The Journal of Geometric Analysis
Volume 12, Number 4, 2002

Reflection Ideals and Mappings Between Generic Submanifolds in Complex Space

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ABSTRACT. Results on finite determination and convergence of formal mappings between smooth generic submanifolds in $\mathbb{C}^N$ are established in this article. The finite determination result gives sufficient conditions to guarantee that a formal map is uniquely determined by its jet, of a preassigned order, at a point. Convergence of formal mappings for real-analytic generic submanifolds under appropriate assumptions is proved, and natural geometric conditions are given to assure that if two germs of such submanifolds are formally equivalent, then, they are necessarily biholomorphically equivalent. It is also shown that if two real-algebraic hypersurfaces in $\mathbb{C}^N$ are biholomorphically equivalent, then, they are algebraically equivalent. All the results are first proved in the more general context of "reflection ideals" associated to formal mappings between formal as well as real-analytic and real-algebraic manifolds.

1. Introduction and main results

In this article, we study formal mappings between smooth generic submanifolds in $\mathbb{C}^N$ and establish results on finite determination, convergence and local biholomorphic, and algebraic equivalence. Our finite determination result gives sufficient conditions to guarantee that a formal map as above is uniquely determined by its jet (of a preassigned order) at a point. For real-analytic generic submanifolds, we prove convergence of formal mappings under appropriate assumptions and also give natural geometric conditions to assure that if two germs of such submanifolds are formally equivalent, then they are necessarily biholomorphically equivalent. If the submanifolds are moreover real-algebraic, we address the question of deciding when biholomorphic equivalence implies algebraic equivalence. In particular, we prove that if two real-algebraic hypersurfaces in $\mathbb{C}^N$ are biholomorphically equivalent, then they are in fact algebraically equivalent. All the results are first proved in the more general context of "reflection ideals" associated to formal mappings between formal as well as real-analytic and real-algebraic manifolds.

We now give precise definitions in order to state some of our main results. Let $p \in \mathbb{C}^N$ and

Math Subject Classifications. 32H02.
Key Words and Phrases. generic submanifold, biholomorphic equivalence, formal equivalence, algebraic equivalence, formal mapping, reflection ideal, finite type, holomorphically nondegenerate.
Acknowledgements and Notes. The first and third authors are partially supported by National Science Foundation grant DMS 01-00330. The second author would like to thank UCSD for its hospitality during the completion of this work.
The authors would like to thank the referee for several valuable comments which have improved the readability of this article.

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ISSN 1050-6926

$p' \in \mathbb{C}^{N'}$. A formal map $H : (\mathbb{C}^N, p) \to (\mathbb{C}^{N'}, p')$ is an N'-vector of formal power series in $Z - p$ with $H(p) = p'$. The map $H(Z) = (H_1(Z), \dots, H_{N'}(Z))$ is called *finite* if the quotient ring $\mathbb{C}[\![Z-p]\!]/(H(Z))$ is finite dimensional as a vector space over $\mathbb{C}$, where $(H(Z))$ is the ideal generated by the $H_j(Z)$ in $\mathbb{C}[\![Z-p]\!]$, $j = 1, \dots, N'$. In the case $N = N'$, H is called *invertible* if its Jacobian determinant does not vanish at p.

Recall that a smooth submanifold $M \subset \mathbb{C}^N$ is called *generic* if it is locally defined by the vanishing of smooth real-valued functions $r_1(Z, \bar{Z}), \dots, r_d(Z, \bar{Z})$ with linearly independent complex differentials $\partial r_1(Z, \bar{Z}), \dots, \partial r_d(Z, \bar{Z})$. A generic submanifold $M \subset \mathbb{C}^N$ is said to be of *finite type* at $p \in M$ in the sense of Kohn [22] and Bloom–Graham [14] if the Lie algebra generated by the (0, 1) and (1, 0) smooth vector fields tangent to M spans the complexified tangent space of M at p.

A (holomorphic) formal vector field at $p \in \mathbb{C}^N$ is given by

$$X = \sum_{k=1}^{N} a_k(Z) \frac{\partial}{\partial Z_k}$$

with $a_k(Z) \in \mathbb{C}[\![Z-p]\!]$, $k = 1, \dots, N$. If M is a generic submanifold of real codimension d as above, and $r_1, \dots, r_d$ are smooth real-valued defining functions of M near $p \in M$, we denote by $\rho(Z, \bar{Z}) = (\rho_1(Z, \bar{Z}), \dots, \rho_d(Z, \bar{Z}))$ the Taylor series of $r_1, \dots, r_d$ at p considered as formal power series in $Z - p$ and $\bar{Z} - \bar{p}$. A holomorphic formal vector field X at $p \in M$ is called *tangent to M* if

$$(X\rho)\left(Z, \bar{Z}\right) = c\left(Z, \bar{Z}\right) \rho\left(Z, \bar{Z}\right) ,$$

where $c(Z, \bar{Z})$ is a $d \times d$ matrix with entries in $\mathbb{C}[\![Z-p, \bar{Z}-\bar{p}]\!]$. Following Stanton [28], we say that the submanifold M is *holomorphically nondegenerate* at $p \in M$ if there is no nontrivial formal holomorphic vector field at p tangent to M (see [6], Section 11.7).

Let $M \subset \mathbb{C}^N$ and $M' \subset \mathbb{C}^{N'}$ be smooth generic submanifolds of codimension d and d' through p and p', respectively and $H : (\mathbb{C}^N, p) \to (\mathbb{C}^{N'}, p')$ a formal map. We say that H maps M into M' and write $H(M) \subset M'$ if

$$\rho'\left(H(Z), \overline{H(Z)}\right) = a\left(Z, \bar{Z}\right) \rho\left(Z, \bar{Z}\right) ,$$

where $\rho(Z, \bar{Z})$ is the d-vector valued formal power series defined as above for (M, p), $\rho'(Z', \bar{Z}')$ is the d'-vector valued corresponding series for (M', p'), and $a(Z, \bar{Z})$ is a $d' \times d$ matrix with entries in $\mathbb{C}[\![Z-p, \bar{Z}-\bar{p}]\!]$.

We are now ready to state some of the main results of this article. We will discuss previous related work towards the end of this introduction. Our first two results deal with finite determination of formal mappings between smooth generic submanifolds in $\mathbb{C}^N$, as well as convergence of such mappings when the submanifolds are real-analytic.

Theorem 1.1. *Let $M, M' \subset \mathbb{C}^N$ be smooth generic submanifolds of the same dimension through p and p', respectively. Assume that M is of finite type at p and that M' is holomorphically nondegenerate at p'. Let $H^0 : (\mathbb{C}^N, p) \to (\mathbb{C}^N, p')$ be a formal finite map sending M into M'. Then, there exists an integer K such that if $H : (\mathbb{C}^N, p) \to (\mathbb{C}^N, p')$ is another formal map sending M into M' with*

$$\partial^\alpha H(p) = \partial^\alpha H^0(p), \quad |\alpha| \le K ,$$

it follows that $H = H^0$.

We should mention that Theorem 1.1 is new even if H is assumed to be holomorphic and M, M' are real-analytic. As an application of Theorem 1.1, it follows for example that if $h^0 : (M, p) \to (M', p')$ is a germ of a smooth CR diffeomorphism with M and M' satisfying the assumptions of Theorem 1.1 and if $h : (M, p) \to (M', p')$ is another smooth CR map whose Taylor polynomial of order K at p agrees with that of h^0, then, necessarily the entire Taylor series at p of h and h^0 are the same.

Theorem 1.2. *Let $M, M' \subset \mathbb{C}^N$ be real-analytic generic submanifolds of the same dimension through p and p', respectively. Assume that M is of finite type at p and that M' is holomorphically nondegenerate at p'. Then, any formal finite map $H : (\mathbb{C}^N, p) \to (\mathbb{C}^N, p')$ sending M into M' is necessarily convergent.*

It is worth mentioning that the holomorphic nondegeneracy condition in Theorems 1.1 and 1.2 is necessary for the conclusions of those theorems to hold (see Section 15 for comments and details).

We say that two germs (M, p) and (M', p') of smooth generic submanifolds in $\mathbb{C}^N$ of the same dimension are *formally equivalent* if there exists a formal invertible map $H : (\mathbb{C}^N, p) \to (\mathbb{C}^N, p')$ sending M into M'. If M and M' are real-analytic and the invertible map H can be chosen to be convergent, we say that (M, p) and (M', p') are *biholomorphically equivalent.* Two formal mappings $H, \check{H} : (\mathbb{C}^N, p) \to (\mathbb{C}^{N'}, p')$ are said to *agree up order κ*, where κ is a positive integer, if their Taylor series at p agree up to order κ. The following theorem may be viewed as an approximation result for formal mappings between real-analytic generic submanifolds by convergent mappings, in the spirit of Artin's approximation theorem [2].

Theorem 1.3. *Let (M, p) and (M', p') be two germs of real-analytic generic submanifolds in $\mathbb{C}^N$ of the same dimension with M of finite type at p. If $H : (\mathbb{C}^N, p) \to (\mathbb{C}^N, p')$ is a formal finite map sending M into M' and if κ is a positive integer, then there exists a convergent map $H^\kappa : (\mathbb{C}^N, p) \to (\mathbb{C}^N, p')$ which sends M into M' and agrees with H up to order κ.*

We should point out that the assumptions of Theorem 1.3 do not imply that the given formal map H is itself convergent. The following, which is an immediate corollary of Theorem 1.3, concerns formal and biholomorphic equivalence.

Corollary 1.4. *Let (M, p) and (M', p') be two germs of real-analytic generic submanifolds in $\mathbb{C}^N$ of the same dimension with M of finite type at p. Then, (M, p) and (M', p') are formally equivalent if and only if they are biholomorphically equivalent.*

A convergent mapping $H : (\mathbb{C}^N, p) \to (\mathbb{C}^{N'}, p')$ is called *algebraic* if each of its components satisfies a non-trivial polynomial equation with holomorphic polynomial coefficients. A germ of a real-analytic generic submanifold (M, p) in $\mathbb{C}^N$ is called *real-algebraic* if it is contained in a real-algebraic subset of $\mathbb{C}^N$ of the same real dimension as that of M. We say that two germs (M, p) and (M', p') of real-algebraic generic submanifolds of $\mathbb{C}^N$ of the same dimension are *algebraically equivalent* if there is a germ of an invertible algebraic map $H : (\mathbb{C}^N, p) \to (\mathbb{C}^N, p')$ sending M into M'. The following theorem can be viewed as an approximation result for local holomorphic mappings between real-algebraic generic submanifolds by algebraic mappings.

Theorem 1.5. *Let $M, M' \subset \mathbb{C}^N$ be two real-algebraic generic submanifolds of the same dimension. Assume that M is connected and of finite type at some point. Let $p \in M$, $p' \in M'$ and $H : (\mathbb{C}^N, p) \to (\mathbb{C}^N, p')$ a germ of a holomorphic map sending M into M' whose Jacobian does not vanish identically. Then, for every positive integer κ, there exists a germ of an algebraic* *holomorphic map* $H^\kappa : (\mathbb{C}^N, p) \to (\mathbb{C}^N, p')$ *which sends* M *into* M' *and agrees with* H *up to order* κ.

One should again note that the assumptions of Theorem 1.5 do not imply that the given holomorphic map H is itself algebraic. Theorem 1.5 immediately implies the following result concerning biholomorphic and algebraic equivalence of generic real-algebraic submanifolds.

Corollary 1.6. *Let* $M, M' \subset \mathbb{C}^N$ *be two real-algebraic generic submanifolds of the same dimension. Assume that* M *is connected and of finite type at some point. Then, for every* $p \in M$ *and every* $p' \in M'$, *the germs* (M, p) *and* (M', p') *are biholomorphically equivalent if and only if they are algebraically equivalent.*

In the case of real-algebraic hypersurfaces, we are able to drop the finite type condition in Corollary 1.6. In fact, we shall prove the following.

Corollary 1.7. *Two germs of real-algebraic hypersurfaces in* $\mathbb{C}^N$ *are biholomorphically equivalent if and only if they are algebraically equivalent.*

For a positive integer k and a point p in $\mathbb{C}^N$, denote by $G^k(\mathbb{C}^N, p)$ the jet group of order k of $\mathbb{C}^N$ at p. An element $j(Z)$ of this group can be viewed as a $\mathbb{C}^N$-valued polynomial in Z of degree at most k, fixing p, and with nonvanishing Jacobian at p. The multiplication of two such elements consist of composition of mappings with the resulting polynomial truncated up to degree k (see e. g., [19]). If (M, p) is a germ of a smooth generic submanifold in $\mathbb{C}^N$, we denote by $\mathcal{F}(M, p)$ the group of formal invertible mappings $H : (\mathbb{C}^N, p) \to (\mathbb{C}^N, p)$ sending M into itself. Moreover, if M is assumed to be real-analytic, then, the subgroup of $\mathcal{F}(M, p)$ consisting of those mappings which are convergent will be denoted by $\mathrm{Aut}(M, p)$, the *stability group* of M at p. For any formal map $H : (\mathbb{C}^N, p) \to (\mathbb{C}^{N'}, p')$, we define its jet $j_p^k H$ to be its Taylor polynomial of degree k at p. If $N = N'$, $p = p'$ and H is invertible, then, $j_p^k H$ may be considered as an element of $G^k(\mathbb{C}^N, p)$. The following corollary is a consequence of Theorem 1.1 and Theorem 1.2.

Corollary 1.8. *Let* $M \subset \mathbb{C}^N$ *be a smooth generic submanifold with* $p \in M$. *If* M *is of finite type and holomorphically nondegenerate at* p, *then there exists a positive integer* K *such that the mapping* $j_p^K : \mathcal{F}(M, p) \to G^K(\mathbb{C}^N, p)$ *is injective. If, in addition,* M *is real-analytic, then,* $\mathcal{F}(M, p) = \mathrm{Aut}(M, p)$.

We shall now briefly mention previous work closely related to the results in this article. For the case of Levi nondegenerate real-analytic hypersurfaces, finite determination by their 2-jets and convergence of formal invertible maps were established in the seminal article of Chern–Moser [16] (see also earlier work of Cartan [15] and Tanaka [29]). The first and third authors, jointly with Ebenfelt [3] recently proved the analogues of Theorems 1.1 and 1.2 under the more restrictive condition that M' is essentially finite at p', rather than just holomorphically nondegenerate. Earlier work by the same authors on these topics appeared in [8, 9, 7]. The second author of this article established Theorem 1.2 (actually the more general version, Theorem 2.6 below) for the case of an invertible map H between real-analytic hypersurfaces [24]. Theorem 2.6 was also proved by the second author for invertible mappings between generic real-analytic submanifolds of any codimension under the additional assumption that one of the manifolds is real-algebraic [25]. In another direction, Ebenfelt [18] obtained results on finite determination (not covered by Theorem 1.1) for smooth CR mappings between smooth hypersurfaces. Lamel [23] proved finite determination and convergence results for certain mappings between generic submanifolds of different dimensions.

It follows from [16] that if two germs of real-analytic Levi nondegenerate hypersurfaces in $\mathbb{C}^N$ are formally equivalent, then they are biholomorphically equivalent. On the other hand, examples due to Moser and Webster [27] show that there are pairs of real-analytic submanifolds which are formally equivalent but are not biholomorphically equivalent. The first and third authors, in joint work with Zaitsev [12] proved that, at "general" points, formal equivalence of real-analytic submanifolds implies biholomorphic equivalence. Corollary 1.4 above establishes this result for points not covered in previous work. A related question for real-algebraic submanifolds is the following, which has been asked in [4]: If two germs of real-algebraic submanifolds are biholomorphically equivalent, are they also algebraically equivalent? It is shown in [13] that at "general" points the answer is positive. Corollaries 1.6 and 1.7 above give further positive results for some classes of submanifolds, including all hypersurfaces. A related question is when a germ of a holomorphic map sending one real-algebraic submanifold into another is itself algebraic. The latter question has a long history. We mention here the work of Webster [30] for invertible maps between Levi nondegenerate hypersurfaces, and, for more recent work, we refer the reader to [20, 10, 26, 31], and [17].

Our approach in the proofs of the results of this article lies in the study of the so-called "reflection ideal" associated to a triple (M, M', H), where M and M' are (germs of) smooth generic submanifolds in $\mathbb{C}^N$ and $\mathbb{C}^{N'}$, respectively, and H is a formal map sending M into M'. Such an ideal lies in the ring of formal power series in $N + N'$ indeterminates. (See Section 2 for precise definitions.) If the source generic submanifold M is of finite type, we establish finite determination of reflection ideals associated to formal mappings (Theorem 2.5 below) with no nondegeneracy condition on the target manifold M'. In fact, we prove such a result in the more general setting of formal manifolds. When the generic submanifolds are real-analytic and the source manifold M is of finite type, we prove (Theorem 2.6 below) that the reflection ideal has a set of convergent generators. If the generic submanifolds M and M' are moreover real-algebraic, the map H is convergent, and the connected source manifold M is of finite type at some point, we prove (Theorem 2.7 below) that the reflection ideal has a set of algebraic generators. An important ingredient for the proofs of the above three theorems is the use of iterated Segre mappings, introduced in [10] (see also [5]), which has already been applied to various mapping problems. Another important tool in the proofs is Artin's approximation theorem [2] and an algebraic version of the latter in [1].

