $\widetilde{Z}$-coordinates for $\widetilde{\mathcal{Y}}$, we see that the conclusion follows from a direct application of Lemma 3.7.

The proof of the conclusion assuming (ii) follows the same lines of the proof assuming condition (i) by making use of Lemma 3.8 (instead of Lemma 3.7). This completes the proof of Proposition 3.10. ■

3.3 Pairs of submersions of finite type and meromorphic extension

We will formulate our main result of this section in terms of pairs of submersions defined on a given complex manifold. The main example of this setting is given by the complexification $\mathcal{M} \subset \mathbb{C}^N \times \mathbb{C}^N$ of a real-analytic generic submanifold $M \subset \mathbb{C}^N$, where a pair of submersions on $\mathcal{M}$ is given by the projections on the first and the last component $\mathbb{C}^N$, respectively.

In general, let $\mathcal{X}$, $\mathcal{Z}$, and $\mathcal{W}$ be complex manifolds and $\lambda : \mathcal{X} \to \mathcal{Z}$, $\mu : \mathcal{X} \to \mathcal{W}$ be holomorphic submersions. Set $\mathcal{X}^{(0)} := \mathcal{X}$ and for any integer $l \geq 1$, define the (odd) fiber product

$$\mathcal{X}^{(l)} := \big\{(z_1, \ldots, z_{2l+1}) \in \mathcal{X}^{2l+1} : \mu(z_{2s-1}) = \mu(z_{2s}),\ \lambda(z_{2s}) = \lambda(z_{2s+1}),\ 1 \leq s \leq l\big\}. \tag{3.18}$$

Analogously, fiber products with even number of factors can be defined, but will not be used in this paper. It is easy to see that $\mathcal{X}^{(l)} \subset \mathcal{X}^{2l+1}$ is a complex submanifold. Let

$$\mathcal{X}^{(l)} \ni (z_1, \ldots, z_{2l+1}) \longmapsto \pi_j^{(l)}(z_1, \ldots, z_{2l+1}) := z_j \in \mathcal{X} \tag{3.19}$$

be the restriction to $\mathcal{X}^{(l)}$ of the natural projection to the jth component, $1 \leq j \leq 2l+1$, and denote by $\widetilde{\lambda} : \mathcal{X}^{(l)} \to \mathcal{Z}$ and $\widetilde{\mu} : \mathcal{X}^{(l)} \to \mathcal{W}$ the maps defined by

$$\widetilde{\lambda} := \lambda \circ \pi_1^{(l)}, \qquad \widetilde{\mu} := \mu \circ \pi_{2l+1}^{(l)}. \tag{3.20}$$

Then, for every $x \in \mathcal{X}$ we set $x^{(l)} := (x, \ldots, x) \in \mathcal{X}^{(l)}$, and

$$D_l(x) := \widetilde{\lambda}^{-1}\big(\widetilde{\lambda}(x^{(l)})\big), \qquad E_l(x) := \widetilde{\mu}^{-1}\big(\widetilde{\mu}(x^{(l)})\big) \tag{3.21}$$

are complex submanifolds of $\mathcal{X}^{(l)}$.

In the above mentioned case, that is, when $\mathcal{X}$ is the complexification of a real-analytic generic submanifold $M \subset \mathbb{C}^N$, the construction of $\mathcal{X}^{(l)}$ yields the iterated complexification $\mathcal{M}^{2l}$ as defined in [25]. In this case the images $\widetilde{\mu}(D_l(x))$ are the Segre sets in the sense of Baouendi, Ebenfelt, and Rothschild [2] and their finite type criterion says that M is of finite type in the sense of Kohn [16] and Bloom and Graham [10] if and only if the Segre sets of sufficiently high order have nonempty interior. The last condition can also be expressed in terms of ranks (see [5]). Motivated by this case, we say in the above general setting that the pair (λ, μ) of submersions is *of finite type* at a point $x_0 \in \mathcal{X}$ if there exists $l_0 \geq 1$ such that the map $\widetilde{\mu}_{l_0}|_{D_{l_0}(x_0)}$ has rank equal to $\dim \mathcal{W}$ at some points of the intersection $D_{l_0}(x_0) \cap E_{l_0}(x_0)$ that are arbitrarily close to $x_0^{(l_0)}$.

The main result of Section 3 is the following meromorphic extension property of ratios of formal power series that was inspired by an analogous result from [19] in a different context. Its proof is however completely different and will consist of repeatedly applying Proposition 3.10.

Theorem 3.13. Let $\mathcal{X}$, $\mathcal{Z}$, $\mathcal{W}$ be complex manifolds and $\lambda : \mathcal{X} \to \mathcal{Z}$, $\mu : \mathcal{X} \to \mathcal{W}$ be a pair of holomorphic submersions of finite type at a point $x_0 \in \mathcal{X}$. Consider formal power series $F(x), G(x)$ on $\mathcal{X}$ centered at x_0 of the form $F(x) = \Phi(Y(\lambda(x)), x)$, $G(x) = \Psi(Y(\lambda(x)), x)$, where Y is a $\mathbb{C}^r$-valued formal power series on $\mathcal{W}$ centered at $\lambda(x_0)$ and Φ, Ψ are convergent power series on $\mathbb{C}^r \times \mathcal{X}$ centered at $(Y(\lambda(x_0)), x_0)$. Suppose that $G \not\equiv 0$ and $\mathcal{L}(F/G) \equiv 0$ holds for any holomorphic vector field $\mathcal{L}$ on $\mathcal{X}$ annihilating μ. Then $(F : G)$ is equivalent to a nontrivial ratio of convergent power series on $\mathcal{X}$ (centered at x_0). □

Remark 3.14. From the proof of Theorem 3.13, it will follow that the ratio $(F : G)$ is even equivalent to a ratio of the form $(\widetilde{\alpha} \circ \mu : \widetilde{\beta} \circ \mu)$, where $\widetilde{\alpha}, \widetilde{\beta}$ are convergent power series on $\mathcal{W}$ centered at $\mu(x_0)$ with $\widetilde{\beta} \not\equiv 0$.

We start by giving several lemmas that will be used in the proof of Theorem 3.13.

Lemma 3.15. Let $\mu : \mathcal{X} \to \mathcal{W}$ be a holomorphic submersion between complex manifolds and let $F(x), G(x)$ be formal power series on $\mathcal{X}$ centered at a point $x_0 \in \mathcal{X}$. Suppose that $G \not\equiv 0$ and that $\mathcal{L}(F/G) \equiv 0$ for any holomorphic vector field $\mathcal{L}$ on $\mathcal{X}$ that annihilates μ. Then there exist formal power series α, β on $\mathcal{W}$ centered at $\mu(x_0)$, with $\beta \not\equiv 0$, such that the ratio $(F : G)$ is equivalent to the ratio $(\alpha \circ \mu : \beta \circ \mu)$. □

The proof of Lemma 3.15 follows from Lemma 3.6 after appropriate choices of local coordinates in $\mathcal{X}$ and $\mathcal{W}$. In the next lemma, we apply the iteration process provided by Proposition 3.10 in the context of Theorem 3.13.

Lemma 3.16. In the setting of Theorem 3.13, the following holds. If, for some nonnegative integer l, the ratio $(F \circ \pi^{(l)}_{2l+1} : G \circ \pi^{(l)}_{2l+1})$ is convergent along $D_l(x_0)$, then the ratio $(F \circ \pi^{(l+1)}_{2l+3} : G \circ \pi^{(l+1)}_{2l+3})$ is convergent along $D_{l+1}(x_0)$. Here, $\pi^{(j)}_{2j+1}$ is the projection given by (3.19) and $D_j(x_0)$ is the submanifold given by (3.21), $j = l, l+1$. □

Proof. In order to apply Proposition 3.10, we first set $\eta := \mu$, $\mathcal{X} := \mathcal{X}$, $\mathcal{C} := \mathcal{W}$, $\mathcal{Y} := \mathcal{X}^{(l)}$, $y_0 := x^{(l)}_0$, $\nu := \pi^{(l)}_{2l+1}$, and $S := D_l(x_0)$, where $\mathcal{X}^{(l)}$ and $\pi^{(l)}_{2l+1}$ are given by (3.18) and (3.19), respectively. Note that ν is a holomorphic submersion and that, by assumption, the pullback under ν of the ratio $(F : G)$ is convergent along S. In view of Lemma 3.15, Proposition 3.10(i) implies that, by setting

$$\begin{aligned} &\mathcal{Y}_1 := \big\{ (z_1, \dots, z_{2l+1}, z_{2l+2}) \in \mathcal{X}^{(l)} \times \mathcal{X} : \mu(z_{2l+1}) = \mu(z_{2l+2}) \big\}, \\ &S_1 := \big\{ (z_1, \dots, z_{2l+1}, z_{2l+2}) \in \mathcal{Y}_1 : \lambda(z_1) = \lambda(x_0) \big\}, \\ &\nu_1 : \mathcal{Y}_1 \ni (z_1, \dots, z_{2l+1}, z_{2l+2}) \longmapsto z_{2l+2} \in \mathcal{X}, \end{aligned} \tag{3.22}$$

the pullback of $(F : G)$ under ν_1 is convergent along S_1. We now want to apply a second time Proposition 3.10. For this, we reset $\eta := \lambda$, $\mathcal{X} := \mathcal{X}$, $\mathcal{C} := \mathcal{Z}$, $\mathcal{Y} := \mathcal{Y}_1$, $y_0 := (x_0, \dots, x_0) \in \mathcal{Y}_1$, $\nu := \nu_1$, and $S := S_1$, where $\mathcal{Y}_1$, S_1, and ν_1 are as in (3.22). Applying Proposition 3.10(ii) in that context, we obtain easily that the pullback of $(F : G)$ under $\pi^{(l+1)}_{2l+3}$ is convergent along $D_{l+1}(x_0)$, the required conclusion. ■

Proof of Theorem 3.13. We first claim that the pullback of $(F : G)$ under $\pi^{(0)}_1$ is convergent along $D_0(x_0)$. Indeed, note that this is equivalent to saying that $(F : G)$ is convergent along $\{x \in \mathcal{X} : \lambda(x) = \lambda(x_0)\}$. Applying Proposition 3.10(ii) with $\eta := \lambda$, $\mathcal{X} := \mathcal{X}$, $\mathcal{C} := \mathcal{Z}$, $\mathcal{Y} := \mathcal{X}$, $y_0 := x_0$, $\nu := \mathrm{Id}_{\mathcal{X}}$, and $S := \{x_0\}$, and using Lemma 3.5(i), we get the desired claim. Applying Lemma 3.16 and using the finite type assumption on the pair (λ, μ), it follows that the ratio $(F \circ \pi^{(l_0)}_{2l_0+1} : G \circ \pi^{(l_0)}_{2l_0+1})$ is convergent along $D_{l_0}(x_0)$, where l_0 is chosen so that $\widetilde{\mu}|_{D_{l_0}(x_0)}$ has rank equal to $\dim \mathcal{W}$ at some points of the intersection $D_{l_0}(x_0) \cap E_{l_0}(x_0)$ that are arbitrarily close to $x^{(l_0)}_0$. Let α and β be power series on $\mathcal{W}$ given by Lemma 3.15. In view of Lemma 3.5(iii), the nontrivial ratio $(\alpha\circ\widetilde{\mu} : \beta\circ\widetilde{\mu})$ is thus convergent along $D_{l_0}(x_0)$, where $\beta\circ\widetilde{\mu}\not\equiv 0$. Since $E_{l_0}=\widetilde{\mu}^{-1}(\{0\})$, we see that Lemma 3.9 implies that the ratio $(\alpha:\beta)$ is equivalent to a nontrivial ratio $(\widetilde{\alpha}:\widetilde{\beta})$ of convergent power series on $\mathcal{W}$ (centered at $\mu(x_0)$). Therefore, it follows from Lemma 3.15 and the fact that μ is a submersion that $(F:G)$ is equivalent to the nontrivial ratio $(\widetilde{\alpha}\circ\mu:\widetilde{\beta}\circ\mu)$. The proof of Theorem 3.13 is complete. ■

4 Applications of Theorem 3.13 to ratios on generic submanifolds

The goal of this section is to apply the meromorphic extension property of ratios of formal power series given by Theorem 3.13 to the context of real-analytic generic submanifolds in $\mathbb{C}^N$, and to deduce some other properties (see Proposition 4.3) which will be useful for the proof of the theorems (Theorems 1.1 and 1.2) mentioned in the introduction.

Let $M\subset\mathbb{C}^N$ be a real-analytic generic submanifold of codimension d through 0, and $\rho(Z,\overline{Z}):=(\rho_1(Z,\overline{Z}),\ldots,\rho_d(Z,\overline{Z}))$ be a real-analytic vector-valued defining function for M defined in a connected neighborhood U of 0 in $\mathbb{C}^N$, satisfying $\partial\rho_1\wedge\cdots\wedge\partial\rho_d\neq 0$ on U. Define the complexification $\mathcal{M}$ of M as follows:

$$\mathcal{M}:=\{(Z,\zeta)\in U\times U^*:\rho(Z,\zeta)=0\},\tag{4.1}$$

where for any subset $V\subset\mathbb{C}^k$, we have denoted $V^*:=\{\overline{w}:w\in V\}$. Clearly, $\mathcal{M}$ is a d-codimensional complex submanifold of $\mathbb{C}^N\times\mathbb{C}^N$. We say that a vector field X defined in a neighborhood of $0\in\mathbb{C}^N\times\mathbb{C}^N$ is a $(0,1)$ vector field if it annihilates the natural projection $\mathbb{C}^N\times\mathbb{C}^N\ni(Z,\zeta)\mapsto Z\in\mathbb{C}^N$. We also say that X is tangent to $\mathcal{M}$ if $X(q)\in T_q\mathcal{M}$ for any $q\in\mathcal{M}$ near the origin. We have the following consequence of Theorem 3.13.

Theorem 4.1. Let $M\subset\mathbb{C}^N$ be a real-analytic generic submanifold through 0 and $\mathcal{M}\subset\mathbb{C}^N_Z\times\mathbb{C}^N_\zeta$ its complexification as given by (4.1). Consider formal power series $F(Z,\zeta)$, $G(Z,\zeta)\in\mathbb{C}[[Z,\zeta]]$ of the form $F(Z,\zeta)=\Phi(Y(\zeta),Z)$, $G(Z,\zeta)=\Psi(Y(\zeta),Z)$, where $Y(\zeta)$ is a $\mathbb{C}^r$-valued formal power series and Φ,Ψ are convergent power series centered at $(Y(0),0)\in\mathbb{C}^r\times\mathbb{C}^N$ with $G(Z,\zeta)\not\equiv 0$ for $(Z,\zeta)\in\mathcal{M}$. Suppose that M is minimal at 0 and that $\mathcal{L}(F/G)\equiv 0$ on $\mathcal{M}$ (i.e., $F\mathcal{L}G-G\mathcal{L}F\equiv 0$ on $\mathcal{M}$) for any $(0,1)$ holomorphic vector field tangent to $\mathcal{M}$. Then there exist convergent power series $\widetilde{F}(Z),\widetilde{G}(Z)\in\mathbb{C}\{Z\}$, with $\widetilde{G}(Z)\not\equiv 0$, such that the ratios $(F:G)$ and $(\widetilde{F}:\widetilde{G})$ are equivalent as formal power series on $\mathcal{M}$. □

For the proof of the theorem, we set $\mathcal{X} := \mathcal{M}$, $\mathcal{Z} = \mathcal{W} := \mathbb{C}^N$ and define the holomorphic submersions

$$\lambda : \mathcal{M} \ni (Z, \zeta) \longmapsto \zeta \in \mathbb{C}^N, \qquad \mu : \mathcal{M} \ni (Z, \zeta) \longmapsto Z \in \mathbb{C}^N. \tag{4.2}$$

Lemma 4.2. In the above setting, the pair (λ, μ) is of finite type at $0 \in \mathcal{M}$ (as defined in Section 3.3) if and only if M is minimal at the origin. □

Proof. For any nonnegative integer l, the fiber product $\mathcal{M}^{(l)}$ is here given by

$$\mathcal{M}^{(l)} = \{((Z_1, \zeta_1), \ldots, (Z_{2l+1}, \zeta_{2l+1})) \in \mathcal{M}^{2l+1} : Z_{2s-1} = Z_{2s},\ \zeta_{2s} = \zeta_{2s+1},\ 1 \le s \le l\} \tag{4.3}$$

and the maps $\widetilde{\lambda}_l : \mathcal{M}^{(l)} \to \mathbb{C}^N$, $\widetilde{\mu} : \mathcal{M}^{(l)} \to \mathbb{C}^N$ by

$$(Z_1, \zeta_1, \ldots, Z_{2l+1}, \zeta_{2l+1}) \longmapsto \zeta_1, \qquad (Z_1, \zeta_1, \ldots, Z_{2l+1}, \zeta_{2l+1}) \longmapsto Z_{2l+1}, \tag{4.4}$$

respectively. We then have $D_l(0) = \{((Z_1, \zeta_1), \ldots, (Z_{2l+1}, \zeta_{2l+1})) \in \mathcal{M}^{(l)} : \zeta_1 = 0\}$ and $E_l(0) = \{((Z_1, \zeta_1), \ldots, (Z_{2l+1}, \zeta_{2l+1})) \in \mathcal{M}^{(l)} : Z_{2l+1} = 0\}$. The reader can check that the map $\widetilde{\mu}_l|_{D_l(0)}$ coincides, up to a parametrization of $\mathcal{M}^{(l)}$, with a suitable Segre mapping v^{2l+1} at 0 as defined in [4, 5]. Therefore, in view of the minimality criterion of [5] (see also [2]), the pair (λ, μ) is of finite type at $0 \in \mathcal{M}$ if and only if M is minimal at 0. The proof of the lemma is complete. ■

Proof of Theorem 4.1. Since $(0, 1)$ holomorphic vector fields tangent to $\mathcal{M}$ coincide with holomorphic vector fields on $\mathcal{M}$ annihilating the submersion μ, in view of Lemma 4.2, we may apply Theorem 3.13 to conclude that the ratio $(F : G)$ is equivalent to a ratio $(\widetilde{F}(Z) : \widetilde{G}(Z))$ of convergent power series on $\mathcal{M}$ with $\widetilde{G}(Z) \not\equiv 0$. (The fact that $\widetilde{F}, \widetilde{G}$ may be chosen independent of ζ, follows from Remark 3.14.) The proof is complete. ■

In what follows, for any ring A, we denote, as usual, by $A[T]$, $T = (T_1, \ldots, T_r)$, the ring of polynomials over A in r indeterminates. An application of Theorem 4.1 is given by the following result, which will be essential for the proof of the main results of this paper.