An outline of the organization of this article is as follows. In Section 2 we state the more general results on reflection ideals from which the theorems stated above in this introduction will follow. Sections 4–9 are devoted to preliminaries needed for the proofs of Theorems 2.5, 2.6 and 2.7, which are given in Sections 10–12. Some remarks and open questions are given in Section 15.

2. Manifold ideals and reflection ideals

For $x = (x_1, \dots, x_k) \in \mathbb{C}^k$, we denote by $\mathbb{C}[\![x]\!]$ the ring of formal power series in x and by $\mathbb{C}\{x\}$ the subring of convergent ones. Moreover, we write $\mathcal{A}\{x\} \subset \mathbb{C}\{x\}$ for the subring of algebraic functions (also called Nash functions). If R is any of the three rings defined above and $I \subset R$ is an ideal generated by $s_1(x), \dots, s_d(x)$, we shall use the notation $s(x) = (s_1(x), \dots, s_d(x))$ and write $I = (s(x))$. An ideal $I \subset R$ is called a *manifold ideal* if it has a set of generators with linearly independent differentials at the origin. Observe that any two sets of such generators have the same number of elements. This number is called the *codimension* of I. The following elementary fact, whose proof is left to the reader, will be used implicitly throughout this article.

Lemma 2.1. *Let $I \subset R$ be a manifold ideal of positive codimension d.*

(i) *Any set of d elements of I whose differentials are linearly independent at the origin generate I.*

(ii) *From any set of generators of I, one may extract a subset of d elements with linearly independent differentials at the origin (which generate I by* (i)*).*

If R is $\mathbb{C}\{x\}$ (respectively $\mathcal{A}\{x\}$) and $\{s_1(x), \dots, s_d(x)\}$ is a set of generators of I in R, with d the codimension of I, then the equations $s_1(x) = \cdots = s_d(x) = 0$ define a germ at 0 of a complex-analytic (resp. complex-algebraic) submanifold Σ of codimension d. In general, we say that a manifold ideal $I \subset \mathbb{C}[\![x]\!]$ of codimension d defines a *formal manifold* $\Sigma \subset \mathbb{C}^k$ of dimension $k-d$ and write $I = \mathcal{I}(\Sigma)$. (We should point out that Σ does not necessarily correspond to a subset of $\mathbb{C}^k$ but we shall use the notation $\Sigma \subset \mathbb{C}^k$ for motivation.) If $\Sigma \subset \mathbb{C}^k$ is a formal manifold of dimension l, a parametrization of Σ is a formal mapping $(\mathbb{C}^l, 0) \ni t \to v(t) \in (\mathbb{C}^k, 0)$ such that for any $h \in \mathcal{I}(\Sigma)$, $h \circ v = 0$ and rk $\partial v/\partial t(0) = l$.

If $I \subset \mathbb{C}[\![x]\!]$ is an ideal and $F : (\mathbb{C}^k_x, 0) \to (\mathbb{C}^{k'}_{x'}, 0)$ is a formal map, then the *pushforward* $F_*(I)$ of I is defined to be the ideal in $\mathbb{C}[\![x']\!]$, $x' \in \mathbb{C}^{k'}$,

$$F_*(I) := \left\{h \in \mathbb{C}\left[\!\left[x'\right]\!\right] : h \circ F \in I\right\}. \tag{2.1}$$

If $\Sigma \subset \mathbb{C}^k$ and $\Sigma' \subset \mathbb{C}^{k'}$ are formal manifolds with $I = \mathcal{I}(\Sigma) \subset \mathbb{C}[\![x]\!]$ and $I' = \mathcal{I}(\Sigma') \subset \mathbb{C}[\![x']\!]$, then we say that F sends Σ into Σ' and write $F(\Sigma) \subset \Sigma'$ if $I' \subset F_*(I)$.

For a formal map $F : (\mathbb{C}^k_x, 0) \to (\mathbb{C}^{k'}_{x'}, 0)$, we denote by Rk F the rank of the Jacobian matrix $\partial F/\partial x$ regarded as a $\mathbb{C}[\![x]\!]$-linear mapping $(\mathbb{C}[\![x]\!])^k \to (\mathbb{C}[\![x]\!])^{k'}$. Hence Rk F is the largest integer r such that there is an $r \times r$ minor of the matrix $\partial F/\partial x$ which is not 0 as a formal power series in x. Note that if F is convergent, then Rk F is the generic rank of the map F.

Definition 2.2. Let $\Sigma \subset \mathbb{C}^k$ and $\Sigma' \subset \mathbb{C}^{k'}$ be two formal manifolds of dimension l, l', respectively and $F : (\mathbb{C}^k, 0) \to (\mathbb{C}^{k'}, 0)$ a formal map sending Σ to Σ'. Then, F is said to be (Σ, Σ')-*nondegenerate* if Rk $F \circ v = l'$ for some (and hence for all) parametrization v of Σ.

A formal vector field V in $\mathbb{C}^k$ is a $\mathbb{C}$-linear derivation of $\mathbb{C}[\![x]\!]$ and hence is given by

$$V = \sum_{j=1}^{k} u_j(x) \frac{\partial}{\partial x_j}, \quad u_j(x) \in \mathbb{C}[\![x]\!],\ j = 1, \dots, k\,.$$

The vector field V is called tangent to a formal manifold $\Sigma \subset \mathbb{C}^k$ or, equivalently, to its ideal $\mathcal{I}(\Sigma)$ if and only if $V(f)$ belongs to $\mathcal{I}(\Sigma)$ for every $f \in \mathcal{I}(\Sigma)$.

Definition 2.3. An ideal $I \subset \mathbb{C}[\![x]\!]$ is said to be *convergent* (resp. *algebraic*) if I has a set of convergent (resp. algebraic) generators.

For $(Z, \zeta) \in \mathbb{C}^N \times \mathbb{C}^N$, we define the involution $\sigma : \mathbb{C}[\![Z, \zeta]\!] \to \mathbb{C}[\![Z, \zeta]\!]$ by $\sigma(f)(Z, \zeta) := \bar{f}(\zeta, Z)$, where $\bar{f}$ is the formal power series obtained from f by taking complex conjugates of the coefficients. An ideal $\mathcal{J} \subset \mathbb{C}[\![Z, \zeta]\!]$ is called *real* if $\sigma(f) \in \mathcal{J}$ for every $f \in \mathcal{J}$. Since σ is also an involution when restricted to $\mathbb{C}\{Z, \zeta\}$ or $\mathcal{A}\{Z, \zeta\}$, a similar definition applies for ideals in these rings. A formal manifold $\mathcal{M} \subset \mathbb{C}^N \times \mathbb{C}^N$ is called *real* if its ideal $\mathcal{I}(\mathcal{M})$ is real. A formal real manifold $\mathcal{M} \subset \mathbb{C}^N \times \mathbb{C}^N$ of codimension d is called *generic* if for some (and hence for any) vector of d generators $\rho(Z, \zeta) = (\rho_1(Z, \zeta), \dots, \rho_d(Z, \zeta))$ of $\mathcal{I}(\mathcal{M})$, the rank of the $d \times N$ matrix $\partial\rho/\partial Z(0)$ is d. To motivate this definition, let $M \subset \mathbb{C}^N$ be a smooth generic submanifold of codimension d through the origin with smooth local defining functions $r(Z,\bar Z)=(r_1(Z,\bar Z),\dots,r_d(Z,\bar Z))$ whose Taylor expansions at zero are $\rho(Z,\bar Z)=(\rho_1(Z,\bar Z),\dots,\rho_d(Z,\bar Z))$. Observe that the d vector-valued formal power series $\rho(Z,\zeta)$ generate a real manifold ideal in $\mathbb{C}[\![Z,\zeta]\!]$ whose formal manifold $\mathcal{M}\subset\mathbb{C}^N\times\mathbb{C}^N$ is generic. If, furthermore, M is real-analytic, then $\mathcal{M}\subset\mathbb{C}^N\times\mathbb{C}^N$ is a germ at 0 of a complex submanifold of codimension d, usually referred to as the *complexification* of M.

For a formal generic manifold $\mathcal{M}\subset\mathbb{C}^N\times\mathbb{C}^N$, we define a manifold ideal $\mathcal{I}_0(\mathcal{M})\subset\mathbb{C}[\![Z]\!]$ as the ideal generated by the $h(Z,0)$ for all $h\in\mathcal{I}(\mathcal{M})$. The formal manifold $S_0(\mathcal{M})\subset\mathbb{C}^N$ associated to this ideal is called the *formal Segre variety* of $\mathcal{M}$ at 0. Observe that when $\mathcal{M}$ is the complexification of a real-analytic generic submanifold $M\subset\mathbb{C}^N$ (through 0), then $S_0(\mathcal{M})$ is the usual Segre variety of M at 0.

For a formal map $H:(\mathbb{C}^N_Z,0)\to(\mathbb{C}^{N'}_{Z'},0)$, we define its *complexification* $\mathcal{H}:(\mathbb{C}^N_Z\times\mathbb{C}^N_\zeta,0)\to(\mathbb{C}^{N'}_{Z'}\times\mathbb{C}^{N'}_{\zeta'},0)$ to be the formal map given by

$$\mathcal{H}(Z,\zeta):=(H(Z),\bar H(\zeta))\,. \tag{2.2}$$

In what follows, given $\mathcal{M}\subset\mathbb{C}^N\times\mathbb{C}^N$ and $\mathcal{M}'\subset\mathbb{C}^{N'}\times\mathbb{C}^{N'}$ two formal generic manifolds, we will consider formal maps $H:(\mathbb{C}^N,0)\to(\mathbb{C}^{N'},0)$ such that their complexifications $\mathcal{H}$, as defined by (2.2), send $\mathcal{M}$ into $\mathcal{M}'$. It is easy to check that if H is such a mapping, then H sends the formal Segre variety $S_0(\mathcal{M})$ into the formal Segre variety $S_0(\mathcal{M}')$.

Definition 2.4. Let $\mathcal{M}\subset\mathbb{C}^N\times\mathbb{C}^N$ and $\mathcal{M}'\subset\mathbb{C}^{N'}\times\mathbb{C}^{N'}$ be two formal generic manifolds and $H:(\mathbb{C}^N,0)\to(\mathbb{C}^{N'},0)$ a formal map such that its complexification $\mathcal{H}$ maps $\mathcal{M}$ into $\mathcal{M}'$. The map H is called *not totally degenerate* if H is $(S_0(\mathcal{M}),S_0(\mathcal{M}'))$-nondegenerate as defined in Definition 2.2.

A formal (1,0)-vector field X in $\mathbb{C}^N_Z\times\mathbb{C}^N_\zeta$ is given by

$$X=\sum_{j=1}^N a_j(Z,\zeta)\frac{\partial}{\partial Z_j},\quad a_j(Z,\zeta)\in\mathbb{C}[\![Z,\zeta]\!],\ j=1,\dots,N\,. \tag{2.3}$$

Similarly, a (0,1)-vector field Y in $\mathbb{C}^N_Z\times\mathbb{C}^N_\zeta$ is given by

$$Y=\sum_{j=1}^N b_j(Z,\zeta)\frac{\partial}{\partial \zeta_j},\quad b_j(Z,\zeta)\in\mathbb{C}[\![Z,\zeta]\!],\ j=1,\dots,N\,. \tag{2.4}$$

For a formal generic manifold $\mathcal{M}\subset\mathbb{C}^N\times\mathbb{C}^N$ of codimension d, we denote by $\mathfrak{g}_{\mathcal{M}}$ the Lie algebra generated by the formal (1,0) and (0,1) vector fields tangent to $\mathcal{M}$. The formal generic manifold $\mathcal{M}$ is said to be *of finite type* if the dimension of $\mathfrak{g}_{\mathcal{M}}(0)$ over $\mathbb{C}$ is $2N-d$, where $\mathfrak{g}_{\mathcal{M}}(0)$ is the vector space obtained by evaluating the vector fields in $\mathfrak{g}_{\mathcal{M}}$ at the origin of $\mathbb{C}^{2N}$. Note that if $M\subset\mathbb{C}^N$ is a smooth generic submanifold through the origin, and if $\mathcal{M}\subset\mathbb{C}^N\times\mathbb{C}^N$ is the associated formal manifold as described above, then $\mathcal{M}$ is of finite type if and only if M is of finite type in the sense of Kohn and Bloom–Graham.

Let $H:(\mathbb{C}^N,0)\to(\mathbb{C}^{N'},0)$ be a formal mapping. For an ideal $J\subset\mathbb{C}[\![Z',\zeta']\!]$, $(Z',\zeta')\in\mathbb{C}^{N'}\times\mathbb{C}^{N'}$, we define $J^H\subset\mathbb{C}[\![Z,\zeta']\!]$ to be the ideal generated by the $h(H(Z),\zeta')$ for all $h\in J$ i. e.,

$$J^H:=\left(h\left(H(Z),\zeta'\right):h\in J\right)\subset\mathbb{C}\left[\!\left[Z,\zeta'\right]\!\right]. \tag{2.5}$$

Note that if J is generated by $s(Z', \zeta') = (s_1(Z', \zeta'), \ldots, s_m(Z', \zeta'))$ in $\mathbb{C}[\![Z', \zeta']\!]$, then J^H is generated by the components of $s(H(Z), \zeta')$ in $\mathbb{C}[\![Z, \zeta']\!]$. If $\mathcal{M}' \subset \mathbb{C}^{N'}_{Z'} \times \mathbb{C}^{N'}_{\zeta'}$ is a formal generic submanifold of codimension d', we write for simplicity of notation

$$\mathcal{I}^H := \mathcal{I}\left(\mathcal{M}'\right)^H \subset \mathbb{C}[\![Z, \zeta']\!] \,. \tag{2.6}$$

where we have used the notation given in (2.5). It is easy to see that $\mathcal{I}^H$ is a manifold ideal of codimension d' in $\mathbb{C}[\![Z, \zeta']\!]$. If $\mathcal{M}'$ and H are as above, then we refer to the ideal $\mathcal{I}^H$ as the *reflection ideal of H* (relative to $\mathcal{M}'$). If $(M', 0)$ is a germ of a real-analytic generic submanifold of $\mathbb{C}^{N'}$ and $H : (\mathbb{C}^N, 0) \to (\mathbb{C}^{N'}, 0)$ is a formal map, we again define $\mathcal{I}^H$ by (2.6), where $\mathcal{M}'$ is the complexification of M'. We should observe that if, in addition, H is convergent, then the reflection ideal defines a germ of a complex manifold which coincides with the zero set of the so-called "reflection function" $Z \mapsto \rho'(H(Z), \zeta')$ for an appropriate choice of defining functions $\rho'(Z', \bar{Z}')$ of M' (see e. g., [11, 21, 24]).

Our first result in this section establishes finite determination of reflection ideals for formal mappings H such that their complexifications $\mathcal{H}$ defined in (2.2) send a formal generic manifold $\mathcal{M}$ into $\mathcal{M}'$. Note that in Theorem 2.5, no nondegeneracy condition is imposed on the formal manifold $\mathcal{M}'$.

Theorem 2.5. *Let $\mathcal{M} \subset \mathbb{C}^N \times \mathbb{C}^N$ and $\mathcal{M}' \subset \mathbb{C}^{N'} \times \mathbb{C}^{N'}$ be formal generic manifolds with $\mathcal{M}$ of finite type. Let $H^0 : (\mathbb{C}^N, 0) \to (\mathbb{C}^{N'}, 0)$ be a formal map such that its complexification $\mathcal{H}^0$ sends $\mathcal{M}$ into $\mathcal{M}'$. Assume furthermore that H^0 is not totally degenerate as in Definition 2.4. Then, there exists a positive integer K_0 such that if $H : (\mathbb{C}^N, 0) \to (\mathbb{C}^{N'}, 0)$ is a formal map with $\mathcal{H}(\mathcal{M}) \subset \mathcal{M}'$ and $j_0^{K_0} H = j_0^{K_0} H^0$, it follows that the corresponding reflection ideals defined by* (2.6) *are the same i. e.,*

$$\mathcal{I}^H = \mathcal{I}^{H^0} \,. \tag{2.7}$$

If $(M, 0)$ and $(M', 0)$ are germs of real-analytic generic submanifolds in $\mathbb{C}^N$ and $\mathbb{C}^{N'}$, respectively, and $H : (\mathbb{C}^N, 0) \to (\mathbb{C}^{N'}, 0)$ is a formal mapping sending M into M' as defined in Section 1, then its complexification $\mathcal{H}$ sends $\mathcal{M}$ into $\mathcal{M}'$, where $\mathcal{M}$ and $\mathcal{M}'$ are the complexifications of M and M', respectively. The second main result of this section establishes convergence of reflection ideals for formal mappings between real-analytic generic submanifolds, with no nondegeneracy condition imposed on the target manifold M'.