Proposition 4.3. Let $M \subset \mathbb{C}^N$ be a minimal real-analytic generic submanifold through 0 and let $\mathcal{M} \subset \mathbb{C}^N_Z \times \mathbb{C}^N_\zeta$ be its complexification given by (4.1). Let $F(Z) := (F_1(Z), \ldots, F_r(Z))$ be a formal power series mapping satisfying one of the following conditions:

(i) there exist $G(\zeta) := (G_1(\zeta), \ldots, G_s(\zeta)) \in (\mathbb{C}[[\zeta]])^s$, $G(0) = 0$, and a polynomial $\mathcal{R}(Z, \zeta, X; T) \in \mathbb{C}\{Z, \zeta, X\}[T]$, $X = (X_1, \ldots, X_s)$, $T = (T_1, \ldots, T_r)$, such that

$\mathcal{R}(Z,\zeta,G(\zeta);T) \not\equiv 0$ for $(Z,\zeta) \in \mathcal{M}$, and such that $\mathcal{R}(Z,\zeta,G(\zeta);F(Z)) \equiv 0$ for $(Z,\zeta) \in \mathcal{M}$;

(ii) there exists a polynomial $\mathcal{P}(Z,\zeta;\widetilde{T},T) \in \mathbb{C}\{Z,\zeta\}[\widetilde{T},T]$, $\widetilde{T} = (\widetilde{T}_1,\ldots,\widetilde{T}_r)$, $T = (T_1, \ldots, T_r)$, such that $\mathcal{P}(Z,\zeta;\widetilde{T},T) \not\equiv 0$ for $(Z,\zeta) \in \mathcal{M}$, and such that $\mathcal{P}(Z,\zeta;\overline{F}(\zeta), F(Z)) \equiv 0$ for $(Z,\zeta) \in \mathcal{M}$.

Then there exists a nontrivial polynomial $\Delta(Z;T) \in \mathbb{C}\{Z\}[T]$ such that $\Delta(Z,F(Z)) \equiv 0$. □

Proof. Let $\mathcal{R}$ be as in (i) such that

$$\mathcal{R}\big(Z,\zeta,G(\zeta);F(Z)\big) \equiv 0, \quad \text{for } (Z,\zeta) \in \mathcal{M}. \tag{4.5}$$

We write $\mathcal{R}$ as a linear combination,

$$\mathcal{R}\big(Z,\zeta,G(\zeta);T\big) = \sum_{j=1}^{l} \delta_j\big(Z,\zeta,G(\zeta)\big) r_j(T), \tag{4.6}$$

where each $\delta_j(Z,\zeta,G(\zeta)) \not\equiv 0$ for $(Z,\zeta) \in \mathcal{M}$, $\delta_j(Z,\zeta,X) \in \mathbb{C}\{Z,\zeta\}[X]$, and r_j is a monomial in T. We prove the desired conclusion by induction on the number l of monomials in (4.6). For $l = 1$, (4.5), (4.6), and the fact that $\delta_1(Z,\zeta,G(\zeta)) \not\equiv 0$ on $\mathcal{M}$ imply that $r_1(F(Z)) \equiv 0$. Since r_1 is a monomial, it follows that $F_j(Z) = 0$ for some j, which yields the required nontrivial polynomial identity.

Suppose now that the desired conclusion holds for any polynomial $\mathcal{R}$ whose number of monomials is strictly less than l and for *any* formal power series mapping $G(\zeta)$. In view of (4.5) and (4.6), we have the following identity (understood in the field of fractions of formal power series):

$$r_l\big(F(Z)\big) + \sum_{j<l} \frac{\delta_j\big(Z,\zeta,G(\zeta)\big)}{\delta_l\big(Z,\zeta,G(\zeta)\big)} r_j\big(F(Z)\big) \equiv 0, \quad (Z,\zeta) \in \mathcal{M}. \tag{4.7}$$

Let $\mathcal{L}$ be any $(0,1)$ holomorphic vector field tangent to $\mathcal{M}$. Applying $\mathcal{L}$ to (4.7) and using the fact that $\mathcal{L}(F_j(Z)) \equiv 0$ for any j, we obtain

$$\sum_{j<l} \mathcal{L}\left(\frac{\delta_j\big(Z,\zeta,G(\zeta)\big)}{\delta_l\big(Z,\zeta,G(\zeta)\big)}\right) r_j\big(F(Z)\big) \equiv 0, \quad (Z,\zeta) \in \mathcal{M}. \tag{4.8}$$

We set $Q_j(Z,\zeta) := \delta_j(Z,\zeta,G(\zeta))/\delta_l(Z,\zeta,G(\zeta))$. It is easy to see that each ratio $\mathcal{L}Q_j$ can be written as a ratio of the following form:

$$\frac{\widetilde{\delta}_j\big(Z,\zeta,\widetilde{G}(\zeta)\big)}{\widetilde{\delta}_l\big(Z,\zeta,\widetilde{G}(\zeta)\big)} \tag{4.9}$$

for some $\widetilde{G}(\zeta) \in (\mathbb{C}[[\zeta]])^{\tilde{s}}$ with $\widetilde{G}(0) = 0$ and some $\widetilde{\delta}_j(Z, \zeta, \widetilde{X}), \widetilde{\delta}_l(Z, \zeta, \widetilde{X}) \in \mathbb{C}\{Z, \zeta, \widetilde{X}\}$, $\widetilde{X} \in \mathbb{C}^{\tilde{s}}$, with $\widetilde{\delta}_l(Z, \zeta, \widetilde{G}(\zeta)) \not\equiv 0$ for $(Z, \zeta) \in \mathcal{M}$. From (4.8), we are led to distinguish two cases. If for some $j \in \{1, \ldots, l-1\}$, $\mathcal{L}Q_j$ does not vanish identically on $\mathcal{M}$, then the required conclusion follows from (4.8), (4.9), and the induction hypothesis.

It remains to consider the case when $\mathcal{L}Q_j \equiv 0$ on $\mathcal{M}$ for all j and for all $(0,1)$ holomorphic vector fields $\mathcal{L}$ tangent to $\mathcal{M}$. Then each ratio Q_j satisfies the assumptions of Theorem 4.1, and therefore, there exist $\Phi^j(Z), \Psi^j(Z) \in \mathbb{C}\{Z\}$ with $\Psi^j(Z) \not\equiv 0$ such that $Q_j(Z, \zeta) = \Phi^j(Z)/\Psi^j(Z)$ for $j = 1, \ldots, l-1$. As a consequence, (4.7) can be rewritten as

$$r_l\big(F(Z)\big) + \sum_{j<l} \frac{\Phi^j(Z)}{\Psi^j(Z)} r_j\big(F(Z)\big) \equiv 0. \tag{4.10}$$

This proves the desired final conclusion and completes the proof of the conclusion assuming (i).

For the statement under the assumption (ii), consider a nontrivial polynomial $\mathcal{P}(Z, \zeta; \widetilde{T}, T)$ (on $\mathcal{M}$) such that $\mathcal{P}(Z, \zeta; \overline{F}(\zeta), F(Z)) \equiv 0$ for $(Z, \zeta) \in \mathcal{M}$. We write

$$\mathcal{P}\big(Z, \zeta; \widetilde{T}, T\big) = \sum_{\nu \in \mathbb{N}^r,\, |\nu| \le l} \mathcal{P}_\nu\big(Z, \zeta; \widetilde{T}\big) T^\nu, \tag{4.11}$$

where each $\mathcal{P}_\nu(Z, \zeta; \widetilde{T}) \in \mathbb{C}\{Z, \zeta\}[\widetilde{T}]$ and at least one of the $\mathcal{P}_\nu$s is nontrivial. If there exists $\nu_0 \in \mathbb{N}^r$ such that $\mathcal{P}_{\nu_0}(Z, \zeta; \overline{F}(\zeta)) \not\equiv 0$ for $(Z, \zeta) \in \mathcal{M}$, then it follows that the polynomial $\mathcal{P}(Z, \zeta; \overline{F}(\zeta), T)$ is nontrivial (on $\mathcal{M}$) and satisfies $\mathcal{P}(Z, \zeta; \overline{F}(\zeta), F(Z)) \equiv 0$ for $(Z, \zeta) \in \mathcal{M}$. Then condition (i) is fulfilled and the required conclusion is proved above.

It remains to consider the case when $\mathcal{P}_\nu(Z, \zeta; \overline{F}(\zeta)) \equiv 0$ on $\mathcal{M}$ and for any $\nu \in \mathbb{N}^r$. Fix any ν such that $\mathcal{P}_\nu(Z, \zeta; \widetilde{T})$ is nontrivial for $(Z, \zeta) \in \mathcal{M}$. Write $\mathcal{P}_\nu(Z, \zeta; \widetilde{T}) = \sum_{|\alpha| \le k} c_{\alpha,\nu}(Z, \zeta) \widetilde{T}^\alpha$ with each $c_{\alpha,\nu}(Z, \zeta) \in \mathbb{C}\{Z, \zeta\}$. Set $\overline{\mathcal{P}_\nu}(Z, \zeta; T) := \sum_{|\alpha| \le k} \overline{c_{\alpha,\nu}}(\zeta, Z) T^\alpha$. Then $\overline{\mathcal{P}_\nu}(Z, \zeta; T)$ is a nontrivial polynomial (on $\mathcal{M}$) and satisfies $\overline{\mathcal{P}_\nu}(Z, \zeta; F(Z)) \equiv 0$ for $(Z, \zeta) \in \mathcal{M}$. Here again, condition (i) is fulfilled and the desired conclusion follows. The proof is complete. ■

We conclude by mentioning the following result proved in [22, Theorem 5.1] and which is an immediate consequence of Proposition 4.3(i) and Proposition 2.1.

Corollary 4.4. Let $M \subset \mathbb{C}^N$ be a minimal real-analytic generic submanifold through the origin, $\mathcal{M} \subset \mathbb{C}^N_Z \times \mathbb{C}^N_\zeta$ its complexification as given by (4.1) and $F(Z) \in \mathbb{C}[[Z]]$. Assume that there exist $G(\zeta) := (G_1(\zeta), \ldots, G_s(\zeta)) \in (\mathbb{C}[[\zeta]])^s$ with $G(0) = 0$ and a polynomial $\mathcal{R}(Z, \zeta, X; T) \in \mathbb{C}\{Z, \zeta, X\}[T]$, $X = (X_1, \ldots, X_s)$, $T \in \mathbb{C}$, such $\mathcal{R}(Z, \zeta, G(\zeta); T) \not\equiv 0$ for $(Z, \zeta) \in \mathcal{M}$ and such that $\mathcal{R}(Z, \zeta, G(\zeta); F(Z)) \equiv 0$ for $(Z, \zeta) \in \mathcal{M}$. Then $F(Z)$ is convergent. □

5 Zariski closure of the graph of a formal map

Throughout this section, we let $f : (\mathbb{C}^N_Z, 0) \to (\mathbb{C}^{N'}_{Z'}, 0)$ be a formal map. As in Section 2.2, we associate to f its graph $\Gamma_f \subset \mathbb{C}^N_Z \times \mathbb{C}^{N'}_{Z'}$ seen as a formal submanifold. Given a (germ at $(0, 0) \in \mathbb{C}^N \times \mathbb{C}^{N'}$ of a) holomorphic function $H(Z, Z')$, we say that H vanishes on Γ_f if the formal power series $H(Z, f(Z))$ vanishes identically. If $A \subset \mathbb{C}^N \times \mathbb{C}^{N'}$ is a (germ through the origin of a) complex-analytic subset, we further say that the graph of f is contained in A, and write $\Gamma_f \subset A$, if any (germ at $(0, 0) \in \mathbb{C}^N \times \mathbb{C}^{N'}$ of a) holomorphic function $H(Z, Z')$ that vanishes on A, vanishes also on Γ_f. The goal of this section is to define and give some basic properties of the Zariski closure of the graph $\Gamma_f \subset \mathbb{C}^N \times \mathbb{C}^N$ over the ring $\mathbb{C}\{Z\}[Z']$.

5.1 Definition

For f as above, define the *Zariski closure* of Γ_f with respect to the ring $\mathbb{C}\{Z\}[Z']$ as the germ $\mathcal{Z}_f \subset \mathbb{C}^N \times \mathbb{C}^{N'}$ at $(0, 0)$ of a complex-analytic set defined as the zero set of all elements in $\mathbb{C}\{Z\}[Z']$ vanishing on Γ_f. Note that since $\mathcal{Z}_f$ contains the graph of f, it follows that $\dim_{\mathbb{C}} \mathcal{Z}_f \geq N$. In what follows, we will denote by $\mu(f)$ the dimension of the Zariski closure $\mathcal{Z}_f$. Observe also that since the ring $\mathbb{C}[[Z]]$ is an integral domain, it follows that $\mathcal{Z}_f$ is irreducible over $\mathbb{C}\{Z\}[Z']$.

5.2 Link with transcendence degree

In this section, we briefly discuss a link between the dimension of the Zariski closure $\mu(f)$ defined above and the transcendence degree of a certain field extension. (See [27] for basic notions from field theory used here.) In what follows, if $\mathbb{K} \subset \mathbb{L}$ is a field extension and $(x_1, \ldots, x_l) \in (\mathbb{L})^l$, we write $\mathbb{K}(x_1, \ldots, x_l)$ for the subfield of $\mathbb{L}$ generated by $\mathbb{K}$ and $(x_1, \ldots, x_l)$.

We denote by $\mathbb{M}_N$ the quotient field of the ring $\mathbb{C}\{Z\}$ and consider the field extension $\mathbb{M}_N \subset \mathbb{M}_N(f_1(Z), \ldots, f_{N'}(Z))$ where we write $f(Z) = (f_1(Z), \ldots, f_{N'}(Z))$. We then define the *transcendence degree* of the formal map f, denoted in what follows by $m(f)$, to be the transcendence degree of the above finitely generated field extension. (We should point out that this notion of transcendence degree of a formal map is in general different from the one discussed in [21, 22].) We have the following standard relation between $m(f)$ and $\mu(f)$.

Lemma 5.1. For any formal map $f : (\mathbb{C}^N, 0) \to (\mathbb{C}^{N'}, 0)$, $\mu(f) = N + m(f)$. □

The following well-known proposition shows the relevance of $\mu(f)$ for the study of the convergence of the map f.

Proposition 5.2. Let $f : (\mathbb{C}^N, 0) \to (\mathbb{C}^{N'}, 0)$ be a formal map and $\mu(f)$ be as defined in Section 5.1. Then the following are equivalent:

(i) $\mu(f) = N$;

(ii) f is convergent. □

Proposition 5.2 is a straightforward consequence of Proposition 2.1.

6 Local geometry of the Zariski closure

In this section, we keep the notation of Section 5. Our goal is to study the Zariski closure defined in Section 5.1 near some points of smoothness.

6.1 Preliminaries

Throughout Section 6, we assume that the dimension of the Zariski closure $\mathcal{Z}_f$ satisfies

$$\mu(f) < N + N'. \tag{6.1}$$

In what follows, for an open subset $\Omega \subset \mathbb{C}^k$, we denote by $\mathcal{O}(\Omega)$ the ring of holomorphic functions in Ω. Recall also that we use the notation Ω^* for the subset $\{\overline{q} : q \in \Omega\}$.

In Section 5, we saw that $\mu(f) \geq N$ and $m := m(f) = \mu(f) - N$ coincides with the transcendence degree of the field extension $\mathbb{M}_N \subset \mathbb{M}_N(f_1(Z), \ldots, f_{N'}(Z))$, where $f(Z) = (f_1(Z), \ldots, f_{N'}(Z))$. As a consequence, there exist integers $1 \leq j_1 < \cdots < j_m < N'$ such that $f_{j_1}(Z), \ldots, f_{j_m}(Z)$ form a transcendence basis of $\mathbb{M}_N(f_1(Z), \ldots, f_{N'}(Z))$ over $\mathbb{M}_N$. After renumbering the coordinates $Z' := (z', w') \in \mathbb{C}^m \times \mathbb{C}^{N'-m}$ and setting $m' := N' - m$, we may assume that

$$f = (g, h) \in \mathbb{C}^m_{z'} \times \mathbb{C}^{m'}_{w'}, \tag{6.2}$$

where $g = (g_1, \ldots, g_m)$ forms a transcendence basis of $\mathbb{M}_N(f_1, \ldots, f_{N'})$ over $\mathbb{M}_N$.

Since the components of the formal map $h : (\mathbb{C}^N_Z, 0) \to (\mathbb{C}^{m'}_{w'}, 0)$ are algebraically dependent over $\mathbb{M}_N(g)$, there exist monic polynomials $P_j(T) \in \mathbb{M}_N(g)[T]$, $j = 1, \ldots, m'$, such that if $h = (h_1, \ldots, h_{m'})$, then

$$P_j\big(h_j\big) = 0, \quad j = 1, \ldots, m', \text{ in } \mathbb{M}_N(f). \tag{6.3}$$

As a consequence, there exist nontrivial polynomials $\widehat{P}_j(T) \in \mathbb{C}\{Z\}[g][T]$, $j = 1, \ldots, m'$, such that

$$\widehat{P}_j\big(h_j\big) = 0, \quad j = 1, \ldots, m'. \tag{6.4}$$

For every $j = 1, \dots, m'$, we can write

$$\widehat{P}_j(T) = \sum_{\nu \le k_j} q_{j\nu} T^\nu, \tag{6.5}$$

where each $q_{j\nu} \in \mathbb{C}\{Z\}[g]$, $q_{jk_j} \not\equiv 0$, and $k_j \ge 1$. Since each $q_{j\nu}$ is in $\mathbb{C}\{Z\}[g]$, we can also write

$$q_{j\nu} = q_{j\nu}(Z) = R_{j\nu}\big(Z, g(Z)\big), \tag{6.6}$$

where $R_{j\nu}(z, z') \in \mathbb{C}\{Z\}[z']$.

Let Δ_0^N be a polydisc neighborhood of 0 in $\mathbb{C}^N$ such that the Zariski closure $\mathcal{Z}_f$ can be represented by an irreducible (over the ring $\mathbb{C}\{Z\}[Z']$) closed analytic subset of $\Delta_0^N \times \mathbb{C}^{N'}$ (also denoted by $\mathcal{Z}_f$). We have the inclusion

$$\Gamma_f \subset \mathcal{Z}_f \subset \mathbb{C}^N \times \mathbb{C}^{N'}. \tag{6.7}$$

Define

$$\widetilde{P}_j(Z, z'; T) := \sum_{\nu=0}^{k_j} R_{j\nu}(Z, z') T^\nu \in \mathcal{O}\big(\Delta_0^N\big)[z'][T], \quad j = 1, \dots, m'. \tag{6.8}$$

It follows from (6.4), (6.5), and (6.6) that we have

$$\widetilde{P}_j\big(Z, g(Z); h_j(Z)\big) \equiv 0, \quad \text{in } \mathbb{C}[[Z]],\ j = 1, \dots, m'. \tag{6.9}$$

Here each $R_{j\nu}(Z, z') \in \mathcal{O}(\Delta_0^N)[z']$, $k_j \ge 1$, and

$$R_{jk_j}\big(Z, g(Z)\big) \not\equiv 0. \tag{6.10}$$

Moreover, since $\mathbb{C}\{Z\}[z'][T]$ is a unique factorization domain (see, e.g., [27]), we may assume that the polynomials given by (6.8) are irreducible.