Theorem 2.6. *Let $(M, 0)$ and $(M', 0)$ be germs of real-analytic generic submanifolds in $\mathbb{C}^N$ and $\mathbb{C}^{N'}$, respectively and $H : (\mathbb{C}^N, 0) \to (\mathbb{C}^{N'}, 0)$ a formal mapping sending M into M'. Assume that M is of finite type at* 0 *and H is not totally degenerate. Then, the reflection ideal $\mathcal{I}^H$, as defined by* (2.6), *is convergent.*

The last result of this section establishes algebraicity of reflection ideals for local holomorphic mappings between real-algebraic generic submanifolds, with no nondegeneracy condition imposed on the target manifold M'.

Theorem 2.7. *Let $M, M' \subset \mathbb{C}^N$ be real-algebraic generic submanifolds of codimension d through the origin and $H : (\mathbb{C}^N, 0) \to (\mathbb{C}^N, 0)$ be a germ of a holomorphic map sending M into M'. Assume that the Jacobian of H does not vanish identically and that there is no germ of a nonconstant holomorphic function $h : (\mathbb{C}^N, 0) \to \mathbb{C}$ with $h(M) \subset \mathbb{R}$. Then, the reflection ideal $\mathcal{I}^H$, as defined by* (2.6), *is algebraic.*

In view of Proposition 6.1 (iii) below, Theorem 2.7 in the case where M and M' are real-algebraic hypersurfaces in $\mathbb{C}^N$, is contained in [26].

Remark 2.8. Even if all the assumptions of Theorem 2.6 are satisfied, the fact that the reflection ideal $\mathcal{I}^H$ is convergent does not imply that the formal map H is convergent. For example, let $M = M'$ be the real-algebraic hypersurface of finite type through the origin in $\mathbb{C}^3$ given by

$$\mathrm{Im}\ Z_3 = |Z_1 Z_2|^2 .$$

For any nonconvergent formal power series $h(Z) = h(Z_1, Z_2, Z_3)$ vanishing at the origin, let $H : (\mathbb{C}^3, 0) \to (\mathbb{C}^3, 0)$ be the formal invertible map given by

$$H(Z_1, Z_2, Z_3) = \left(Z_1 e^{h(Z)}, Z_2 e^{-h(Z)}, Z_3 \right) .$$

In this example, the formal map H sends M into itself and is not convergent, but one can easily check that its reflection ideal $\mathcal{I}^H$ is convergent. (This fact also follows from Theorem 2.6.) Similar considerations can be made in the algebraic case relative to Theorem 2.7. Proposition 2.12 below gives an additional condition on M' which guarantees that the convergence of $\mathcal{I}^H$ implies that H is convergent.

The following proposition, which justifies the notion of convergent reflection ideals introduced here, will be used for the proofs of Theorems 1.3 and 1.5.

Proposition 2.9. *Let $(M', 0)$ be a germ of a generic real-analytic (resp. real-algebraic) submanifold of codimension d' in $\mathbb{C}^{N'}$ and $H : (\mathbb{C}^N, 0) \to (\mathbb{C}^{N'}, 0)$ a formal map. Then, the reflection ideal $\mathcal{I}^H$ is convergent (resp. algebraic) if and only if there exists a convergent (resp. algebraic) map $\check{H} : (\mathbb{C}^N, 0) \to (\mathbb{C}^{N'}, 0)$ such that $\mathcal{I}^H = \mathcal{I}^{\check{H}}$. More precisely, if $\mathcal{I}^H$ is convergent (resp. algebraic), then for any positive integer κ, there exists a convergent (resp. algebraic) map $H^\kappa : (\mathbb{C}^N, 0) \to (\mathbb{C}^{N'}, 0)$ agreeing up to order κ with H such that $\mathcal{I}^H = \mathcal{I}^{H^\kappa}$.*

If $\mathcal{M} \subset \mathbb{C}^N \times \mathbb{C}^N$ is a formal generic manifold, we say that $\mathcal{M}$ is *holomorphically nondegenerate* if there is no nontrivial (1,0) vector field of the form (2.3) tangent to $\mathcal{M}$ with coefficients $a_j(Z, \zeta) = a_j(Z)$ independent of ζ for $j = 1, \dots, N$. Note that if $(M, 0)$ is a germ of a smooth generic submanifold in $\mathbb{C}^N$, then M is holomorphically nondegenerate in the sense defined in Section 1 if and only if its associated formal generic manifold $\mathcal{M}$ is holomorphically nondegenerate as defined here. If $\mathcal{M}$ is the complexification of a germ of a real-analytic generic submanifold $(M, 0)$ in $\mathbb{C}^N$, then $\mathcal{M}$ is holomorphically nondegenerate as defined here if and only if there is no germ of a nontrivial (1,0) vector field of the form (2.3) tangent to $\mathcal{M}$ with convergent coefficients $a_j(Z, \zeta) = a_j(Z)$ independent of ζ for $j = 1, \dots, N$ (see e. g., [6]).

Theorem 2.5 will be used in conjunction with the following finite determination result to prove Theorem 1.1.

Proposition 2.10. *Let $\mathcal{M}' \subset \mathbb{C}^{N'} \times \mathbb{C}^{N'}$ be a holomorphically nondegenerate formal generic manifold and $H^0 : (\mathbb{C}^N, 0) \to (\mathbb{C}^{N'}, 0)$ a formal map with* Rk $H^0 = N'$. *Then, there exists a positive integer K such that if $H : (\mathbb{C}^N, 0) \to (\mathbb{C}^{N'}, 0)$ is a formal map with $j_0^K H = j_0^K H^0$ and $\mathcal{I}^H = \mathcal{I}^{H^0}$* [*as defined in* (2.6)], *it follows that $H = H^0$.*

In the case of a real-analytic generic submanifold and holomorphic mappings, we have the following geometric interpretation of the equality (2.7) of reflection ideals. In view of Proposition 2.11 below, Theorem 2.5 can, then, be seen as a finite determination result for Segre varieties.

Proposition 2.11. *Let $(M', 0)$ be a germ of a real-analytic generic submanifold in $\mathbb{C}^{N'}$ with real-analytic local defining functions $r'(Z', \bar{Z}')$. Assume that $H, H^0 : (\mathbb{C}^N, 0) \to (\mathbb{C}^{N'}, 0)$ are germs of holomorphic mappings. Then, the following two conditions are equivalent:*

(i) $\mathcal{I}^H = \mathcal{I}^{H^0}$, *where the ideals* $\mathcal{I}^H$ *and* $\mathcal{I}^{H^0}$ *are defined by* (2.6).

(ii) *For Z near the origin, the Segre varieties of M' relative to the points $H(Z)$ and $H^0(Z)$ are the same. More precisely, there exists open neighborhoods of 0, U and U' in $\mathbb{C}^N$ and $\mathbb{C}^{N'}$, respectively such that for all $Z \in U$,*

$$S_{H(Z)} = S_{H^0(Z)} \ , \tag{2.8}$$

where $S_{H(Z)} = \{Z' \in U' : r'(Z', \overline{H(Z)}) = 0\}$, *with a similar definition for* $S_{H^0(Z)}$.

Proposition 2.11 will not be used in the remainder of the article and its proof is left to the reader. The last result of this section connects the convergence of the reflection ideal $\mathcal{I}^H$ to the convergence of the mapping H.

Proposition 2.12. *Let $(M', 0)$ be a germ of a generic real-analytic holomorphically nondegenerate submanifold of codimension d' in $\mathbb{C}^{N'}$. If $H : (\mathbb{C}^N, 0) \to (\mathbb{C}^{N'}, 0)$ is a formal map with* $\operatorname{Rk} H = N'$ *such that its reflection ideal $\mathcal{I}^H$, as defined by* (2.6), *is convergent, then H is convergent.*

Remark 2.13. We should point out that a statement similar to Proposition 2.12 holds in the algebraic case. Indeed, if $(M', 0)$ is a germ of a generic real-algebraic holomorphically nondegenerate submanifold of codimension d' in $\mathbb{C}^{N'}$ and if $H : (\mathbb{C}^N, 0) \to (\mathbb{C}^{N'}, 0)$ is a formal map with $\operatorname{Rk} H = N'$ such that its reflection ideal $\mathcal{I}^H$ is algebraic, then H is algebraic. This fact will not be used in this article.

The proofs of Propositions 2.9, 2.10, and 2.12 will be given in Section 13.

3. Further results on finite determination, convergence, and approximation of mappings

The following finite determination result, which is a generalization of Theorem 1.1, will be a consequence of Theorem 2.5 and Proposition 2.10.

Theorem 3.1. *Let $\mathcal{M} \subset \mathbb{C}^N \times \mathbb{C}^N$ and $\mathcal{M}' \subset \mathbb{C}^{N'} \times \mathbb{C}^{N'}$ be formal generic manifolds with $\mathcal{M}$ of finite type and $\mathcal{M}'$ holomorphically nondegenerate. Let $H^0 : (\mathbb{C}^N, 0) \to (\mathbb{C}^{N'}, 0)$ be a formal map such that its complexification $\mathcal{H}^0$ sends $\mathcal{M}$ into $\mathcal{M}'$. Assume furthermore that H^0 is not totally degenerate as in Definition 2.4 and that* $\operatorname{Rk} H^0 = N'$. *Then, there exists a positive integer K such that if $H : (\mathbb{C}^N, 0) \to (\mathbb{C}^{N'}, 0)$ is a formal map with $\mathcal{H}(\mathcal{M}) \subset \mathcal{M}'$ and $j_0^K H = j_0^K H^0$, it follows that $H = H^0$.*

Similarly, the following convergence result, which is a generalization of Theorem 1.2, will be a consequence of Theorem 2.6 and Proposition 2.12.

Theorem 3.2. *Let $(M, 0)$ and $(M', 0)$ be germs of real-analytic generic submanifolds in $\mathbb{C}^N$ and $\mathbb{C}^{N'}$, respectively and $H : (\mathbb{C}^N, 0) \to (\mathbb{C}^{N'}, 0)$ a formal mapping sending M into M'. Assume that M is of finite type at 0 and that M' is holomorphically nondegenerate at 0. If H is not totally degenerate and* $\operatorname{Rk} H = N'$, *then H is convergent.*

Remark 3.3. We should point out that the assumptions of Theorem 3.2 are less restrictive than those of Theorem 1.2, even in the case where M and M' are real-analytic hypersurfaces in the same space $\mathbb{C}^N$. (The same can also be said about Theorems 3.1 and 1.1.) For instance, given a nontrivial convergent power series $h : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$, consider the following hypersurfaces in $\mathbb{C}^3$:

$$\begin{aligned} M &:= \left\{ Z \in \mathbb{C}^3 : \operatorname{Im} Z_3 = \left| Z_1^2 Z_2 \right|^2 + |h(Z_1)|^2 \right\} , \\ M' &:= \left\{ Z' \in \mathbb{C}^3 : \operatorname{Im} Z_3' = \left| Z_1' Z_2' \right|^2 + \left| h\left(Z_1'\right) \right|^2 \right\} . \end{aligned} \tag{3.1}$$

Observe that the convergent mapping $(\mathbb{C}^3, 0) \ni Z \mapsto H(Z) := (Z_1, Z_1 Z_2, Z_3) \in (\mathbb{C}^3, 0)$ sends M into M'. Moreover, M and M' are of finite type and holomorphically nondegenerate at the origin. Note also that H is not totally degenerate and $\operatorname{Rk} H = 3$ but H is not finite. We should point out that the convergence of formal mappings between M and M' satisfying the latter conditions follows from Theorem 3.2, but does not follow from Theorem 1.2 nor from previously known results. (Indeed, since M and M' are not essentially finite at the origin, the result in [3] does not apply, nor does the one in [25] if the function h is chosen not be algebraic.)

The following approximation result generalizes Theorem 1.3.

Theorem 3.4. *Let $(M, 0)$ and $(M', 0)$ be two germs of real-analytic generic submanifolds in $\mathbb{C}^N$ and $\mathbb{C}^{N'}$, respectively, with M of finite type at 0. If $H : (\mathbb{C}^N, 0) \to (\mathbb{C}^{N'}, 0)$ is a not totally degenerate formal map sending M into M' and if κ is a positive integer, then there exists a convergent map $H^\kappa : (\mathbb{C}^N, 0) \to (\mathbb{C}^{N'}, 0)$ which sends M into M' and agrees with H up to order κ.*

4. Ideals in jet spaces

Given nonnegative integers l, k, r, with $k, r \geq 1$, we denote by $J_0^l(\mathbb{C}^k, \mathbb{C}^r)$ the jet space at the origin of order l of holomorphic mappings from $\mathbb{C}^k$ to $\mathbb{C}^r$. An element j of $J_0^l(\mathbb{C}^k, \mathbb{C}^r)$ can be written as a polynomial mapping

$$j(X) = \sum_{\alpha \in \mathbb{N}^k,\ 0 \leq |\alpha| \leq l} \frac{\Lambda_\alpha}{\alpha!} X^\alpha, \quad \Lambda_\alpha \in \mathbb{C}^r . \tag{4.1}$$

We think of the coefficients $\Lambda := (\Lambda_\alpha)_{0 \leq |\alpha| \leq l}$, $\Lambda_\alpha \in \mathbb{C}^r$, as linear coordinates in the finite dimensional vector space $J_0^l(\mathbb{C}^k, \mathbb{C}^r)$ and we identify j with Λ. We write $\Lambda_\alpha = (\Lambda_{\alpha,i})_{1 \leq i \leq r}$ for any $\alpha \in \mathbb{N}^k$, $|\alpha| \leq l$. We also use the splitting

$$\Lambda = \left(\Lambda_0, \hat{\Lambda}\right), \quad \hat{\Lambda} = (\Lambda_\alpha)_{1 \leq |\alpha| \leq l} . \tag{4.2}$$

Using the coordinates Λ, we identify $J_0^l(\mathbb{C}^k, \mathbb{C}^r)$ with $\mathbb{C}^m_\Lambda$ where $m = \dim_{\mathbb{C}} J_0^l(\mathbb{C}^k, \mathbb{C}^r)$. For a formal map $F : (\mathbb{C}^k, 0) \to (\mathbb{C}^r, 0)$, we write $j_x^l F$ and $\hat{j}_x^l F$ for the vectors of formal series

$$j_x^l F := \left(\partial^\nu F(x)\right)_{0 \leq |\nu| \leq l}, \quad \hat{j}_x^l F := \left(\partial^\nu F(x)\right)_{1 \leq |\nu| \leq l} . \tag{4.3}$$

Here, for $\nu \in \mathbb{N}^k$, $\partial^\nu F(x) \in (\mathbb{C}[\![x]\!])^r$ and $x \in \mathbb{C}^k$. If s is another positive integer and $\eta : (J_0^l(\mathbb{C}^k, \mathbb{C}^r), 0) \to (J_0^l(\mathbb{C}^k, \mathbb{C}^s), 0)$ is a formal map, we take coordinates $\Lambda = (\Lambda_\nu)_{|\nu| \leq l}$ and $\Lambda' = (\Lambda'_\nu)_{|\nu| \leq l}$ for $J_0^l(\mathbb{C}^k, \mathbb{C}^r)$ and $J_0^l(\mathbb{C}^k, \mathbb{C}^s)$, respectively. Here, $\Lambda_\nu \in \mathbb{C}^r$, $\Lambda'_\nu \in \mathbb{C}^s$. We then write $\eta_\nu = \Lambda'_\nu \circ \eta$; that is, the map η is given by $\Lambda' = \eta(\Lambda)$. Hence, for $\nu \in \mathbb{N}^k$, $|\nu| \le l$, η_ν is the ν-th component of η, i. e.,

$$\eta_\nu : \left(J_0^l\left(\mathbb{C}^k, \mathbb{C}^r\right), 0\right) \to \left(\mathbb{C}^s, 0\right), \quad \eta = (\eta_\nu)_{|\nu|\le l} \, . \tag{4.4}$$

If R is a ring and $T \in \mathbb{C}^q$, as usual we denote by $R[T]$ the ring of polynomials in T with coefficients in R. If $\Lambda = (\Lambda_0, \hat\Lambda)$ are coordinates in $J_0^l(\mathbb{C}^k, \mathbb{C}^r)$ as in (4.1) and (4.2), the subring $\mathbb{C}[\![\Lambda_0]\!][\hat\Lambda] := (\mathbb{C}[\![\Lambda_0]\!])[\hat\Lambda]$ of the ring $\mathbb{C}[\![\Lambda]\!]$ will play a crucial role in the rest of this article. For instance, if $u \in \mathbb{C}[\![\Lambda_0]\!][\hat\Lambda]$ and $F : (\mathbb{C}^k, 0) \to (\mathbb{C}^r, 0)$ is a formal map, then $u(j_x^l F)$ is a well defined formal power series in $\mathbb{C}[\![x]\!]$ while for a general $u \in \mathbb{C}[\![\Lambda]\!]$, one cannot define it.[1]

We have the following uniqueness result.