Consider the complex-analytic variety $\mathcal{V}_f \subset \mathbb{C}^N \times \mathbb{C}^{N'}$ through $(0, 0)$ defined by

$$\mathcal{V}_f := \big\{\big(Z, z', w'\big) \in \Delta_0^N \times \mathbb{C}^m \times \mathbb{C}^{m'} : \widetilde{P}_j\big(Z, z'; w_j'\big) = 0,\ j = 1, \dots, m'\big\}. \tag{6.11}$$

By (6.9), $\mathcal{V}_f$ contains the graph Γ_f and hence the Zariski closure $\mathcal{Z}_f$. In fact, since by Lemma 5.1, $\dim_{\mathbb{C}} \mathcal{Z}_f = \mu_p(f) = N + m$, it follows from the construction that $\mathcal{Z}_f$ is the (unique) irreducible component of $\mathcal{V}_f$ (over $\mathbb{C}\{Z\}[Z']$) containing Γ_f. Note that $\mathcal{V}_f$ is not irreducible in general and, moreover, may have a dimension larger than $\mu(f)$.

For $j = 1, \ldots, m'$, let $\widetilde{D_j}(Z, z') \in \mathcal{O}(\Delta_0^N)[z']$ be the discriminant of the polynomial $\widetilde{P_j}(Z, z'; T)$ (with respect to T). Consider the complex-analytic set

$$\widetilde{\mathcal{D}} := \cup_{j=1}^{m'} \{ (Z, z') \in \Delta_0^N \times \mathbb{C}^m : \widetilde{D_j}(Z, z') = 0 \}. \tag{6.12}$$

By the irreducibility of each polynomial $\widetilde{P_j}(Z, z'; T)$, we have $\widetilde{D_j}(Z, z') \not\equiv 0$ in $\Delta_0^N \times \mathbb{C}^m$, for $j = 1, \ldots, m'$. Therefore from the algebraic independence of the components of the formal map g over $\mathbb{M}_N$, it follows that the graph of g is not (formally) contained in $\widetilde{\mathcal{D}}$, that is,

$$\widetilde{D_j}(Z, g(Z)) \not\equiv 0, \quad \text{for } j = 1, \ldots, m'. \tag{6.13}$$

We also set

$$\mathcal{E} := \cup_{j=1}^{m'} \{ (Z, z') \in \Delta_0^N \times \mathbb{C}^m : R_{jk_j}(Z, z') = 0 \}. \tag{6.14}$$

It is well known that $\mathcal{E} \subset \widetilde{\mathcal{D}}$, and hence the graph of g is not contained in $\mathcal{E}$ too.

6.2 Description near smooth points

By the implicit function theorem, for any point $(Z_0, Z_0') \in \mathcal{V}_f$, $Z_0' = (z_0', w_0') \in \mathbb{C}^m \times \mathbb{C}^{m'}$, with $(Z_0, z_0') \notin \widetilde{\mathcal{D}}$, there exist polydisc neighborhoods of Z_0, z_0', and w_0', denoted by $\Delta_{Z_0}^N \subset \Delta_0^N \subset \mathbb{C}^N$, $\Delta_{z_0'}^m \subset \mathbb{C}^m$, $\Delta_{w_0'}^{m'} \subset \mathbb{C}^{m'}$, respectively, and a holomorphic map

$$\theta(Z_0, Z_0'; \cdot) : \Delta_{Z_0}^N \times \Delta_{z_0'}^m \longrightarrow \Delta_{w_0'}^{m'}, \tag{6.15}$$

such that for $(Z, z', w') \in \Delta_{Z_0}^N \times \Delta_{z_0'}^m \times \Delta_{w_0'}^{m'}$,

$$(Z, z', w') \in \mathcal{V}_f \Longleftrightarrow w' = \theta(Z_0, Z_0'; Z, z'). \tag{6.16}$$

Note that if moreover $(Z_0, Z_0') \in \mathcal{Z}_f$, then (6.16) is equivalent to $(Z, z', w') \in \mathcal{Z}_f$. For any point $(Z_0, Z_0') \in \mathcal{Z}_f$ with $(Z_0, z_0') \notin \widetilde{\mathcal{D}}$, consider the complex submanifold $\mathcal{Z}_f(Z_0, Z_0')$ defined by setting

$$\mathcal{Z}_f(Z_0, Z_0') := \mathcal{Z}_f \cap (\Delta_{Z_0}^N \times \Delta_{z_0'}^m \times \Delta_{w_0'}^{m'}). \tag{6.17}$$

Note that for any point (Z_0, Z_0') as above, by making the holomorphic change of coordinates $(\widetilde{Z}, \widetilde{Z}') = (Z, \varphi(Z, Z')) \in \mathbb{C}^N \times \mathbb{C}^{N'}$ where

$$\varphi(Z, Z') = \varphi(Z, (z', w')) := (z', w' - \theta(Z_0, Z_0'; Z, z')), \tag{6.18}$$

the submanifold $\mathcal{Z}_f(Z_0, Z_0')$ is given in these new coordinates by

$$\mathcal{Z}_f(Z_0, Z_0') = \{(\widetilde{Z}, \widetilde{Z}') \in \Delta^N_{Z_0} \times \Delta^m_{z_0'} \times \mathbb{C}^{m'} : \widetilde{Z}'_{m+1} = \cdots = \widetilde{Z}'_{N'} = 0\}, \tag{6.19}$$

where we write $\widetilde{Z}' = (\widetilde{Z}'_1, \ldots, \widetilde{Z}'_{N'})$.

We summarize the above in the following proposition.

Proposition 6.1. Let $f : (\mathbb{C}^N, 0) \to (\mathbb{C}^{N'}, 0)$ be a formal map and $\mathcal{Z}_f$ the Zariski closure of the graph of f as defined in Section 5.1. Suppose that $\mu(f) < N + N'$. Then for any point $(Z_0, Z_0') \in \mathcal{Z}_f$ with $(Z_0, z_0') \notin \widetilde{\mathcal{D}}$, where $\widetilde{\mathcal{D}}$ is given by (6.12), there exists a holomorphic change of coordinates near (Z_0, Z_0') of the form $(\widetilde{Z}, \widetilde{Z}') = (Z, \varphi(Z, Z')) \in \mathbb{C}^N \times \mathbb{C}^{N'}$ such that the complex submanifold $\mathcal{Z}_f$ is given near (Z_0, Z_0') by (6.19), with $m = \mu(f) - N$. □

For $(Z_0, Z_0') \in \mathcal{V}_f$ with $(Z_0, z_0') \notin \widetilde{\mathcal{D}}$ and $(\zeta, \chi') \in (\Delta^N_{Z_0})^* \times (\Delta^m_{z_0'})^*$, we define the $\mathbb{C}^{m'}$-valued holomorphic map

$$\overline{\theta}(Z_0, Z_0'; \zeta, \chi') := \overline{\theta(Z_0, Z_0'; \overline{\zeta}, \overline{\chi'})}, \tag{6.20}$$

where $\theta(Z_0, Z_0'; \cdot)$ is given by (6.15). The following lemma will be important for the proof of Theorem 7.1.

Lemma 6.2. With the above notation, for any polynomial $r(Z', \zeta') \in \mathbb{C}[Z', \zeta']$, $(Z', \zeta') \in \mathbb{C}^{N'} \times \mathbb{C}^{N'}$, there exists a nontrivial polynomial $\mathcal{R}_0(Z, \zeta, z', \chi'; T) \in \mathcal{O}(\Delta^N_0 \times \Delta^N_0)[z', \chi'][T]$, $T \in \mathbb{C}$, such that for any point $(Z_0, Z_0') \in \mathcal{V}_f$ with $(Z_0, z_0') \notin \widetilde{\mathcal{D}}$,

$$\mathcal{R}_0(Z, \zeta, z', \chi'; r(z', \theta(Z_0, Z_0'; Z, z'), \chi', \overline{\theta}(Z_0, Z_0'; \zeta, \chi'))) \equiv 0, \tag{6.21}$$

for $(Z, z') \in \Delta^N_{Z_0} \times \Delta^m_{z_0'}$ and $(\zeta, \chi') \in (\Delta^N_{Z_0})^* \times (\Delta^m_{z_0'})^*$. Moreover, $\mathcal{R}_0$ can be chosen with the following property: for any real-analytic generic submanifold $M \subset \mathbb{C}^N$ through the origin, $\mathcal{R}_0(Z, \zeta, z', \chi'; T) \not\equiv 0$ for $((Z, \zeta), z', \chi', T) \in (\mathcal{M} \cap (\Delta^N_0 \times \Delta^N_0)) \times \mathbb{C}^m \times \mathbb{C}^m \times \mathbb{C}$, where $\mathcal{M}$ is the complexification of M as defined by (4.1). □

Proof. For $(Z, z') \in \Delta^N_0 \times \mathbb{C}^m$ with $(Z, z') \notin \mathcal{E}$, where $\mathcal{E}$ is given by (6.14), and for $j = 1, \ldots, m'$, we denote by $\sigma^{(j)}_1(Z, z'), \ldots, \sigma^{(j)}_{k_j}(Z, z')$ the k_j roots (counted with multiplicity) of the polynomial $\widetilde{P}_j(Z, z'; T)$ given by (6.8). Similarly, for $(\zeta, \chi') \in \Delta^N_0 \times \mathbb{C}^m$ with $(\zeta, \chi') \notin \mathcal{E}$, $\overline{\sigma}^{(j)}_1(\zeta, \chi')$, $\ldots, \overline{\sigma}^{(j)}_{k_j}(\zeta, \chi')$ denote the k_j roots of the polynomial $\overline{\widetilde{P}}_j(\zeta, \chi'; T) := \sum_{\nu=0}^{k_j} \overline{R_{j\nu}}(\zeta, \chi') T^\nu$ (obtained from (6.8)). (Note that for any $j = 1, \ldots, m'$ and for any $1 \le \nu \le k_j$ we have $\overline{\sigma}^{(j)}_\nu(\zeta, \chi') = \overline{\sigma^{(j)}_\nu(\overline{\zeta}, \overline{\chi'})}$, which justifies the slight abuse of notation made here.) Fix

$r(Z', \zeta') \in \mathbb{C}[Z', \zeta']$ and set for (Z, z') and (ζ, χ') as above

$$\mathcal{R}_1\big(Z, \zeta, z', \chi'; T\big) := \prod_{l_1=1}^{k_1} \cdots \prod_{l_{m'}=1}^{k_{m'}} \prod_{n_1=1}^{k_1} \cdots \prod_{n_{m'}=1}^{k_{m'}} \Big(T - r\big(z', \sigma_{n_1}^{(1)}(Z, z'), \ldots, \sigma_{n_{m'}}^{(m')}(Z, z'), \chi', \overline{\sigma}_{l_1}^{(1)}(\zeta, \chi'), \ldots, \overline{\sigma}_{l_{m'}}^{(m')}(\zeta, \chi')\big)\Big). \tag{6.22}$$

It follows from Newton's theorem that (6.22) may be rewritten as

$$\mathcal{R}_1(Z, \zeta, z', \chi'; T) = T^{\delta} + \sum_{\nu < \delta} A_\nu(Z, \zeta, z', \chi') T^{\nu}, \tag{6.23}$$

for some positive integer δ, and where A_ν is of the form

$$A_\nu(Z, \zeta, z', \chi') = B_\nu\left(z', \chi', \left(\left(\frac{R_{j\alpha}(Z, z')}{R_{jk_j}(Z, z')}\right)_{\alpha \le k_j}, \left(\frac{\overline{R_{j\beta}}(\zeta, \chi')}{\overline{R_{jk_j}}(\zeta, \chi')}\right)_{\beta \le k_j}\right)_{0 \le j \le m'}\right), \tag{6.24}$$

with B_ν being polynomials in their arguments (depending only on the coefficients of $r(Z', \zeta')$). In view of (6.23) and (6.24), it is clear that there exists $C(Z, \zeta, z', \chi') \in \mathcal{O}(\Delta_0^N \times \Delta_0^N)[z', \chi']$ with $C(Z, \zeta, z', \chi') \not\equiv 0$ such that

$$\mathcal{R}_0(Z, \zeta, z', \chi'; T) := C(Z, \zeta, z', \chi') \cdot \mathcal{R}_1(Z, \zeta, z', \chi'; T) \in \mathcal{O}\big(\Delta_0^N \times \Delta_0^N\big)[z', \chi'][T]. \tag{6.25}$$

(C is obtained by clearing denominators in (6.24) for all $\nu < \delta$, and hence is a product of two nonzero terms, one in the ring $\mathcal{O}(\Delta_0^N)[z']$ and the other in $\mathcal{O}(\Delta_0^N)[\chi']$.) Since for any fixed $(Z_0, Z_0') \in \mathcal{V}_f$ with $(Z_0, z_0') \notin \widetilde{\mathcal{D}}$ and for any $(Z, z') \in \Delta_{Z_0}^N \times \Delta_{z_0'}^m$, $(Z, z') \notin \mathcal{E}$ and the jth component of $\theta(Z_0, Z_0'; Z, z')$ is a root of the polynomial $\widetilde{P}_j(Z, z'; T)$ by (6.16) and (6.11), it follows that $\mathcal{R}_0$ satisfies (6.21). Finally, the last desired property of $\mathcal{R}_0$ is easily seen from the explicit construction of the polynomial, that is, from the fact that $C(Z, \zeta, z', \chi')$ cannot vanish identically when restricted to $(\mathcal{M} \cap (\Delta_0^N \times \Delta_0^N)) \times \mathbb{C}^m \times \mathbb{C}^m$. The proof of Lemma 6.2 is complete. ■

6.3 Approximation by convergent maps

Since the graph of the formal map f is contained in $\mathcal{Z}_f$, by applying Artin's approximation theorem [1], for any nonnegative integer κ, there exists a convergent map $f^\kappa : (\mathbb{C}^N, 0) \to (\mathbb{C}^{N'}, 0)$ agreeing with f at 0 up to order κ such that the graph of f^κ is contained in $\mathcal{Z}_f$. We may assume that the maps f^κ are convergent in a polydisc neighborhood $\Delta^N_{0,\kappa} \subset \Delta^N_0$ of 0 in $\mathbb{C}^N$. Following the splitting (6.2), we write $f^\kappa = (g^\kappa, h^\kappa)$ and set

$$\begin{aligned} \Sigma^\kappa &:= \{Z \in \Delta^N_{0,\kappa} : (Z, g^\kappa(Z)) \notin \widetilde{\mathcal{D}}\}, \\ \Gamma_{f^\kappa} &:= \{(Z, f^\kappa(Z)) : Z \in \Delta^N_{0,\kappa}\} \subset \mathbb{C}^N \times \mathbb{C}^{N'}. \end{aligned} \tag{6.26}$$

Observe that since Γ_g is not contained in $\widetilde{\mathcal{D}}$ (see Section 6.1), it follows that for κ large enough, say, $\kappa \geq \widetilde{\kappa}$, the graph of g^κ is not contained in $\widetilde{\mathcal{D}}$ too, and therefore $\Delta^N_{0,\kappa} \setminus \Sigma^\kappa$ is dense in $\Delta^N_{0,\kappa}$. We may therefore, in what follows, assume that $\widetilde{\kappa} = 0$. Note also, that since the graph of f^κ is contained in $\mathcal{Z}_f$, in view of (6.16), we have for any $Z_0 \in \Delta^N_{0,\kappa} \setminus \Sigma^\kappa$

$$h^\kappa(Z) = \theta(Z_0, f^\kappa(Z_0); Z, g^\kappa(Z)), \tag{6.27}$$

for all Z in some (connected) neighborhood $\Omega^\kappa_{Z_0} \subset \Delta^N_{Z_0} \cap \Delta^N_{0,\kappa}$ of Z_0.

7 Main technical result

With all the tools defined in Sections 5 and 6 at our disposal, we are now ready to prove the following statement from which all theorems mentioned in the introduction will follow. In what follows, we keep the notation introduced in Sections 5 and 6.

Theorem 7.1. Let $f : (\mathbb{C}^N, 0) \to (\mathbb{C}^{N'}, 0)$ be a formal map, $\mathcal{Z}_f$ the Zariski closure of Γ_f as defined in Section 5.1, and $(f^\kappa)_{\kappa \geq 0}$ the convergent maps given in Section 6.3 (associated to f and $\mathcal{Z}_f$). Let $M \subset \mathbb{C}^N$ be a minimal real-analytic generic submanifold through the origin. Assume that f sends M into M' where $M' \subset \mathbb{C}^{N'}$ is a proper real-algebraic subset through the origin. Then, shrinking M around the origin if necessary, there exist a positive integer κ_0 and an appropriate union Z_f of local real-analytic irreducible components of $\mathcal{Z}_f \cap (M \times \mathbb{C}^{N'})$ such that the following hold:

(i) $\mu(f) < N + N'$ for $\mu(f) = \dim \mathcal{Z}_f$;

(ii) for any $\kappa \geq \kappa_0$, $\Gamma_{f^\kappa} \cap (M \times \mathbb{C}^{N'}) \subset Z_f \subset M \times M'$, where Γ_{f^κ} is given by (6.26);

(iii) Z_f satisfies the following straightening property: for any $\kappa \geq \kappa_0$, there exists a neighborhood M^κ of 0 in M such that for any point Z_0 in a dense open subset of M^κ, there exist a neighborhood $U^\kappa_{Z_0}$ of $(Z_0, f^\kappa(Z_0))$ in $\mathbb{C}^N \times \mathbb{C}^{N'}$ and a holomorphic change of coordinates in $U^\kappa_{Z_0}$ of the form $(\widetilde{Z}, \widetilde{Z}') = \Phi^\kappa(Z, Z') = (Z, \varphi^\kappa(Z, Z')) \in \mathbb{C}^N \times \mathbb{C}^{N'}$ such that

$$Z_f \cap U^\kappa_{Z_0} = \{(Z, Z') \in U^\kappa_{Z_0} : Z \in M,\ \widetilde{Z}'_{m+1} = \cdots = \widetilde{Z}'_{N'} = 0\}, \tag{7.1}$$

where $m = \mu(f) - N$. □

For the proof of the above result, we will need the following key proposition.