Lemma 4.1. *If $\Lambda = (\Lambda_0, \hat\Lambda)$ are coordinates in $J_0^l(\mathbb{C}^k, \mathbb{C}^r)$ as in* (4.1) *and* (4.2) *and if $u \in \mathbb{C}[\![\Lambda_0]\!][\hat\Lambda]$ is a formal power series satisfying*

$$u\left(j_x^l F\right) = 0, \quad \textit{in } \mathbb{C}[\![x]\!] \, , \tag{4.5}$$

for any formal power series mapping $F : (\mathbb{C}^k, 0) \to (\mathbb{C}^r, 0)$, then $u = 0$ (in $\mathbb{C}[\![\Lambda_0]\!][\hat\Lambda]$).

Proof. We shall define a polynomial map

$$\varphi : \left(J_0^l\left(\mathbb{C}^k, \mathbb{C}^r\right) \times \mathbb{C}^k, 0\right) \to \left(J_0^l\left(\mathbb{C}^k, \mathbb{C}^r\right), 0\right) \tag{4.6}$$

as follows. If $A = (A_\nu)_{|\nu|\le l}$, $\nu \in \mathbb{N}^k$, $A_\nu \in \mathbb{C}^r$, are coordinates on the source jet space $J_0^l(\mathbb{C}^k, \mathbb{C}^r)$ as in (4.1), $x = (x_1, \ldots, x_k) \in \mathbb{C}^k$ and $\Lambda = (\Lambda_\alpha)_{|\alpha|\le l}$ are coordinates on the target jet space $J_0^l(\mathbb{C}^k, \mathbb{C}^r)$, then φ is defined by

$$\Lambda = \varphi(A, x) := \left(\partial_x^\alpha\left(x_1 \sum_{0\le|\nu|\le l} A_\nu x^\nu\right)\right)_{0\le|\alpha|\le l} \, . \tag{4.7}$$

We claim that the generic rank of φ, Rk φ, is equal to m, the dimension of $J_0^l(\mathbb{C}^k, \mathbb{C}^r)$ over $\mathbb{C}$. For this, let

$$\tilde\varphi(A, x) := \left(x_1 \partial_x^\alpha\left(\sum_{0\le|\nu|\le l} A_\nu x^\nu\right)\right)_{0\le|\alpha|\le l} \, .$$

First note that Rk φ is greater or equal to the generic rank (in x) of the $m \times m$ matrix $\Delta(x) := \dfrac{\partial\varphi}{\partial A}(A, x)$. Moreover, it is not difficult to see that the generic rank of the matrix $\Delta(x)$ is the same as that of the matrix $\tilde\Delta(x) := \dfrac{\partial\tilde\varphi}{\partial A}(A, x)$. Since for $x_1 \ne 0$, the rank of $\tilde\Delta(x)$ is clearly m, it follows that Rk $\varphi = m$. This proves the claim.

[1]Indeed, given two formal power series $f(x) \in \mathbb{C}[\![x - x_0]\!]$ and $g(y) \in \mathbb{C}[\![y - y_0]\!]$, the composition $g \circ f$ as a power series in x is well defined provided $f(x_0) = y_0$. However, if g is a polynomial then the composition is always well defined without the assumption $f(x_0) = y_0$.

Given a formal power series $v \in \mathbb{C}[\![\Lambda]\!]$, since $\varphi(0) = 0$, we can consider the composition $(v \circ \varphi)(A, x)$ as a formal power series in $\mathbb{C}[\![A, x]\!]$. We write

$$(v \circ \varphi)(A, x) = \sum_{\beta \in \mathbb{N}^k} v_\beta(A) x^\beta, \quad v_\beta \in \mathbb{C}[\![A]\!] \,. \tag{4.8}$$

Observe that if v is in the subring $\mathbb{C}[\![\Lambda_0]\!][\hat{\Lambda}] \subset \mathbb{C}[\![\Lambda]\!]$ then for each $\beta \in \mathbb{N}^k$, v_β is a polynomial in A, i. e., $(v \circ \varphi)(A, x) \in \mathbb{C}[A][\![x]\!]$. Let u be as in Lemma 4.1 satisfying (4.5). For any vector $a = (a_\nu)_{|\nu| \le l} \in \mathbb{C}^m \cong J_0^l(\mathbb{C}^k, \mathbb{C}^r)$, by (4.5) with $F(x) = x_1 \sum_{0 \le |\nu| \le l} a_\nu x^\nu$, we obtain

$$u\left(j_x^l F\right) = u(\varphi(a, x)) = \sum_{\beta \in \mathbb{N}^k} u_\beta(a) x^\beta = 0, \text{ in } \mathbb{C}[\![x]\!] \,. \tag{4.9}$$

As a consequence, we have $u_\beta(a) =$ for any $\beta \in \mathbb{N}^k$ and any vector a in $\mathbb{C}^m$. Since u_β is a polynomial, it follows that $u_\beta \equiv 0$ and hence the formal power series $(u \circ \varphi)(A, x)$ is zero in $\mathbb{C}[A][\![x]\!] \subset \mathbb{C}[\![A, x]\!]$. To conclude that u is identically zero, by e. g., Proposition 5.3.5 of [6], it suffices to use the fact that Rk $\varphi = m$. This completes the proof of Lemma 4.1. □

Proposition 4.2. *Let l, r, s be nonnegative integers with $r, s \ge 1$, and let $\phi : (\mathbb{C}^r, 0) \to (\mathbb{C}^s, 0)$ be a formal map. Then, there exists a unique formal map*

$$\phi^{(l)} : \left(J_0^l\left(\mathbb{C}^k, \mathbb{C}^r\right), 0\right) \to \left(J_0^l\left(\mathbb{C}^k, \mathbb{C}^s\right), 0\right) \tag{4.10}$$

whose components are in $\mathbb{C}[\![\Lambda_0]\!][\hat{\Lambda}]$, with $\Lambda = (\Lambda_0, \hat{\Lambda})$ the coordinates of $J_0^l(\mathbb{C}^k, \mathbb{C}^r)$ introduced in (4.1) *and* (4.2)*, such that for any formal map $F : (\mathbb{C}^k, 0) \to (\mathbb{C}^r, 0)$*

$$j_x^l(\phi \circ F) = \phi^{(l)}\left(j_x^l F\right) \,. \tag{4.11}$$

Moreover, if we write $\phi^{(l)}(\Lambda) = (\phi_\nu^{(l)}(\Lambda))_{\nu \in \mathbb{N}^k, |\nu| \le l}$, then for each ν, $\phi_\nu^{(l)}(\Lambda)$ depends only on $(\Lambda_\alpha)_{\alpha \le \nu}$. Finally, if $r = s$ and $\phi : (\mathbb{C}^r, 0) \to (\mathbb{C}^r, 0)$ is invertible, then so is $\phi^{(l)} : (J_0^l(\mathbb{C}^k, \mathbb{C}^r), 0) \to (J_0^l(\mathbb{C}^k, \mathbb{C}^r), 0)$ and $(\phi^{(l)})^{-1} = (\phi^{-1})^{(l)}$.

Proof. The existence of the map $\phi^{(l)}$ and its properties follow easily from the chain rule. The uniqueness of such a map is a consequence of Lemma 4.1. The proof of the last statement of the proposition is straightforward and left to the reader. □

Remark 4.3. Let ϕ and $\phi^{(l)}$ be as in Proposition 4.2. It follows from (4.11) and the other properties of $\phi^{(l)}$ that for any formal map $G : (\mathbb{C}_x^k \times \mathbb{C}_t^q, 0) \to (\mathbb{C}_y^r, 0)$, we have the equality of vector valued formal power series in $\mathbb{C}[\![x, t]\!]$

$$j_x^l(\phi(G(x, t))) = \phi^{(l)}\left(j_x^l G(x, t)\right) \,. \tag{4.12}$$

Here, as in (4.3), $j_x^l G(x, t) = (\partial_x^\nu G(x, t))_{|\nu| \le l}$. Hence, (4.11) appears as a special case of (4.12), without an additional formal parameter t.

For any ideal $I \subset \mathbb{C}[\![y]\!]$, $y \in \mathbb{C}^r$, and any nonnegative integers k, l, with $k \ge 1$, we define an ideal $I^{(l)} \subset \mathbb{C}[\![\Lambda_0]\!][\hat{\Lambda}]$, where $\Lambda = (\Lambda_0, \hat{\Lambda})$ are coordinates on $J_0^l(\mathbb{C}^k, \mathbb{C}^r)$ as in (4.1) and (4.2), as follows:

$$\begin{aligned} I^{(l)} := \Big\{ h \in \mathbb{C}[\![\Lambda_0]\!]\big[\hat{\Lambda}\big] : h\big(j_x^l F\big) = 0 \text{ for all } F : \big(\mathbb{C}_x^k, 0\big) \to \big(\mathbb{C}_y^r, 0\big) \\ \text{such that } u \circ F = 0, \quad \text{for all } u \in I \Big\} \,. \end{aligned} \tag{4.13}$$

We have the following proposition.

Proposition 4.4. *If $I \subset \mathbb{C}[\![y]\!]$ is a manifold ideal of codimension d, then the ideal $I^{(l)} \subset \mathbb{C}[\![\Lambda_0]\!][\hat{\Lambda}]$ defined by (4.13) is also a manifold ideal. Moreover, if the manifold ideal I is generated by $\rho_1(y), \ldots, \rho_d(y)$ in $\mathbb{C}[\![y]\!]$, then the ideal $I^{(l)}$ in $\mathbb{C}[\![\Lambda_0]\!][\hat{\Lambda}]$ is generated by the components of $\rho_1^{(l)}(\Lambda), \ldots, \rho_d^{(l)}(\Lambda)$, where $\rho_j^{(l)}$ is given by Proposition 4.2.*

Proof. Recall by Proposition 4.2 that an invertible formal map $\psi : (\mathbb{C}^r_y, 0) \to (\mathbb{C}^r_{y'}, 0)$ induces a formal invertible map $\psi^{(l)} : (J_0^l(\mathbb{C}^k, \mathbb{C}^r), 0) \to (J_0^l(\mathbb{C}^k, \mathbb{C}^r), 0)$. We leave it to the reader to check that the equality

$$\left(\psi^{(l)}\right)_*(I^{(l)}) = (\psi_*(I))^{(l)} \tag{4.14}$$

follows from Proposition 4.2, where the pushforward of an ideal is given by (2.1). If $\rho_1(y), \ldots, \rho_d(y)$ are generators of the manifold ideal I, we may choose a formal invertible map $\psi : (\mathbb{C}^r_y, 0) \to (\mathbb{C}^r_{y'}, 0)$ such that $y'_j = \psi_j(y) = \rho_j(y)$ for $j = 1, \ldots, d$ and hence the manifold ideal $\psi_*(I) \subset \mathbb{C}[\![y']\!]$ is generated by the coordinate functions $y'_1, \ldots, y'_d$. We take $\Lambda = (\Lambda_0, \hat{\Lambda})$ for coordinates in the source jet space $J_0^l(\mathbb{C}^k, \mathbb{C}^r)$ and $\Lambda' = (\Lambda'_0, \hat{\Lambda}')$ for coordinates in the target one, as in (4.1) and (4.2). It is, then, easy to check that the ideal $(\psi_*(I))^{(l)} \subset \mathbb{C}[\![\Lambda'_0]\!][\hat{\Lambda}']$ is the manifold ideal generated by the coordinate functions $(\Lambda'_{\alpha,i})$ for $0 \le |\alpha| \le l$ and $i = 1, \ldots, d$. It follows from (4.14) that $I^{(l)} \subset \mathbb{C}[\![\Lambda_0]\!][\hat{\Lambda}]$ is a manifold ideal and is generated by the $\psi^{(l)}_{\alpha,i}(\Lambda)$ for $0 \le |\alpha| \le l$ and $i = 1, \ldots, d$. Since by construction $\psi_i(y) = \rho_i(y)$, $i = 1, \ldots, d$, the last part of the proposition follows. □

5. Generators of the ideal $\mathcal{I}(\mathcal{M}')^{(l)}$

In this section, we consider a formal generic manifold $\mathcal{M}' \subset \mathbb{C}^{N'}_{Z'} \times \mathbb{C}^{N'}_{\zeta'}$ of codimension d'. Let

$$\rho' : \left(\mathbb{C}^{N'}_{Z'} \times \mathbb{C}^{N'}_{\zeta'}, 0\right) \to \left(\mathbb{C}^{d'}, 0\right)$$

be a formal mapping such that $\mathcal{I}(\mathcal{M}') = (\rho'(Z', \zeta')) = (\rho'_1(Z', \zeta'), \ldots, \rho'_{d'}(Z', \zeta'))$ in $\mathbb{C}[\![Z', \zeta']\!]$. We define

$$\tilde{\rho}'\left(Z', \zeta'\right) := \overline{\rho'}\left(\zeta', Z'\right) . \tag{5.1}$$

Since $\mathcal{I}(\mathcal{M}')$ is real, the ideal $\mathcal{I}(\mathcal{M}') \subset \mathbb{C}[\![Z', \zeta']\!]$ is also generated by the components of $\tilde{\rho}'(Z', \zeta')$.

Given a formal map $H : (\mathbb{C}^N_Z, 0) \to (\mathbb{C}^{N'}_{Z'}, 0)$, we define two formal mappings ${}^H\rho' : (\mathbb{C}^N_Z \times \mathbb{C}^{N'}_{\zeta'}, 0) \to (\mathbb{C}^{d'}, 0)$ and $\rho'^H : (\mathbb{C}^{N'}_{Z'} \times \mathbb{C}^N_{\zeta}, 0) \to (\mathbb{C}^{d'}, 0)$ as follows

$${}^H\rho'\left(Z, \zeta'\right) := \rho'\left(H(Z), \zeta'\right), \quad \rho'^H\left(Z', \zeta\right) := \rho'\left(Z', \overline{H}(\zeta)\right) . \tag{5.2}$$

Similarly, we define

$${}^H\tilde{\rho}'\left(Z, \zeta'\right) := \tilde{\rho}'\left(H(Z), \zeta'\right), \quad \tilde{\rho}'^H\left(Z', \zeta\right) := \tilde{\rho}'\left(Z', \overline{H}(\zeta)\right) . \tag{5.3}$$

Note that by the reality condition, we have

$${}^H\tilde{\rho}'\left(Z, \zeta'\right) = \overline{\rho'^H}\left(\zeta', Z\right) . \tag{5.4}$$

Observe also that the components of ${}^{H}\rho'(Z,\zeta')$ generate the reflection ideal $\mathcal{I}^{H}\subset\mathbb{C}[\![Z,\zeta']\!]$ as defined by (2.6).