Proposition 7.2. Under the assumptions of Theorem 7.1, shrinking M around the origin if necessary, the following hold:

(i) $\mu(f) < N + N'$;

(ii) there exists a positive integer κ_0 such that for all $\kappa \geq \kappa_0$ and for all points $Z_0 \in (M \cap \Delta^N_{0,\kappa}) \setminus \Sigma^\kappa$, the real-analytic submanifold $\mathcal{Z}_f(Z_0, f^\kappa(Z_0)) \cap (M \times \mathbb{C}^{N'})$ is contained in $M \times M'$. Here Σ^κ and $\mathcal{Z}_f((Z_0, f^\kappa(Z_0)))$ are given by (6.26) and (6.17), respectively, and $\Delta^N_{0,\kappa}$ is a polydisc of convergence of f^κ. □

Proof of Proposition 7.2(i). Since M' is a proper real-algebraic subset of $\mathbb{C}^{N'}$, there exists a nontrivial polynomial $\rho'(Z', \overline{Z'}) \in \mathbb{C}[Z', \overline{Z'}]$ vanishing on M'. By assumption, f sends M into M' and therefore we have $\rho'(f(Z), \overline{f(Z)}) \equiv 0$ for $Z \in M$, or, equivalently,

$$\rho'\big(f(Z), \bar{f}(\zeta)\big) = 0, \quad (Z, \zeta) \in \mathcal{M}, \tag{7.2}$$

where $\mathcal{M}$ is the complexification of M as given by (4.1). It follows from Proposition 4.3(ii) (applied to $F(Z) := f(Z) = (f_1(Z), \ldots, f_{N'}(Z))$) that the components $f_1(Z), \ldots, f_{N'}(Z)$ satisfy a nontrivial polynomial identity with coefficients in $\mathbb{C}\{Z\}$. This implies that $\mu(f) < N + N'$. The proof of Proposition 7.2(i) is complete. ■

By Proposition 7.2(i), we may now assume that (6.1) holds and hence the arguments of Section 6 apply. Since M' is a real-algebraic subset of $\mathbb{C}^{N'}$, it is given by

$$M' := \big\{Z' \in \mathbb{C}^{N'} : \rho'_1(Z', \overline{Z'}) = \cdots = \rho'_l(Z', \overline{Z'}) = 0\big\}, \tag{7.3}$$

where each $\rho'_j(Z', \overline{Z'})$, for $j = 1, \ldots, l$, is a real-valued polynomial in $\mathbb{C}[Z', \overline{Z'}]$.

Proof of Proposition 7.2(ii). By shrinking M around the origin, we may assume that M is connected and is contained in Δ^N_0. We proceed by contradiction. Then, in view of (6.16), (6.17), and (7.3), there exists $j_0 \in \{1, \ldots, l\}$ and a subsequence $(f^{s_k})_{k \geq 0}$ of $(f^\kappa)_{\kappa \geq 0}$ such that for any k, there exists $Z^k \in M \cap \Delta^N_{0,s_k}$ such that

$$\rho'_{j_0}\big(z', \theta\big(Z^k, f^{s_k}(Z^k); Z, z'\big), \overline{z'}, \overline{\theta\big(Z^k, f^{s_k}(Z^k); Z, z'\big)}\big) \not\equiv 0, \tag{7.4}$$

for $(Z, z') \in (M \cap \Delta^N_{Z^k}) \times \Delta_{g^{s_k}(Z^k)}$. After complexification of (7.4), we obtain

$$\rho'_{j_0}\big(z', \theta\big(Z^k, f^{s_k}(Z^k); Z, z'\big), \chi', \bar{\theta}\big(Z^k, f^{s_k}(Z^k); \zeta, \chi'\big)\big) \not\equiv 0, \tag{7.5}$$

for $(Z,\zeta)\in\mathcal{M}\cap(\Delta^N_{Z^k}\times(\Delta^N_{Z^k})^*)$ and $(z',\chi')\in\Delta_{g^{s_k}(Z^k)}\times(\Delta_{g^{s_k}(Z^k)})^*$. By Lemma 6.2 applied to $r(Z',\zeta'):=\rho'_{j_0}(Z',\zeta')$, there exists a nontrivial polynomial $\mathcal{R}_0(Z,\zeta,z',\chi';T)\in\mathcal{O}(\Delta^N_0\times\Delta^N_0)[z',\chi'][T]$ such that for any positive integer k we have

$$\mathcal{R}_0\big(Z,\zeta,z',\chi';\rho'_{j_0}\big(z',\theta\big(Z^k,f^{s_k}\big(Z^k\big);Z,z'\big),\chi',\overline{\theta}\big(Z^k,f^{s_k}\big(Z^k\big);\zeta,\chi'\big)\big)\big)\equiv 0, \tag{7.6}$$

for (Z,ζ) and (z',χ') as above. By Lemma 6.2, $\mathcal{R}_0$ does not vanish identically when restricted to $(\mathcal{M}\cap(\Delta^N_0\times\Delta^N_0))\times\mathbb{C}^m\times\mathbb{C}^m\times\mathbb{C}$ and therefore we may write

$$\mathcal{R}_0(Z,\zeta,z',\chi';T)=T^\eta\cdot\mathcal{R}_{00}(Z,\zeta,z',\chi';T), \tag{7.7}$$

for $((Z,\zeta),z',\chi',T)$ as above and for some integer η and some polynomial $\mathcal{R}_{00}(Z,\zeta,z',\chi';T)\in\mathcal{O}(\Delta^N_0\times\Delta^N_0)[z',\chi'][T]$ satisfying

$$\mathcal{R}_{00}(Z,\zeta,z',\chi';0)\not\equiv 0,\quad \big((Z,\zeta),z',\chi'\big)\in\mathcal{M}\times\mathbb{C}^m\times\mathbb{C}^m. \tag{7.8}$$

We also write

$$\mathcal{R}_{00}(Z,\zeta,z',\chi';T)=\mathcal{R}_{00}(Z,\zeta,z',\chi';0)+T\cdot\mathcal{P}_{00}(Z,\zeta,z',\chi';T), \tag{7.9}$$

with $\mathcal{P}_{00}(Z,\zeta,z',\chi';T)\in\mathcal{O}(\Delta^N_0\times\Delta^N_0)[z',\chi'][T]$. In view of (7.6), (7.5), and (7.7), we obtain

$$\mathcal{R}_{00}\big(Z,\zeta,z',\chi';\rho'_{j_0}\big(\big(z',\theta\big(Z^k,f^{s_k}\big(Z^k\big);Z,z'\big)\big)\big),\big(\chi',\overline{\theta}\big(Z^k,f^{s_k}\big(Z^k\big);\zeta,\chi'\big)\big)\big)\equiv 0, \tag{7.10}$$

for $(Z,\zeta)\in\mathcal{M}\cap(\Delta^N_{Z^k}\times(\Delta^N_{Z^k})^*)$ and $(z',\chi')\in\Delta_{g^{s_k}(Z^k)}\times(\Delta_{g^{s_k}(Z^k)})^*$. Setting $z'=g^{s_k}(Z)$ and $\chi'=\overline{g^{s_k}}(\zeta)$ in (7.10), we obtain, in view of (6.27)

$$\mathcal{R}_{00}\big(Z,\zeta,g^{s_k}(Z),\overline{g^{s_k}}(\zeta);\rho'_{j_0}\big(f^{s_k}(Z),\overline{f^{s_k}}(\zeta)\big)\big)\equiv 0, \tag{7.11}$$

for (Z,ζ) in some neighborhood of $(Z^k,\overline{Z^k})$ in $\mathcal{M}$ and hence, by unique continuation, for all $(Z,\zeta)\in\mathcal{M}\cap(\Delta^N_{0,s_k}\times(\Delta^N_{0,s_k})^*)$. In view of (7.9), (7.11) leads to the equality

$$\begin{aligned}&\mathcal{R}_{00}\big(Z,\zeta,g^{s_k}(Z),\overline{g^{s_k}}(\zeta);0\big)\\&\quad=-\rho'_{j_0}\big(f^{s_k}(Z),\overline{f^{s_k}}(\zeta)\big)\cdot\mathcal{P}_{00}\big(Z,\zeta,g^{s_k}(Z),\overline{g^{s_k}}(\zeta);\rho'_{j_0}\big(f^{s_k}(Z),\overline{f^{s_k}}(\zeta)\big)\big),\end{aligned} \tag{7.12}$$

for (Z,ζ) as above. Since $f(M) \subset M'$, we have the formal identity $\rho'_{j_0}(f(Z)), \bar{f}(\zeta) = 0$ for $(Z,\zeta) \in \mathcal{M}$. Therefore, since $f^{s_k}(Z)$ approximates $f(Z)$ up to order $s_k \geq k$ at 0, it follows that

$$\rho'_{j_0}\left(f^{s_k}(Z), \overline{f^{s_k}}(\zeta)\right) = O(k), \quad (Z,\zeta) \in \mathcal{M}. \tag{7.13}$$

In view of (7.12), (7.13) implies that

$$\mathcal{R}_{00}\left(Z, \zeta, g^{s_k}(Z), \overline{g^{s_k}}(\zeta); 0\right) = O(k) \tag{7.14}$$

for $(Z,\zeta) \in \mathcal{M}$. Since for any k, $g^{s_k}(Z)$ approximates $g(Z)$ up to order $s_k \geq k$ at 0, the only possibility for (7.14) to hold is that

$$\mathcal{R}_{00}\left(Z, \zeta, g(Z), \bar{g}(\zeta); 0\right) \equiv 0, \quad (Z,\zeta) \in \mathcal{M}. \tag{7.15}$$

In view of (7.8) and (7.15), condition (ii) in Proposition 4.3 is satisfied for the components $g_1(Z), \ldots, g_m(Z)$ of $g(Z)$. By Proposition 4.3, there exists a nontrivial polynomial $\Delta(Z, z') \in \mathbb{C}\{Z\}[z']$ such $\Delta(Z, g(Z)) \equiv 0$. This contradicts the fact that $g(Z) = (g_1(Z), \ldots, g_m(Z))$ is a transcendence basis of $\mathbb{M}_N(f(Z))$ over $\mathbb{M}_N$. This completes the proof of Proposition 7.2. ■

Proof of Theorem 7.1. In view of Proposition 7.2(i), we just need to prove parts (ii) and (iii) of the theorem. We choose the integer κ_0 given by Proposition 7.2(ii) and define Z_f to be the union of all local real-analytic irreducible components of $\mathcal{Z}_f \cap (M \times \mathbb{C}^{N'})$ at $(0,0)$ that contain the germ of $\Gamma_{f^\kappa} \cap (M \times \mathbb{C}^{N'})$ for some $\kappa \geq \kappa_0$. The inclusion $Z_f \subset M \times M'$ follows from the construction of Z_f and Proposition 7.2(ii). This shows Theorem 7.1(ii). Finally, by setting for any $\kappa \geq \kappa_0$, $M^\kappa := M \cap \Delta^N_{0,\kappa}$, Theorem 7.1(iii) follows from Propositions 7.2(ii), 6.1, and the fact that the subset Σ^κ is nowhere dense in $\Delta^N_{0,\kappa}$. The proof of Theorem 7.1 is complete. ■

8 Proofs of Theorems 1.1 and 1.2

Proof of Theorem 1.1. Without loss of generality, we may assume that p and p' are the origin in $\mathbb{C}^N$ and $\mathbb{C}^{N'}$, respectively. In the case where M is generic, Theorem 1.1 is then an immediate consequence of Theorem 7.1(ii). It remains to consider the nongeneric case. If M is not generic, using the intrinsic complexification of M, we may assume, after a local holomorphic change of coordinates near 0, that $M = \widetilde{M} \times \{0\} \subset \mathbb{C}^{N-r}_z \times \mathbb{C}^r_w$, for some $1 \leq r \leq N-1$ and some real-analytic generic minimal submanifold $\widetilde{M}$ (see, e.g., [5]). By the generic case treated above, for any positive integer k, there exists a local holomorphic map $g^k : (\mathbb{C}^{N-r}, 0) \to (\mathbb{C}^{N'}, 0)$ defined in a neighborhood of 0 in $\mathbb{C}^{N-r}$, sending $\widetilde{M}$ into M' and for which the Taylor series mapping at $0 \in \mathbb{C}^{N-r}$ agrees with $z \mapsto f(z, 0)$ up to order k. Let $h^k : (\mathbb{C}^N, 0) \to (\mathbb{C}^{N'}, 0)$ be the polynomial mapping obtained by taking the Taylor polynomial of order k at 0 of each component of the formal map $f(z, w) - f(z, 0)$. Then by setting for every nonnegative integer k, $f^k(z, w) := g^k(z, 0) + h^k(z, w)$, the reader can easily check that the convergent map $f^k : (\mathbb{C}^N, 0) \to (\mathbb{C}^{N'}, 0)$ satisfies all the desired properties. The proof of Theorem 1.1 is complete. ∎

Proof of Theorem 1.2. Without loss of generality, we may assume that p and p' are the origin in $\mathbb{C}^N$ and $\mathbb{C}^{N'}$, respectively. Suppose that f is not convergent. Let $\mu(f)$ and $(f^\kappa)_{\kappa \geq 0}$ be given by Theorem 7.1. By Proposition 5.2, we have $m = \mu(f) - N > 0$. Therefore Theorem 7.1(iii) implies that for κ large enough, f^κ maps a dense subset of a neighborhood of 0 (which may depend on κ) into the subset $\mathcal{E}'$. Since $\mathcal{E}'$ is closed in M' (see, e.g., [12]), f^κ maps actually a whole neighborhood of 0 in M to $\mathcal{E}'$. Since for any κ, f^κ agrees with f up to order κ at 0, it follows that f sends M into $\mathcal{E}'$ as defined in Section 1. This completes the proof. ∎

Acknowledgment

The first author was partially supported by Swiss National Science Foundation (NSF) grant 2100-063464.00/1.

References

[1] M. Artin, *On the solutions of analytic equations*, Invent. Math. **5** (1968), 277–291.

[2] M. S. Baouendi, P. Ebenfelt, and L. P. Rothschild, *Algebraicity of holomorphic mappings between real algebraic sets in* $\mathbf{C}^n$, Acta Math. **177** (1996), no. 2, 225–273.

[3] ———, *Parametrization of local biholomorphisms of real analytic hypersurfaces*, Asian J. Math. **1** (1997), no. 1, 1–16.

[4] ———, *Rational dependence of smooth and analytic CR mappings on their jets*, Math. Ann. **315** (1999), no. 2, 205–249.

[5] ———, *Real Submanifolds in Complex Space and Their Mappings*, Princeton Mathematical Series, vol. 47, Princeton University Press, New Jersey, 1999.

[6] ———, *Convergence and finite determination of formal CR mappings*, J. Amer. Math. Soc. **13** (2000), no. 4, 697–723.

[7] ———, *Local geometric properties of real submanifolds in complex space*, Bull. Amer. Math. Soc. (N.S.) **37** (2000), no. 3, 309–336.

[8] M. S. Baouendi, N. Mir, and L. P. Rothschild, *Reflection ideals and mappings between generic submanifolds in complex space*, J. Geom. Anal. **12** (2002), no. 4, 543–580.

[9] M. S. Baouendi, L. P. Rothschild, and D. Zaitsev, *Equivalences of real submanifolds in complex space*, J. Differential Geom. **59** (2001), 301–351.

[10] T. Bloom and I. Graham, *On "type" conditions for generic real submanifolds of* $\mathbf{C}^n$, Invent. Math. **40** (1977), no. 3, 217–243.

[11] S. S. Chern and J. K. Moser, *Real hypersurfaces in complex manifolds*, Acta Math. **133** (1974), 219–271.

[12] J. P. D'Angelo, *Finite type and the intersection of real and complex subvarieties*, Several Complex Variables and Complex Geometry, Part 3 (Santa Cruz, Calif, 1989), Proc. Sympos. Pure Math., vol. 52, Part 3, American Mathematical Society, Rhode Island, 1991, pp. 103–117.

[13] F. Forstnerič, *Extending proper holomorphic mappings of positive codimension*, Invent. Math. **95** (1989), no. 1, 31–61.

[14] X. Huang, *Schwarz reflection principle in complex spaces of dimension two*, Comm. Partial Differential Equations **21** (1996), no. 11-12, 1781–1828.

[15] ———, *On some problems in several complex variables and CR geometry*, First International Congress of Chinese Mathematicians (Beijing, 1998), AMS/IP Stud. Adv. Math., vol. 20, American Mathematical Society, Rhode Island, 2001, pp. 383–396.

[16] J. J. Kohn, *Boundary behavior of* $\bar{\partial}$ *on weakly pseudo-convex manifolds of dimension two*, J. Differential Geometry **6** (1972), 523–542.

[17] B. Lamel, *Holomorphic maps of real submanifolds in complex spaces of different dimensions*, Pacific J. Math. **201** (2001), no. 2, 357–387.

[18] L. Lempert, *On the boundary behavior of holomorphic mappings*, Contributions to Several Complex Variables, Aspects Math., vol. E9, Vieweg, Braunschweig, 1986, pp. 193–215.

[19] F. Meylan, N. Mir, and D. Zaitsev, *Analytic regularity of CR-mappings*, Math. Res. Lett. **9** (2002), no. 1, 73–93.

[20] ———, *Holomorphic extension of smooth CR-mappings between real-analytic and real-algebraic CR-manifolds*, preprint, 2002, http://www.arxiv.org/abs/math.CV/0201267.

[21] N. Mir, *Formal biholomorphic maps of real analytic hypersurfaces*, Math. Res. Lett. **7** (2000), no. 2-3, 343–359.

[22] ———, *On the convergence of formal mappings*, Comm. Anal. Geom. **10** (2002), no. 1, 23–59.

[23] J. K. Moser and S. M. Webster, *Normal forms for real surfaces in* $\mathbf{C}^2$ *near complex tangents and hyperbolic surface transformations*, Acta Math. **150** (1983), no. 3-4, 255–296.

[24] A. E. Tumanov, *Extension of CR-functions into a wedge from a manifold of finite type*, Mat. Sb. (N.S.) **136**(**178**) (1988), no. 1, 128–139, translation in Math. USSR-Sb. **64** (1989), no. 1, 129–140.

[25] D. Zaitsev, *Germs of local automorphisms of real-analytic CR structures and analytic dependence on* k*-jets*, Math. Res. Lett. **4** (1997), no. 6, 823–842.

[26] ———, *Algebraicity of local holomorphisms between real-algebraic submanifolds of complex spaces*, Acta Math. **183** (1999), no. 2, 273–305.

[27] O. Zariski and P. Samuel, *Commutative Algebra*, *Volume I*, The University Series in Higher Mathematics, Springer-Verlag, New York, 1958.

Francine Meylan: Institut de Mathématiques, Université de Fribourg, 1700 Perolles, Fribourg, Switzerland
E-mail address: francine.meylan@unifr.ch

Nordine Mir: Université de Rouen, Laboratoire de Mathématiques Raphaël Salem, UMR 6085 CNRS, 76821 Mont-Saint-Aignan Cedex, France
E-mail address: Nordine.Mir@univ-rouen.fr

Dmitri Zaitsev: Mathematisches Institut, Eberhard-Karls-Universität Tübingen, Auf der Morgenstelle 10, 72076 Tübingen, Germany
E-mail address: dmitri.zaitsev@uni-tuebingen.de