Throughout the rest of this section and Sections 6–9, we fix a nonnegative integer l. Since

$$\rho',\ \tilde\rho' : \left(\mathbb{C}^{N'}_{Z'}\times\mathbb{C}^{N'}_{\zeta'},0\right)\to\left(\mathbb{C}^{d'},0\right)$$

are formal mappings, by Proposition 4.2 there exist unique formal mappings

$$\rho'^{(l)},\ \tilde\rho'^{(l)} : \left(J^l_0\left(\mathbb{C}^N,\mathbb{C}^{N'}\times\mathbb{C}^{N'}\right),0\right)\to\left(J^l_0\left(\mathbb{C}^N,\mathbb{C}^{d'}\right),0\right) \tag{5.5}$$

such that for every formal mapping

$$F=\left(F^1,F^2\right):\left(\mathbb{C}^N_Z,0\right)\to\left(\mathbb{C}^{N'}_{Z'}\times\mathbb{C}^{N'}_{\zeta'},0\right)\ , \tag{5.6}$$

one has

$$j^l_Z\left(\rho'\circ F\right)=\rho'^{(l)}\left(j^l_Z F\right),\quad j^l_Z\left(\tilde\rho'\circ F\right)=\tilde\rho'^{(l)}\left(j^l_Z F\right). \tag{5.7}$$

If $\Lambda=(\Lambda_\alpha)_{|\alpha|\le l}$, $\alpha\in\mathbb{N}^N$, are the coordinates given by (4.1) on the jet space

$$J^l_0\left(\mathbb{C}^N_Z,\mathbb{C}^{N'}_{Z'}\times\mathbb{C}^{N'}_{\zeta'}\right)=J^l_0\left(\mathbb{C}^N_Z,\mathbb{C}^{N'}_{Z'}\right)\times J^l_0\left(\mathbb{C}^N_Z,\mathbb{C}^{N'}_{\zeta'}\right)\ , \tag{5.8}$$

then, we write $\Lambda=(\Lambda^1,\Lambda^2)$ according to the splitting (5.8). Thus, we have $\Lambda^i=(\Lambda^i_\alpha)_{|\alpha|\le l}$, $i=1,2$, $\alpha\in\mathbb{N}^N$. As in (4.2), we continue to use the splitting $\Lambda^i=(\Lambda^i_0,\hat\Lambda^i)$ with $\hat\Lambda^i=(\Lambda^i_\alpha)_{1\le|\alpha|\le l}$, $i=1,2$. Since $I(\mathcal{M}')$ is generated either by the components of $\rho'(Z',\zeta')$ or by those of $\tilde\rho'(Z',\zeta')$, it follows from Proposition 4.4 that the ideal $I(\mathcal{M}')^{(l)}$ in $\mathbb{C}[\![\Lambda_0]\!][\hat\Lambda]$ is generated either by the components of $\rho'^{(l)}(\Lambda)$ or by the components of $\tilde\rho'^{(l)}(\Lambda)$.

We shall now give a more explicit expression for $\rho'^{(l)}(\Lambda)$. As in (4.4), we write $\rho'^{(l)}=(\rho'^{(l)}_\nu)_{|\nu|\le l}$ and $\tilde\rho'^{(l)}=(\tilde\rho'^{(l)}_\nu)_{|\nu|\le l}$. For any formal mapping $F(Z)$ as in (5.6), by (5.7), the chain rule, and (5.2), one has for any $\nu\in\mathbb{N}^N$, $|\nu|\le l$,

$$\begin{aligned}\rho'^{(l)}_\nu\left(j^l_Z F^1,j^l_Z F^2\right)&=\frac{\partial^\nu}{\partial Z^\nu}\left[\rho'\left(F^1(Z),F^2(Z)\right)\right]=\frac{\partial^\nu}{\partial Z^\nu}\left[\rho'^{\overline{F^2}}\left(F^1(Z),Z\right)\right]\\&=\sum_{\substack{\alpha\in\mathbb{N}^{N'},\ \beta\in\mathbb{N}^N\\|\beta|+|\alpha|\le|\nu|,\ \beta\le\nu}}P_{\nu\alpha\beta}\left(\left(\partial^\mu F^1(Z)\right)_{1\le|\mu|\le|\nu|}\right)\rho'^{\overline{F^2}}_{Z'^\alpha\zeta^\beta}\left(F^1(Z),Z\right)\ ,\end{aligned} \tag{5.9}$$

where the $P_{\nu\alpha\beta}$ are universal scalar polynomials depending only on N and N' (independent of F and ρ'). Note that we also have

$$P_{\nu0\nu}\equiv 1\ . \tag{5.10}$$

As in (5.2), one should regard $\rho'^{\overline{F^2}}$ as a power series mapping of the indeterminates (Z',ζ); this is the meaning of the derivative $\rho'^{\overline{F^2}}_{Z'^\alpha\zeta^\beta}$ in (5.9). For any $\alpha\in\mathbb{N}^{N'}$, any $\beta\in\mathbb{N}^N$ and for any formal map $F^2:(\mathbb{C}^N,0)\to(\mathbb{C}^{N'},0)$, we have, again by the chain rule,

$$\rho'^{\overline{F^2}}_{Z'^\alpha\zeta^\beta}\left(Z',\zeta\right)=\sum_{\mu\in\mathbb{N}^{N'},\ |\mu|\le|\beta|}R_{\beta\mu}\left(\left(\partial^\delta F^2(\zeta)\right)_{1\le|\delta|\le|\beta|}\right)\ \rho'_{Z'^\alpha\zeta'^\mu}\left(Z',F^2(\zeta)\right)\ , \tag{5.11}$$

where the $R_{\beta\mu}$ are universal scalar polynomials depending only on N and N' (independent of F^2 and ρ'). Again, as in (5.1), one should regard ρ' as a power series mapping of the indeterminates (Z', ζ'). Moreover, one has $R_{00} = 1$ and $R_{\beta 0} = 0$ for all $\beta \neq 0$. As a consequence of (5.9) and (5.11) and using the notation (4.3), we have for any formal mapping $F(Z)$ as in (5.6)

$$\rho'^{(l)}_\nu\left(j^l_Z F^1, j^l_Z F^2\right) = \sum_{\substack{\alpha\in\mathbb{N}^{N'},\ \beta\in\mathbb{N}^N \\ |\beta|+|\alpha|\le|\nu|,\ \beta\le\nu}} P_{\nu\alpha\beta}\left(\hat{j}^l_Z F^1\right) \sum_{\substack{\mu\in\mathbb{N}^{N'} \\ |\mu|\le|\beta|}} R_{\beta\mu}\left(\hat{j}^l_Z F^2\right)\ \rho'_{Z'^\alpha\zeta'^\mu}\left(F^1(Z), F^2(Z)\right)\ . \tag{5.12}$$

Hence, by the uniqueness in Proposition 4.2, we have in $\mathbb{C}[\![\Lambda_0]\!][\hat{\Lambda}]$, $\Lambda = (\Lambda^1, \Lambda^2)$, for $\nu \in \mathbb{N}^N$, $|\nu| \le l$,

$$\rho'^{(l)}_\nu\left(\Lambda^1, \Lambda^2\right) = \sum_{\substack{\alpha\in\mathbb{N}^{N'},\ \beta\in\mathbb{N}^N \\ |\beta|+|\alpha|\le|\nu|,\ \beta\le\nu}} P_{\nu\alpha\beta}\left(\hat{\Lambda}^1\right) \sum_{\substack{\mu\in\mathbb{N}^{N'} \\ |\mu|\le|\beta|}} R_{\beta\mu}\left(\hat{\Lambda}^2\right)\ \rho'_{Z'^\alpha\zeta'^\mu}\left(\Lambda^1_0, \Lambda^2_0\right)\ . \tag{5.13}$$

Using ${}^{F^1}\rho'(Z, F^2(Z))$ [given by (5.2)] instead of $\rho'^{\overline{F^2}}(F^1(Z), Z)$ in carrying out the calculation in (5.9), one is led to the following expression of $\rho'^{(l)}_\nu(\Lambda^1, \Lambda^2)$:

$$\rho'^{(l)}_\nu\left(\Lambda^1, \Lambda^2\right) = \sum_{\substack{\alpha\in\mathbb{N}^{N'},\ \beta\in\mathbb{N}^N \\ |\beta|+|\alpha|\le|\nu|,\ \beta\le\nu}} P_{\nu\alpha\beta}\left(\hat{\Lambda}^2\right) \sum_{\substack{\mu\in\mathbb{N}^{N'} \\ |\mu|\le|\beta|}} R_{\beta\mu}\left(\hat{\Lambda}^1\right)\ \rho'_{Z'^\mu\zeta'^\alpha}\left(\Lambda^1_0, \Lambda^2_0\right)\ , \tag{5.14}$$

where the polynomials $P_{\nu\alpha\beta}$ and $R_{\beta\mu}$ are the same as those in (5.13). Of course, the expressions (5.13) and (5.14) also hold for ρ' replaced by $\tilde{\rho}'$ as well, since the components of $\tilde{\rho}'$ are also generators of $\mathcal{I}(\mathcal{M}')$.

We summarize the above in the following lemma.

Lemma 5.1. *Let $\mathcal{M}'$, ρ' and $\tilde{\rho}'$ be as above. Then, the ideal in $\mathbb{C}[\![\Lambda_0]\!][\hat{\Lambda}]$ generated by the components of $\rho'^{(l)}(\Lambda)$ is the same as the ideal generated by the components of $\tilde{\rho}'^{(l)}(\Lambda)$, and both coincide with $\mathcal{I}(\mathcal{M}')^{(l)}$. Furthermore, the components $\rho'^{(l)}_\nu(\Lambda)$ are given either by* (5.13) *or by* (5.14).

We should mention that in what follows, we will use the expression (5.13) for $\tilde{\rho}'$ and the expression (5.14) for ρ', for a specific choice of ρ'.

Remark 5.2. As in (5.11), for any $\alpha \in \mathbb{N}^{N'}$, any $\beta \in \mathbb{N}^N$ and for any formal map $F^1 : (\mathbb{C}^N, 0) \to (\mathbb{C}^{N'}, 0)$, we have

$${}^{F^1}\rho'_{Z^\beta\zeta'^\alpha}\left(Z, \zeta'\right) = \sum_{\mu\in\mathbb{N}^{N'},\ |\mu|\le|\beta|} R_{\beta\mu}\left(\left(\partial^\delta F^1(Z)\right)_{1\le|\delta|\le|\beta|}\right)\ \rho'_{Z'^\mu\zeta'^\alpha}\left(F^1(Z), \zeta'\right)\ , \tag{5.15}$$

where the universal polynomials $R_{\beta\mu}$ are the same as those in (5.11). Observe that (5.15) has already been used in provin (5.14).

6. Properties of reflection ideals and their generators

As in Section 5, we consider a formal generic manifold $\mathcal{M}' \subset \mathbb{C}^{N'}_{Z'} \times \mathbb{C}^{N'}_{\zeta'}$ of codimension d'. Since $\mathcal{M}'$ is generic, we may assume by using the formal implicit function theorem that

$Z' = (z', w') \in \mathbb{C}^{n'} \times \mathbb{C}^{d'}$, $\zeta' = (\chi', \tau') \in \mathbb{C}^{n'} \times \mathbb{C}^{d'}$ with $n' = N' - d'$, and that the ideal $\mathcal{I}(\mathcal{M}')$ in $\mathbb{C}[\![Z', \zeta']\!]$ is given by

$$\mathcal{I}\left(\mathcal{M}'\right) = \left(w' - Q'\left(z', \zeta'\right)\right) , \tag{6.1}$$

where $Q' : (\mathbb{C}^{n'}_{z'} \times \mathbb{C}^{N'}_{\zeta'}, 0) \to (\mathbb{C}^{d'}, 0)$ is a formal mapping. Note that since $\mathcal{M}'$ is real, we also have

$$\mathcal{I}\left(\mathcal{M}'\right) = \left(\tau' - \bar{Q}'\left(\chi', Z'\right)\right) . \tag{6.2}$$

For the rest of this article, we make the following choice of generators for $\mathcal{I}(\mathcal{M}')$

$$\rho'\left(Z', \zeta'\right) := \tau' - \bar{Q}'\left(\chi', Z'\right) . \tag{6.3}$$

Hence, in view of (5.1), we have

$$\tilde{\rho}'\left(Z', \zeta'\right) = w' - Q'\left(z', \zeta'\right) . \tag{6.4}$$

We have the following proposition which holds for this choice of generators of $\mathcal{I}(\mathcal{M}')$.

Proposition 6.1. *Let $\mathcal{M}' \subset \mathbb{C}^{N'}_{Z'} \times \mathbb{C}^{N'}_{\zeta'}$ be a formal generic manifold of codimension d' and $H, H^0 : (\mathbb{C}^N, 0) \to (\mathbb{C}^{N'}, 0)$ be two formal mappings. Let $\mathcal{I}^H$ be the reflection ideal defined by* (2.6) *and ${}^H\rho'$ be the formal map given by* (5.2) *with the choice of ρ' given by* (6.3). *Then, the following hold.*

(i) $\mathcal{I}^H = \mathcal{I}^{H^0} \iff {}^H\rho'(Z, \zeta') = {}^{H^0}\rho'(Z, \zeta')$.

(ii) *The reflection ideal $\mathcal{I}^H$ is convergent (as in Definition 2.3) if and only if the components of ${}^H\rho'(Z, \zeta')$ are convergent power series.*

(iii) *The reflection ideal $\mathcal{I}^H$ is algebraic (as in Definition 2.3) if and only if the components of ${}^H\rho'(Z, \zeta')$ are algebraic functions.*

Proof. (i) Since $\mathcal{I}^H = \left({}^H\rho'(Z, \zeta')\right)$, it follows that ${}^H\rho'(Z, \zeta') = {}^{H^0}\rho'(Z, \zeta')$ implies the equality of the ideals $\mathcal{I}^H$ and $\mathcal{I}^{H^0}$. Conversely, if $\mathcal{I}^H = \mathcal{I}^{H^0}$, then there exists a $d' \times d'$ matrix $a(Z, \zeta')$ with entries in $\mathbb{C}[\![Z, \zeta']\!]$ such that

$${}^H\rho'\left(Z, \zeta'\right) = a\left(Z, \zeta'\right)\ {}^{H^0}\rho'\left(Z, \zeta'\right) . \tag{6.5}$$

Putting $\tau' = \bar{Q}'(\chi', H^0(Z))$ in (6.5) and making use of (5.2) and (6.3), we obtain that $\bar{Q}'(\chi', H^0(Z)) = \bar{Q}'(\chi', H(Z))$ and hence ${}^H\rho'(Z, \zeta') = {}^{H^0}\rho'(Z, \zeta')$.

(ii) Since $\mathcal{I}^H = \left({}^H\rho'(Z, \zeta')\right)$, if the components of ${}^H\rho'(Z, \zeta')$ are convergent, then $\mathcal{I}^H$ is convergent. Conversely, if $\mathcal{I}^H$ is convergent, then, by Definition 2.3 and Lemma 2.1 (ii), there exist $r_j(Z, \zeta') \in \mathbb{C}\{Z, \zeta'\}$, $j = 1, \ldots, d'$, with linearly independent differentials at 0 such that $\mathcal{I}^H = (r) = (r_1, \ldots, r_{d'})$ in $\mathbb{C}[\![Z, \zeta']\!]$. As a consequence, there exist a $d' \times d'$ invertible matrix $a(Z, \zeta')$ with entries in $\mathbb{C}[\![Z, \zeta']\!]$ such that

$$r\left(Z, \zeta'\right) = a\left(Z, \zeta'\right) {}^H\rho'\left(Z, \zeta'\right) = a\left(Z, \zeta'\right)\left(\tau' - \bar{Q}'\left(\chi', H(Z)\right)\right) , \tag{6.6}$$

and hence $\partial r / \partial \tau'(0)$ is invertible. By the implicit function theorem, one sees that the equation $r(Z, \chi', \tau') = 0$ has a unique convergent solution $\tau' = u(Z, \chi')$. It follows from (6.6) that $\bar{Q}'(\chi', H(Z)) = u(Z, \chi')$ and hence that ${}^H\rho'(Z, \zeta')$ is a convergent power series mapping. This completes the proof of (ii).

(iii) The proof of this case is similar to that of part (ii) above by making use of the algebraic version of the implicit function theorem. □

7. Ideals associated to formal generic manifolds and mappings

In this section, we consider two formal generic manifolds $\mathcal{M} \subset \mathbb{C}^N_Z \times \mathbb{C}^N_\zeta$ and $\mathcal{M}' \subset \mathbb{C}^{N'}_{Z'} \times \mathbb{C}^{N'}_{\zeta'}$ of codimension d and d' respectively. We write $N = n + d$ and $N' = n' + d'$. As in Section 6, we continue to use the choice of generators of $\mathcal{I}(\mathcal{M}')$ given by (6.3) and (6.4).

By the implicit function theorem, there exists a formal mapping

$$\gamma : \left(\mathbb{C}^N_\zeta \times \mathbb{C}^n_t, 0\right) \to \left(\mathbb{C}^N_Z, 0\right), \quad \operatorname{rk} \frac{\partial \gamma}{\partial t}(0) = n \ , \tag{7.1}$$

such that for any $h \in \mathcal{I}(\mathcal{M})$,

$$h(\gamma(\zeta, t), \zeta) = 0 \ ,$$

and hence, by the reality of $\mathcal{M}$, we also have

$$h\left(Z, \bar{\gamma}(Z, t)\right) = 0 \ . \tag{7.2}$$

Observe that each of the formal mappings $(\mathbb{C}^N \times \mathbb{C}^n, 0) \ni (Z, t) \mapsto (Z, \bar{\gamma}(Z, t))$ and $(\mathbb{C}^N \times \mathbb{C}^n, 0) \ni (\zeta, t) \mapsto (\gamma(\zeta, t), \zeta)$ is a parametrization of the formal generic manifold $\mathcal{M}$. If, moreover, $\mathcal{M} \subset \mathbb{C}^N \times \mathbb{C}^N$ is the complexification of a generic real-analytic (resp. real-algebraic) submanifold through the origin in $\mathbb{C}^N$, then one can choose γ to be convergent (resp. algebraic). As in [5], we shall call a formal map γ satisfying the above properties a *Segre variety mapping* relative to $\mathcal{M}$. Note that the formal map $(\mathbb{C}^n, 0) \ni t \mapsto \gamma(0, t)$ is a parametrization of $\mathcal{S}_0(\mathcal{M})$, the formal Segre variety of $\mathcal{M}$ at 0 as defined in Section 2. In the rest of this article, we shall fix such a map γ.

For a formal map $H : (\mathbb{C}^N, 0) \to (\mathbb{C}^{N'}, 0)$ and the fixed nonnegative integer l, we define two formal mappings

$$\varphi^{[l]}(H; \cdot),\ \tilde{\varphi}^{[l]}(H; \cdot) : \left(J^l_0\left(\mathbb{C}^N, \mathbb{C}^{N'}\right) \times \mathbb{C}^N_Z \times \mathbb{C}^n_t, 0\right) \to J^l_0\left(\mathbb{C}^N, \mathbb{C}^{d'}\right) \ ,$$

as follows. Consider $\rho'^{(l)}(\Lambda^1, \Lambda^2)$ and $\tilde{\rho}'^{(l)}(\Lambda^1, \Lambda^2)$ as defined in (5.5), with the choice of ρ' and $\tilde{\rho}'$ made in (6.3) and (6.4). Taking $\Lambda^2 = \partial^\alpha_Z(\bar{H}(\bar{\gamma}(Z, t)))_{|\alpha| \le l}$ we set

$$\varphi^{[l]}\left(H; \Lambda^1, Z, t\right) := \rho'^{(l)}\left(\Lambda^1, \left(\partial^\alpha_Z\left(\bar{H}\left(\bar{\gamma}(Z, t)\right)\right)_{|\alpha| \le l}\right) \ , \tag{7.3}$$

and

$$\tilde{\varphi}^{[l]}\left(H; \Lambda^1, Z, t\right) := \tilde{\rho}'^{(l)}\left(\Lambda^1, \left(\partial^\alpha_Z(\bar{H}\left(\bar{\gamma}(Z, t)\right)\right)_{|\alpha| \le l}\right) \ . \tag{7.4}$$

Observe that each component of the right-hand side of (7.3) and (7.4) is a formal power series which is in $\mathbb{C}[\![\Lambda^1_0, Z, t]\!][\hat{\Lambda}^1]$. Here we recall that $\Lambda^1 = (\Lambda^1_0, \hat{\Lambda}^1)$ are coordinates on $J^l_0(\mathbb{C}^N, \mathbb{C}^{N'})$ as in (4.1) and (4.2).

We shall write, as in (4.4), $\varphi^{[l]}(H; \cdot) = (\varphi^{[l]}_\nu(H; \cdot))_{|\nu| \le l}$ and use a similar notation for $\tilde{\varphi}^{[l]}(H; \cdot)$. We shall now compute the ν-th component $\tilde{\varphi}^{[l]}_\nu(H; \cdot)$. It follows from (7.4) and (5.13), with ρ' replaced by $\tilde{\rho}'$, that

$$\begin{aligned}\tilde{\varphi}^{[l]}_\nu&\left(H; \Lambda^1, Z, t\right)\\ &= \sum_{\substack{\alpha \in \mathbb{N}^{N'},\ \beta \in \mathbb{N}^N \\ |\beta| + |\alpha| \le |\nu|,\ \beta \le \nu}} P_{\nu\alpha\beta}\left(\hat{\Lambda}^1\right) \sum_{\substack{\mu \in \mathbb{N}^{N'} \\ |\mu| \le |\beta|}} R_{\beta\mu}\Big(\hat{j}^l_Z\left(\bar{H}\left(\bar{\gamma}(Z, t)\right)\right)\Big)\ \tilde{\rho}'_{Z'^\alpha \zeta'^\mu}\left(\Lambda^1_0, \bar{H}\left(\bar{\gamma}(Z, t)\right)\right) \ ,\end{aligned} \tag{7.5}$$

where $\hat{j}^l_Z(\bar{H}(\bar{\gamma}(Z,t))) = (\partial^\delta_Z[\bar{H}(\bar{\gamma}(Z,t))])_{1\le|\delta|\le l}$. By the chain rule, a computation similar to (5.11) shows that one has

$$\frac{\partial^{|\beta|}}{\partial Z^\beta}\left[\tilde{\rho}'^H{}_{Z'^\alpha}\left(Z',\bar{\gamma}(Z,t)\right)\right] = \sum_{\substack{\mu\in\mathbb{N}^{N'}\\ |\mu|\le|\beta|}} R_{\beta\mu}\left(\hat{j}^l_Z\left(\bar{H}\left(\bar{\gamma}(Z,t)\right)\right)\right)\tilde{\rho}'_{Z'^\alpha\zeta'^\mu}\left(Z',\bar{H}\left(\bar{\gamma}(Z,t)\right)\right), \tag{7.6}$$

where the universal polynomials $R_{\beta\mu}$ are the same as those in (5.11). On the other hand, by the chain rule (again considering $\tilde{\rho}'^H$ as a power series mapping of the indeterminates (Z',ζ)), we also have

$$\frac{\partial^{|\beta|}}{\partial Z^\beta}\left[\tilde{\rho}'^H_{Z'^\alpha}\left(Z',\bar{\gamma}(Z,t)\right)\right] = \sum_{|\delta|\le|\beta|} c_{\beta\delta}(Z,t)\tilde{\rho}'^H_{Z'^\alpha\zeta^\delta}\left(Z',\bar{\gamma}(Z,t)\right). \tag{7.7}$$

Here, the formal power series maps $c_{\beta\delta}:(\mathbb{C}^N_Z\times\mathbb{C}^n_t,0)\to\mathbb{C}$ depend only on the Segre variety mapping γ and not on the mapping H. Moreover, if γ is convergent (resp. algebraic), then the $c_{\beta\delta}$ are also convergent (resp. algebraic). As a consequence of (7.5), (7.6), and (7.7), we obtain

$$\tilde{\varphi}^{[l]}_\nu\left(H;\Lambda^1,Z,t\right) = \sum_{\substack{\alpha\in\mathbb{N}^{N'},\ \beta\in\mathbb{N}^N\\ |\beta|+|\alpha|\le|\nu|,\ \beta\le\nu}} P_{\nu\alpha\beta}\left(\hat{\Lambda}^1\right)\sum_{|\delta|\le|\beta|} c_{\beta\delta}(Z,t)\tilde{\rho}'^H_{Z'^\alpha\zeta^\delta}\left(\Lambda^1_0,\bar{\gamma}(Z,t)\right). \tag{7.8}$$

If $H:(\mathbb{C}^N_Z,0)\to(\mathbb{C}^{N'}_{Z'},0)$ is a formal map such that its complexification $\mathcal{H}:(\mathbb{C}^N_Z\times\mathbb{C}^N_\zeta,0)\to(\mathbb{C}^{N'}_{Z'}\times\mathbb{C}^{N'}_{\zeta'},0)$ given by (2.2) sends $\mathcal{M}$ into $\mathcal{M}'$, then it follows from (7.2) that

$$h'\left(H(Z),\bar{H}\left(\bar{\gamma}(Z,t)\right)\right)=0,\quad \forall h'\in\mathcal{I}\left(\mathcal{M}'\right). \tag{7.9}$$

Taking h' in (7.9) to be any of the components of $\rho'(Z',\zeta')$ or $\tilde{\rho}'(Z',\zeta')$ and making use of Remark 4.3, we obtain

$$\rho'^{(l)}\left(j^l_Z H\left(Z\right), j^l_Z\left(\bar{H}\left(\bar{\gamma}(Z,t)\right)\right)\right)=0,\quad \tilde{\rho}'^{(l)}\left(j^l_Z H(Z), j^l_Z\left(\bar{H}\left(\bar{\gamma}(Z,t)\right)\right)\right)=0. \tag{7.10}$$

Hence $\Lambda^1 = j^l_Z H$ is a solution of each of the systems of equations

$$\varphi^{[l]}\left(H;\Lambda^1,Z,t\right)=0,\quad \tilde{\varphi}^{[l]}\left(H;\Lambda^1,Z,t\right)=0, \tag{7.11}$$

where $\varphi^{[l]}(H;\cdot)$ and $\tilde{\varphi}^{[l]}(H;\cdot)$ are defined by (7.3) and (7.4) respectively.

We summarize the above in the following lemma.

Lemma 7.1. *Let $H:(\mathbb{C}^N_Z,0)\to(\mathbb{C}^{N'}_{Z'},0)$ and $\varphi^{[l]}(H;\cdot)$ and $\tilde{\varphi}^{[l]}(H;\cdot)$ be the formal series given by* (7.3) *and* (7.4) *respectively. Then, the ideal in $\mathbb{C}[\![\Lambda^1_0,Z,t]\!][\hat{\Lambda}^1]$ generated by the components of $\varphi^{[l]}(H;\Lambda^1,Z,t)$ is the same as that generated by the components of $\tilde{\varphi}^{[l]}(H;\Lambda^1,Z,t)$. Moreover, the components of $\tilde{\varphi}^{[l]}(H;\Lambda^1,Z,t)$ are given by formula* (7.8). *If, in addition, the complexification $\mathcal{H}$ of H, as given by* (2.2), *maps $\mathcal{M}$ into $\mathcal{M}'$, then $\Lambda^1 = j^l_Z H$ is a solution of each of the two systems of equations in* (7.11).

8. Iterated Segre mappings and associated ideals

In this section, we assume that $\mathcal{M}$ and $\mathcal{M}'$ are given formal generic manifolds as in Section 7. We continue to use the choice of generators $\rho'(Z',\zeta')$ and $\tilde{\rho}'(Z',\zeta')$ given in (6.3) and (6.4) for the ideal $\mathcal{I}(\mathcal{M}')$. If γ is a Segre variety mapping relative to $\mathcal{M}$ as defined in (7.1), we define, as in [5], the *iterated Segre mappings* (relative to $\mathcal{M}$) as follows. First, we set $v^0 := 0 \in \mathbb{C}^N$. For any positive integer j, $v^j : (\mathbb{C}^{nj},0) \to (\mathbb{C}^N,0)$ is the formal mapping defined inductively by

$$v^j\left(t^1,\dots,t^j\right) := \gamma\left(\bar{v}^{j-1}\left(t^1,\dots,t^{j-1}\right),t^j\right), \quad t^1,\dots,t^j \in \mathbb{C}^n . \tag{8.1}$$

In what follows, it will be convenient to introduce for a given positive integer j the notation

$$t^{[j]} := \left(t^1,\dots,t^j\right) ,$$

considered as a variable in $\mathbb{C}^{nj}$. With this notation, we may rewrite (8.1) in the form

$$v^j\left(t^{[j]}\right) = \gamma\left(\bar{v}^{j-1}\left(t^{[j-1]}\right),t^j\right) .$$

It follows from (7.2) and (8.1) that for any $h \in \mathcal{I}(\mathcal{M})$ and any nonnegative integer j, we have

$$h\left(v^j\left(t^{[j]}\right),\bar{v}^{j+1}\left(t^{[j+1]}\right)\right) = 0 . \tag{8.2}$$

If $H : (\mathbb{C}^N,0) \to (\mathbb{C}^{N'},0)$ is a formal mapping and j is a fixed nonnegative integer, we define two formal mappings

$$\psi^{[l,j]}(H;\cdot),\ \tilde{\psi}^{[l,j]}(H;\cdot) : \left(J_0^l\left(\mathbb{C}^N,\mathbb{C}^{N'}\right)\times\mathbb{C}^{(j+2)n},0\right) \to J_0^l\left(\mathbb{C}^N,\mathbb{C}^{d'}\right) ,$$

as follows:

$$\psi^{[l,j]}\left(H;\Lambda^1,t^{[j+2]}\right) := \varphi^{[l]}\left(H;\Lambda^1,v^{j+1}\left(t^{[j+1]}\right),t^{j+2}\right) \tag{8.3}$$

and similarly,

$$\tilde{\psi}^{[l,j]}\left(H;\Lambda^1,t^{[j+2]}\right) := \tilde{\varphi}^{[l]}\left(H;\Lambda^1,v^{j+1}\left(t^{[j+1]}\right),t^{j+2}\right) . \tag{8.4}$$

Here we recall that the formal mappings $\varphi^{[l]}(H;\cdot)$ and $\tilde{\varphi}^{[l]}(H;\cdot)$ are given by (7.3) and (7.4), respectively. Hence the components of $\psi^{[l,j]}(H;\cdot)$ and $\tilde{\psi}^{[l,j]}(H;\cdot)$ are formal power series in the ring $\mathbb{C}[\![\Lambda_0^1,t^{[j+2]}]\!][\hat{\Lambda}^1] = \mathbb{C}[\![\Lambda_0^1,t^1,\dots,t^{j+2}]\!][\hat{\Lambda}^1]$. It follows from the definition of $\tilde{\psi}^{[l,j]}(H;\cdot)$ and from (7.8) that one has the following identity for every $\nu \in \mathbb{N}^N$, $|\nu| \le l$,

$$\begin{aligned}\tilde{\psi}_\nu^{[l,j]}&\left(H;\Lambda^1,t^{[j+2]}\right)\\ &= \sum_{\substack{\alpha\in\mathbb{N}^{N'},\ \beta\in\mathbb{N}^N\\ |\beta|+|\alpha|\le|\nu|,\ \beta\le\nu}} P_{\nu\alpha\beta}\left(\hat{\Lambda}^1\right)\sum_{|\delta|\le|\beta|} u_{\beta\delta}^j\left(t^{[j+2]}\right)\tilde{\rho}'^H_{Z'^\alpha\zeta^\delta}\left(\Lambda_0^1,\bar{v}^{j+2}\left(t^{[j+2]}\right)\right) .\end{aligned} \tag{8.5}$$

Here we have used (8.1) in the form $\bar{v}^{j+2}(t^{[j+2]}) = \bar{\gamma}(v^{j+1}(t^{[j+1]}),t^{j+2})$ and set $u_{\beta\delta}^j(t^{[j+2]}) := c_{\beta\delta}(v^{j+1}(t^{[j+1]}),t^{j+2})$, where the $c_{\beta\delta}$ are as in (7.8). Note that since the $c_{\beta\delta}$ are independent of H, so are the $u_{\beta\delta}^j$. Moreover, if $\mathcal{M}$ is the complexification of a real-analytic (resp. real-algebraic) generic submanifold of $\mathbb{C}^N$ through the origin, then we may assume that the formal power series

$u^j_{\beta\delta}$ in (8.5) are convergent (resp. algebraic). The following lemma is then a consequence of Lemmas 5.1 and 7.1 as well as the above construction.

Lemma 8.1. *The ideals* $\left(\psi^{[l,j]}(H;\Lambda^1,t^{[j+2]})\right)$ *and* $\left(\tilde{\psi}^{[l,j]}(H;\Lambda^1,t^{[j+2]})\right)$ *in* $\mathbb{C}[\![\Lambda^1_0,t^{[j+2]}]\!][\hat{\Lambda}^1]$ *are the same. In particular, let*

$$S:\left(\mathbb{C}^{n(j+2)},0\right)\to J^l_0\left(\mathbb{C}^N,\mathbb{C}^{N'}\right)\ ,$$

be a formal map with $S(t^{[j+2]})=(S_0(t^{[j+2]}),\hat{S}(t^{[j+2]}))$ *as in* (4.2) *and* $S_0(0)=0$. *Then,* $\Lambda^1=S(t^{[j+2]})$ *is a solution of* $\psi^{[l,j]}(H;\Lambda^1,t^{[j+2]})=0$ *if and only if it is a solution of* $\tilde{\psi}^{[l,j]}(H;\Lambda^1,t^{[j+2]})=0$. *Moreover, if* $H:(\mathbb{C}^N,0)\to(\mathbb{C}^{N'},0)$ *is a formal map such that its complexification* $\mathcal{H}$ *sends* $\mathcal{M}$ *into* $\mathcal{M}'$, *then*

$$\Lambda^1=\left(\left(\partial^\alpha H\right)\left(v^{j+1}\left(t^{[j+1]}\right)\right)\right)_{|\alpha|\le l} \tag{8.6}$$

is a solution of the systems of equations

$$\psi^{[l,j]}\left(H;\Lambda^1,t^{[j+2]}\right)=0,\quad \tilde{\psi}^{[l,j]}\left(H;\Lambda^1,t^{[j+2]}\right)=0\ . \tag{8.7}$$

We need the following lemma concerning the iterated Segre mappings of $\mathcal{M}$.

Lemma 8.2. *Let* $\gamma(\zeta,t)$ *be a Segre variety mapping relative to the generic formal manifold* $\mathcal{M}\subset\mathbb{C}^N\times\mathbb{C}^N$ *as defined in* (7.1) *and* v^j *the iterated Segre mappings as defined in* (8.1). *Then, for every nonnegative integer* j, *there exists a unique formal mapping* $\xi^j:(\mathbb{C}^{n(j+1)},0)\to(\mathbb{C}^n,0)$ *such that*

$$v^{j+2}\left(t^{[j+1]},\xi^j\left(t^{[j+1]}\right)\right)=v^j\left(t^{[j]}\right)\ . \tag{8.8}$$

Moreover, if the formal mapping γ *is convergent (resp. algebraic), then* ξ^j *is convergent (resp. algebraic).*

Proof. Since $\mathcal{M}$ is generic, by making use of the implicit function theorem we can assume that $Z=(z,w)\in\mathbb{C}^n\times\mathbb{C}^d$, where d is the codimension of $\mathcal{M}$ and $n=N-d$, and that $\mathcal{I}(\mathcal{M})$ is generated by the components of $w-Q(z,\zeta)$ where $Q:(\mathbb{C}^{n+N},0)\to(\mathbb{C}^d,0)$ is a formal mapping. If $\mathcal{M}$ is the complexification of a real-analytic (resp. real-algebraic) generic submanifold $M\subset\mathbb{C}^N$, then the formal map Q is convergent (resp. algebraic). By the reality of $\mathcal{M}$, one has

$$Q\left(z,\chi,\bar{Q}(\chi,z,w)\right)=w\ . \tag{8.9}$$

Corresponding to the splitting $Z=(z,w)$, we may write $\gamma(\zeta,t)=(\mu(\zeta,t),\nu(\zeta,t))$, with $\mu:(\mathbb{C}^{N+n},0)\to(\mathbb{C}^n,0)$ and $\nu:(\mathbb{C}^{N+n},0)\to(\mathbb{C}^d,0)$. By the definition of a Segre variety mapping γ, we necessarily have

$$\gamma(\zeta,t)=(\mu(\zeta,t),Q(\mu(\zeta,t),\zeta))\ . \tag{8.10}$$

Since rk $\partial\gamma/\partial t(0)=n$, the matrix $\partial\mu/\partial t(0)$ is invertible. As a consequence of the implicit function theorem, there exist a formal mapping $\pi:(\mathbb{C}^{N+n},0)\to(\mathbb{C}^n,0)$ such that

$$\mu\left(\bar{\gamma}(Z,t),\pi(Z,t)\right)=z,\qquad \text{where } Z=(z,w)\ . \tag{8.11}$$

It follows from (8.9), (8.10), and (8.11) that $\gamma(\bar{\gamma}(Z,t),\pi(Z,t)) = Z$. The lemma follows by taking $\xi^j(t^{[j+1]}) := \pi(v^j(t^{[j]}), t^{j+1})$.[2] □

If j is a nonnegative integer, l is the previously fixed nonnegative integer, and $H : (\mathbb{C}^N, 0) \to (\mathbb{C}^{N'}, 0)$ is a formal map, then for $\nu \in \mathbb{N}^N$, $|\nu| \le l$, and $\epsilon \in \mathbb{N}^n$, we define formal mappings

$$\Theta^{[l,j]}_{\nu,\epsilon}(H;\cdot),\ \tilde{\Theta}^{[l,j]}_{\nu,\epsilon}(H;\cdot) : \left(J_0^l\left(\mathbb{C}^N, \mathbb{C}^{N'}\right) \times \mathbb{C}^{n(j+1)}, 0\right) \to \mathbb{C}^{d'} \tag{8.12}$$

by

$$\begin{aligned} \Theta^{[l,j]}_{\nu,\epsilon}\left(H;\Lambda^1, t^{[j+1]}\right) &:= \partial^{\epsilon}_{t^{j+2}} \psi^{[l,j]}_{\nu}\left(H;\Lambda^1, t^{[j+2]}\right)\Big|_{t^{j+2}=\bar{\xi}^j(t^{[j+1]})}, \\ \tilde{\Theta}^{[l,j]}_{\nu,\epsilon}\left(H;\Lambda^1, t^{[j+1]}\right) &:= \partial^{\epsilon}_{t^{j+2}} \tilde{\psi}^{[l,j]}_{\nu}\left(H;\Lambda^1, t^{[j+2]}\right)\Big|_{t^{j+2}=\bar{\xi}^j(t^{[j+1]})}, \end{aligned} \tag{8.13}$$

where $\psi^{[l,j]}(H;\cdot) = (\psi^{[l,j]}_{\nu}(H;\cdot))_{|\nu|\le l}$ and $\tilde{\psi}^{[l,j]}(H;\cdot) = (\tilde{\psi}^{[l,j]}_{\nu}(H;\cdot))_{|\nu|\le l}$ are defined by (8.3) and (8.4), respectively, and the map ξ^j is given by Lemma 8.2. Observe that each component of $\Theta^{[l,j]}_{\nu,\epsilon}(H;\cdot)$ and $\tilde{\Theta}^{[l,j]}_{\nu,\epsilon}(H;\cdot)$ is a formal power series in $\mathbb{C}[\![\Lambda^1_0, t^{[j+1]}]\!][\hat{\Lambda}^1]$. We have the following lemma concerning the formal power series mapping $\tilde{\Theta}^{[l,j]}_{\nu,\epsilon}(H;\cdot)$.

Lemma 8.3. *For any $\nu \in \mathbb{N}^N$, $|\nu| \le l$ and any $\epsilon \in \mathbb{N}^n$, the following holds.*

$$\tilde{\Theta}^{[l,j]}_{\nu,\epsilon}\left(H;\Lambda^1, t^{[j+1]}\right) = \sum_{\substack{|\alpha|\le l \\ |\delta|\le l+|\epsilon|}} \omega^j_{\nu\epsilon\alpha\delta}\left(\hat{\Lambda}^1, t^{[j+1]}\right) \tilde{\rho}'^H_{Z'^\alpha\zeta^\delta}\left(\Lambda^1_0, \bar{v}^j\left(t^{[j]}\right)\right), \tag{8.14}$$

where each $\omega^j_{\nu\epsilon\alpha\delta}(\hat{\Lambda}^1, t^{[j+1]}) \in \mathbb{C}[\![t^{[j+1]}]\!][\hat{\Lambda}^1]$ is independent of the formal mapping H. Here, $\tilde{\rho}'^H(Z',\zeta)$ is considered as a formal power series mapping in the indeterminates (Z',ζ). Moreover, if the Segre variety mapping γ relative to $\mathcal{M}$ is convergent (resp. algebraic), then each formal power series $\omega^j_{\nu\epsilon\alpha\delta}(\hat{\Lambda}^1, t^{[j+1]})$ is in $\mathbb{C}\{t^{[j+1]}\}[\hat{\Lambda}^1]$ (resp. $\mathcal{A}\{t^{[j+1]}\}[\hat{\Lambda}^1]$).

Proof. The proof is an immediate consequence of (8.5), the definition of the $\tilde{\Theta}^{[l,j]}_{\nu,\epsilon}(H;\cdot)$ given in (8.13), and (8.8). □

The formal power series given by (8.13) will not be used until Section 10. Their importance lies in the following remark.

Remark 8.4. Let $S : (\mathbb{C}^{n(j+1)}, 0) \to J_0^l(\mathbb{C}^N, \mathbb{C}^{N'})$ be a formal mapping such that $S_0(0) = 0$ where $S(t^{[j+1]}) = (S_\nu(t^{[j+1]}))_{|\nu|\le l} = (S_0(t^{[j+1]}), \hat{S}(t^{[j+1]}))$ as in (4.2). Then, $\Lambda^1 = S(t^{[j+1]})$ is a solution of the system of equations $\tilde{\psi}^{[l,j]}(H;\Lambda^1, t^{[j+2]}) = 0$ if and only if it is a solution of the system of equations $\tilde{\Theta}^{[l,j]}_{\nu,\epsilon}(H;\Lambda^1, t^{[j+1]}) = 0$ for all $\nu \in \mathbb{N}^N$, $|\nu| \le l$, and all $\epsilon \in \mathbb{N}^n$. This is an immediate consequence of the fact that $S(t^{[j+1]})$ is independent of the indeterminate t^{j+2} and the definition (8.13) of the $\tilde{\Theta}^{[l,j]}_{\nu,\epsilon}(H;\cdot)$.

9. Properties of solutions of the system $\psi^{[l,j]}(H;\Lambda^1, t^{[j+2]}) = 0$

The following technical lemma will be essential for the proofs of Theorems 2.5, 2.6, and 2.7.

[2] If one takes $\mu(\zeta,t) = t$, where $\mu(\zeta,t)$ is the component of $\gamma(\zeta,t)$ as in (8.10), then the reader can check that one has $\xi^j(t^{[j+1]}) = t^j$.

Lemma 9.1. *Let $\mathcal{M} \subset \mathbb{C}^N \times \mathbb{C}^N$ and $\mathcal{M}' \subset \mathbb{C}^{N'} \times \mathbb{C}^{N'}$ be formal generic manifolds and $H : (\mathbb{C}^N, 0) \to (\mathbb{C}^{N'}, 0)$ a formal map such that its complexification $\mathcal{H} : (\mathbb{C}^N \times \mathbb{C}^N, 0) \to (\mathbb{C}^{N'} \times \mathbb{C}^{N'}, 0)$ sends $\mathcal{M}$ into $\mathcal{M}'$. Assume that H is not totally degenerate as in Definition 2.4. Let l, j be nonnegative integers and $\psi^{[l,j]}(H; \cdot)$ the formal map given by (8.3). Let $S : (\mathbb{C}^{n(j+1)}, 0) \to J_0^l(\mathbb{C}^N, \mathbb{C}^{N'})$, $S_0(0) = 0$, be a formal map and assume that $\Lambda^1 = S(t^{[j+1]}) = (S_\nu(t^{[j+1]}))_{|\nu| \le l} = (S_0(t^{[j+1]}), \hat{S}(t^{[j+1]}))$ is a formal solution of the system*

$$\psi^{[l,j]}\left(H; \Lambda^1, t^{[j+2]}\right) = 0 \,. \tag{9.1}$$

Then, the following holds. For every $\nu \in \mathbb{N}^N$, $|\nu| \le l$,

$${}^H\!\rho'\left(v^{j+1}\left(t^{[j+1]}\right), \zeta'\right) = \sum_{|\mu| \le |\nu|} R_{\nu\mu}\left(\hat{S}\left(t^{[j+1]}\right)\right) \rho'_{Z'^\mu}\left(S_0\left(t^{[j+1]}\right), \zeta'\right) \,, \tag{9.2}$$

where ρ' and ${}^H\!\rho'$ are given by (6.3) *and* (5.2), *respectively, and the $R_{\nu\mu}$ are the universal polynomials given in* (5.11). *Here $\rho'(Z', \zeta')$ and ${}^H\!\rho'(Z, \zeta')$ are considered as formal power series mappings in the indeterminates (Z', ζ') and (Z, ζ') respectively. If, moreover, $S(t^{[j+1]}) = \left((\partial^\alpha H^0)(v^{j+1}(t^{[j+1]}))\right)_{|\alpha| \le l}$ for some formal map $H^0 : (\mathbb{C}^N, 0) \to (\mathbb{C}^{N'}, 0)$, then* (9.2) *for $|\nu| \le l$ is equivalent to*

$${}^H\!\rho'_{Z^\nu}\left(v^{j+1}\left(t^{[j+1]}\right), \zeta'\right) = {}^{H^0}\!\rho'_{Z^\nu}\left(v^{j+1}\left(t^{[j+1]}\right), \zeta'\right) \,. \tag{9.3}$$

Proof. In what follows, we use the coordinates $Z' = (z', w')$, $\zeta' = (\chi', \tau')$ as in the beginning of Section 6 and write

$$H(Z) = (f(Z), g(Z)), \qquad \text{with } z' = f(Z) \text{ and } w' = g(Z) \,. \tag{9.4}$$

For the proof of (9.2), we proceed by induction on $|\nu|$ and we start first by proving (9.2) for $\nu = 0$. Note that since $\Lambda^1 = S(t^{[j+1]})$ is a solution of the system (9.1), it follows that $\psi_0^{[l,j]}(H; S(t^{[j+1]}), t^{[j+2]}) = 0$. The latter equation is equivalent to

$$\rho'\left(S_0\left(t^{[j+1]}\right), \bar{H}\left(\bar{v}^{j+2}\left(t^{[j+2]}\right)\right)\right) = 0 \,. \tag{9.5}$$

Observe that since $\mathcal{H}$ maps $\mathcal{M}$ into $\mathcal{M}'$, we have by making use of (7.9) and (8.1) that

$$\rho'\left(H\left(v^{j+1}\left(t^{[j+1]}\right)\right), \bar{H}\left(\bar{v}^{j+2}\left(t^{[j+2]}\right)\right)\right) = 0 \,. \tag{9.6}$$

It follows from (9.4), (9.5), (9.6), and (6.3) that

$$\bar{Q}'\left(\bar{f}\left(\bar{v}^{j+2}\left(t^{[j+2]}\right)\right), S_0(t^{[j+1]})\right) = \bar{Q}'\left(\bar{f}\left(\bar{v}^{j+2}\left(t^{[j+2]}\right)\right), H\left(v^{j+1}\left(t^{[j+1]}\right)\right)\right) \,. \tag{9.7}$$

To show that (9.2) holds for $\nu = 0$, in view of (5.2) and (6.3), we must show that

$$\bar{Q}'\left(\chi', S_0\left(t^{[j+1]}\right)\right) = \bar{Q}'\left(\chi', H\left(v^{j+1}\left(t^{[j+1]}\right)\right)\right) \,. \tag{9.8}$$

For this, by e. g., Proposition 5.3.5 of [6], it suffices to show that Rk B, the rank of the formal map

$$B : \left(\mathbb{C}^{n(j+2)}_{t^{[j+2]}}, 0\right) \to \left(\mathbb{C}^{n(j+1)} \times \mathbb{C}^{n'}, 0\right) \tag{9.9}$$

given by $B(t^{[j+2]}) := (t^{[j+1]}, \bar{f}(\bar{v}^{[j+2]}(t^{[j+2]})))$, is $n(j+1)+n'$. The latter follows from the fact that H is not totally degenerate. Indeed, since $v^1(t) = \gamma(0,t)$ is a parametrization of the formal Segre variety $S_0(\mathcal{M})$, it follows from Definition 2.4 that Rk $H \circ v^1 = n'$. Hence we also have Rk $f \circ v^1 = n'$. From this, we easily obtain that Rk $B = n(j+1)+n'$. This completes the proof of (9.2) for $\nu = 0$.

It follows from (5.14) and the definition of the $\psi^{[l,j]}(H;\cdot)$ given in (8.3) that the following identity holds for all $\nu \in \mathbb{N}^N$, $|\nu| \le l$,

$$\begin{aligned}\psi_\nu^{[l,j]}&\left(H;\Lambda^1,t^{[j+2]}\right)\\ &= \sum_{\substack{\alpha\in\mathbb{N}^{N'},\ \beta\in\mathbb{N}^N\\ |\beta|+|\alpha|\le|\nu|,\ \beta\le\nu}} P_{\nu\alpha\beta}\left(\widehat{m}\left(t^{[j+2]}\right)\right)\sum_{\substack{\mu\in\mathbb{N}^{N'}\\ |\mu|\le|\beta|}} R_{\beta\mu}\left(\hat{\Lambda}^1\right)\,\rho'_{Z'^\mu\zeta'^\alpha}\left(\Lambda_0^1, m_0\left(t^{[j+2]}\right)\right)\,,\end{aligned} \tag{9.10}$$

where

$$\begin{aligned} m_\alpha\left(t^{[j+2]}\right) &:= \partial_Z^\alpha\left[\bar{H}\left(\bar{\gamma}(Z,t)\right)\right]\Big|_{Z=v^{j+1}\left(t^{[j+1]}\right),\ t=t^{j+2}}\,,\\ \widehat{m}\left(t^{[j+2]}\right) &:= \left(m_\alpha\left(t^{[j+2]}\right)\right)_{1\le|\alpha|\le l}\,. \end{aligned} \tag{9.11}$$

In view of (5.10), we may rewrite (9.10) as follows

$$\begin{aligned}\psi_\nu^{[l,j]}\left(H;\Lambda^1,t^{[j+2]}\right) &= \sum_{\substack{\mu\in\mathbb{N}^{N'}\\ |\mu|\le|\nu|}} R_{\nu\mu}\left(\hat{\Lambda}^1\right)\,\rho'_{Z'^\mu}\left(\Lambda_0^1, m_0\left(t^{[j+2]}\right)\right)\\ &+ \sum_{\substack{\alpha\in\mathbb{N}^{N'},\ \beta\in\mathbb{N}^N\\ |\beta|+|\alpha|\le|\nu|,\ \beta<\nu}} P_{\nu\alpha\beta}\left(\widehat{m}\left(t^{[j+2]}\right)\right)\sum_{\substack{\mu\in\mathbb{N}^{N'}\\ |\mu|\le|\beta|}} R_{\beta\mu}\left(\hat{\Lambda}^1\right)\,\rho'_{Z'^\mu\zeta'^\alpha}\left(\Lambda_0^1, m_0\left(t^{[j+2]}\right)\right)\end{aligned} \tag{9.12}$$

Let $\epsilon \in \mathbb{N}^N$, $0 < |\epsilon| \le l$, and assume that (9.2) holds for all $\nu \in \mathbb{N}^N$ with $|\nu| < |\epsilon|$. We now show that (9.2) holds for $\nu = \epsilon$. Since $\Lambda^1 = S(t^{[j+1]})$ is a solution of the system (9.1), it follows from (9.12), with ν replaced by ϵ, that we have

$$\begin{aligned}&\sum_{\substack{\mu\in\mathbb{N}^{N'}\\ |\mu|\le|\epsilon|}} R_{\epsilon\mu}\left(\hat{S}\left(t^{[j+1]}\right)\right)\,\rho'_{Z'^\mu}\left(S_0\left(t^{[j+1]}\right), m_0\left(t^{[j+2]}\right)\right)\\ &\quad= -\sum_{\substack{|\beta|+|\alpha|\le|\epsilon|\\ \beta<\epsilon}} P_{\epsilon\alpha\beta}\left(\widehat{m}\left(t^{[j+2]}\right)\right)\\ &\qquad\sum_{\substack{\mu\in\mathbb{N}^{N'}\\ |\mu|\le|\beta|}} R_{\beta\mu}\left(\hat{S}\left(t^{[j+1]}\right)\right)\,\rho'_{Z'^\mu\zeta'^\alpha}\left(S_0\left(t^{[j+1]}\right), m_0\left(t^{[j+2]}\right)\right)\,.\end{aligned} \tag{9.13}$$

On the other hand, using the notation

$$e_\alpha\left(t^{[j+1]}\right) := \left(\partial^\alpha H\right)\left(v^{j+1}\left(t^{[j+1]}\right)\right),\quad \hat{e}\left(t^{[j+1]}\right) := \left(e_\alpha\left(t^{[j+1]}\right)\right)_{1\le|\alpha|\le l}\,, \tag{9.14}$$

it follows from Lemma 8.1 that $\Lambda^1 = (e_\alpha(t^{[j+1]}))_{|\alpha|\le l}$ is also a solution of (9.1) and hence, from (9.12), we obtain

$$\sum_{\substack{\mu\in\mathbb{N}^{N'}\\ |\mu|\le|\epsilon|}} R_{\epsilon\mu}\left(\hat{e}\left(t^{[j+1]}\right)\right)\ \rho'_{Z'^{\mu}}\left(e_0\left(t^{[j+1]}\right), m_0\left(t^{[j+2]}\right)\right)\right)$$

$$= -\sum_{\substack{|\beta|+|\alpha|\le|\epsilon|\\ \beta<\epsilon}} P_{\epsilon\alpha\beta}\left(\widehat{m}\left(t^{[j+2]}\right)\right)$$

$$\sum_{\substack{\mu\in\mathbb{N}^{N'}\\ |\mu|\le|\beta|}} R_{\beta\mu}\left(\hat{e}\left(t^{[j+1]}\right)\right)\rho'_{Z'^{\mu}\zeta'^{\alpha}}\left(e_0\left(t^{[j+1]}\right), m_0\left(t^{[j+2]}\right)\right)\right)\ . \tag{9.15}$$

By (5.15) with $F^1 = H$, Z replaced by $v^{j+1}(t^{[j+1]})$ and ζ' replaced by $m_0(t^{[j+2]})$, we have for any $\beta \in \mathbb{N}^N$, $|\beta| \le l$ and any $\alpha \in \mathbb{N}^{N'}$,

$$^{H}\rho'_{Z^{\beta}\zeta'^{\alpha}}\left(v^{j+1}\left(t^{[j+1]}\right), m_0\left(t^{[j+2]}\right)\right)$$

$$= \sum_{\mu\in\mathbb{N}^{N'},\ |\mu|\le|\beta|} R_{\beta\mu}\left(\hat{e}\left(t^{[j+1]}\right)\right)\ \rho'_{Z'^{\mu}\zeta'^{\alpha}}\left(e_0\left(t^{[j+1]}\right), m_0\left(t^{[j+2]}\right)\right)\ . \tag{9.16}$$

By the induction hypothesis, since $\beta < \epsilon$ in the right-hand side of (9.15), we have, after differentiating (9.2) (with $\nu = \beta$) with respect to ζ' and replacing ζ' by $m_0(t^{[j+2]})$,

$$^{H}\rho'_{Z^{\beta}\zeta'^{\alpha}}\left(v^{j+1}\left(t^{[j+1]}\right), m_0\left(t^{[j+2]}\right)\right)$$

$$= \sum_{\mu\in\mathbb{N}^{N'},\ |\mu|\le|\beta|} R_{\beta\mu}\left(\hat{S}\left(t^{[j+1]}\right)\right)\ \rho'_{Z'^{\mu}\zeta'^{\alpha}}\left(S_0\left(t^{[j+1]}\right), m_0\left(t^{[j+2]}\right)\right)\ . \tag{9.17}$$

It follows from (9.13), (9.15), (9.16), and (9.17) that

$$\sum_{\substack{\mu\in\mathbb{N}^{N'}\\ |\mu|\le|\epsilon|}} R_{\epsilon\mu}\left(\hat{S}\left(t^{[j+1]}\right)\right)\rho'_{Z'^{\mu}}\left(S_0\left(t^{[j+1]}\right), m_0\left(t^{[j+2]}\right)\right)\right)$$

$$= \sum_{\substack{\mu\in\mathbb{N}^{N'}\\ |\mu|\le|\epsilon|}} R_{\epsilon\mu}\left(\hat{e}\left(t^{[j+1]}\right)\right)\ \rho'_{Z'^{\mu}}\left(e_0\left(t^{[j+1]}\right), m_0\left(t^{[j+2]}\right)\right)\right)\ . \tag{9.18}$$

Using (9.16) with $\beta = \epsilon$ and $\alpha = 0$, we obtain that (9.18) implies

$$^{H}\rho'_{Z^{\epsilon}}\left(v^{j+1}\left(t^{[j+1]}\right), m_0\left(t^{[j+2]}\right)\right) =$$

$$\sum_{\substack{\mu\in\mathbb{N}^{N'}\\ |\mu|\le|\epsilon|}} R_{\epsilon\mu}\left(\hat{S}\left(t^{[j+1]}\right)\right)\rho'_{Z'^{\mu}}\left(S_0\left(t^{[j+1]}\right), m_0\left(t^{[j+2]}\right)\right)\right)\ . \tag{9.19}$$

To prove (9.2) for $\nu = \epsilon$, we must show that (9.19) still holds if $m_0(t^{[j+2]}) = \bar{H}(\bar{v}^{j+2}(t^{[j+2]}))$ is replaced by an arbitrary $\zeta' = (\chi', \tau') \in \mathbb{C}^{N'}$. Observe that for $\mu \in \mathbb{N}^{N'}$, $|\mu| > 0$, we have in view of (6.3), $\rho'_{Z'^{\mu}}(Z', \zeta') = -\bar{Q}'_{Z'^{\mu}}(\chi', Z') := a_{\mu}(Z', \chi')$ and since $|\epsilon| > 0$, $^{H}\rho'_{Z^{\epsilon}}(Z, \zeta') = -\partial_Z^{\epsilon}\left[\bar{Q}'(\chi', H(Z))\right] := b_{\epsilon}(Z, \chi')$. Recall also that since $\epsilon \ne 0$, $R_{\epsilon 0} = 0$ (see Section 5). Hence, (9.19) may be rewritten in the form

$$b_{\epsilon}\left(v^{j+1}\left(t^{[j+1]}\right), \bar{f}(\bar{v}^{j+2}\left(t^{[j+2]}\right)\right)\right)$$

$$= \sum_{\substack{\mu\in\mathbb{N}^{N'}\\ 0<|\mu|\le|\epsilon|}} R_{\epsilon\mu}\left(\hat{S}\left(t^{[j+1]}\right)\right) a_{\mu}\left(S_0\left(t^{[j+1]}\right), \bar{f}\left(\bar{v}^{j+2}\left(t^{[j+2]}\right)\right)\right)\ , \tag{9.20}$$

and we must show that (9.20) still holds with $\bar{f}(\bar{v}^{j+2}(t^{[j+2]}))$ replaced by an arbitrary $\chi' \in \mathbb{C}^{n'}$. For this, one can apply the same rank argument using the map B defined in (9.9) which was already used in the case $\nu = 0$. This completes the proof of (9.2).

To complete the proof of Lemma 9.1, it suffices to observe that the equivalence of (9.2) and (9.3) follows from (5.15). □

10. Proof of Theorem 2.5

For the proof of Theorem 2.5, we shall need Proposition 10.1 given below. We assume that $\mathcal{M}$, $\mathcal{M}'$ and the iterated Segre mappings v^j are as in Sectin 8 and continue to use the notation of that section. In particular, we still assume that $\rho'(Z',\zeta')$ and $\tilde{\rho}'(Z',\zeta')$ are the special choice of generators of $\mathcal{I}(\mathcal{M}')$ given by (6.3) and (6.4).

Proposition 10.1. *Let $\mathcal{M} \subset \mathbb{C}^N \times \mathbb{C}^N$ and $\mathcal{M}' \subset \mathbb{C}^{N'} \times \mathbb{C}^{N'}$ be formal generic manifolds. Let $H^0 : (\mathbb{C}^N,0) \to (\mathbb{C}^{N'},0)$ be a formal mapping such its complexification $\mathcal{H}^0$ sends $\mathcal{M}$ into $\mathcal{M}'$. Assume that H^0 is not totally degenerate (as in Definition 2.4). Then, for every pair of nonnegative integers l, j, there exists a positive integer $K = K(H^0,l,j)$ such that if $H : (\mathbb{C}^N,0) \to (\mathbb{C}^{N'},0)$ is a formal map whose complexification $\mathcal{H}$ maps $\mathcal{M}$ into $\mathcal{M}'$ and such that*

$$ {}^{H^0}\rho'_{Z^\delta}\left(v^j\left(t^{[j]}\right),\zeta'\right) = {}^{H}\rho'_{Z^\delta}\left(v^j\left(t^{[j]}\right),\zeta'\right), \quad |\delta| \le K\ , \tag{10.1} $$

then,

$$ {}^{H^0}\rho'_{Z^\delta}\left(v^{j+1}\left(t^{[j+1]}\right),\zeta'\right) = {}^{H}\rho'_{Z^\delta}\left(v^{j+1}\left(t^{[j+1]}\right),\zeta'\right), \quad |\delta| \le l\ . \tag{10.2} $$

Here ${}^{H}\rho'(Z,\zeta')$ and ${}^{H^0}\rho'(Z,\zeta')$ are the formal mappings given by (5.2) with the choice (6.3) of ρ'.

Proof. We fix the pair of nonnegative integers l, j. In the ring $R := \mathbb{C}[\![\Lambda_0^1, t^{[j+1]}]\!][\hat{\Lambda}^1]$, where $\Lambda^1 = (\Lambda_0^1, \hat{\Lambda}^1)$ are coordinates on $J_0^l(\mathbb{C}^N,\mathbb{C}^{N'})$ as in (4.1) and (4.2), we consider the ideal $\mathcal{J}$ generated by the components of the formal mappings

$$ \tilde{\Theta}^{[l,j]}_{\nu,\epsilon}\left(H^0;\Lambda^1,t^{[j+1]}\right), \quad \nu \in \mathbb{N}^N,\ |\nu| \le l,\ \epsilon \in \mathbb{N}^n\ , $$

where the $\tilde{\Theta}^{[l,j]}_{\nu,\epsilon}(H^0;\cdot)$ are given by (8.13). Since R is Noetherian, there exists a positive integer $L = L(H^0,l,j)$ such that the ideal $\mathcal{J}$ is generated by the components of the formal mappings

$$ \tilde{\Theta}^{[l,j]}_{\nu,\epsilon}\left(H^0;\Lambda^1,t^{[j+1]}\right), \quad \nu \in \mathbb{N}^N,\ |\nu| \le l,\ \epsilon \in \mathbb{N}^n,\ |\epsilon| \le L\ . $$

We claim that the conclusion of Proposition 10.1 holds with $K := L + l$. Indeed, let $H : (\mathbb{C}^N,0) \to (\mathbb{C}^{N'},0)$ be a formal map whose complexification sends $\mathcal{M}$ into $\mathcal{M}'$ and such that (10.1) holds (with this choice of K). We must prove that (10.2) holds. By (5.4), we have

$$ \tilde{\rho}'^{H}\left(Z',\zeta\right) = \overline{{}^{H}\rho'}\left(\zeta,Z'\right), \quad \tilde{\rho}'^{H^0}\left(Z',\zeta\right) = \overline{{}^{H^0}\rho'}\left(\zeta,Z'\right)\ , \tag{10.3} $$

and hence it follows from (10.1) that

$$ \tilde{\rho}'^{H^0}_{Z'^\alpha\zeta^\delta}\left(Z',\bar{v}^j\left(t^{[j]}\right)\right) = \tilde{\rho}'^{H}_{Z'^\alpha\zeta^\delta}\left(Z',\bar{v}^j\left(t^{[j]}\right)\right), \quad \alpha \in \mathbb{N}^{N'},\ |\delta| \le K\ , \tag{10.4} $$

where we have considered $\tilde{\rho}'^{H}(Z',\zeta)$ and $\tilde{\rho}'^{H^0}(Z',\zeta)$ as formal mappings in the indeterminates (Z',ζ) as in (5.3). As a consequence of (10.4), (8.14) and the choice of K, it follows that

$$\tilde{\Theta}^{[l,j]}_{\nu,\epsilon}\left(H;\Lambda^1,t^{[j+1]}\right)=\tilde{\Theta}^{[l,j]}_{\nu,\epsilon}\left(H^0;\Lambda^1,t^{[j+1]}\right),\quad |\nu|\le l,\quad |\epsilon|\le L\,. \tag{10.5}$$

By Lemma 8.1,

$$\Lambda^1=\left(\left(\partial^\alpha H\right)\left(v^{j+1}\left(t^{[j+1]}\right)\right)\right)_{|\alpha|\le l} \tag{10.6}$$

is a formal solution of the system of equations $\tilde{\psi}^{[l,j]}(H;\Lambda^1,t^{[j+2]})=0$, and hence by Remark 8.4 (since (10.6) is independent of the indeterminate t^{j+2}), it is also a solution of the system of equations $\tilde{\Theta}^{[l,j]}_{\nu,\epsilon}(H;\Lambda^1,t^{[j+1]})=0$ for $|\nu|\le l$ and all $\epsilon\in\mathbb{N}^n$. From (10.5), we conclude that (10.6) is also a solution of the system of equations

$$\tilde{\Theta}^{[l,j]}_{\nu,\epsilon}\left(H^0;\Lambda^1,t^{[j+1]}\right)=0,\quad |\nu|\le l,\quad |\epsilon|\le L\,.$$

By the choice of L, it follows that the formal power series mapping given by (10.6) is a solution of the system of equations

$$\tilde{\Theta}^{[l,j]}_{\nu,\epsilon}\left(H^0;\Lambda^1,t^{[j+1]}\right)=0,\quad |\nu|\le l,\quad \forall\epsilon\in\mathbb{N}^n\,.$$

Again making use of Remark 8.4, we conclude that (10.6) is a formal solution of the system of equations $\tilde{\psi}^{[l,j]}(H^0;\Lambda^1,t^{[j+2]})=0$ and hence, by Lemma 8.1, also a solution of the system of equations $\psi^{[l,j]}(H^0;\Lambda^1,t^{[j+2]})=0$. We may now apply Lemma 9.1 with H and H^0 interchanged and with $S(t^{[j+1]})=\left((\partial^\alpha H)(v^{j+1}(t^{[j+1]}))\right)_{|\alpha|\le l}$ to conclude that (9.3) holds, which is the desired conclusion (10.2) of Proposition 10.1. □

Proof of Theorem 2.5. Since $\mathcal{M}$ is of finite type, it follows from Theorem 2.3 in [5] (see also [7]) and the definition of finite type given in Section 2, that there exists an integer j_0, $2\le j_0\le d+1$, where d is the codimension of $\mathcal{M}$ such that Rk $v^{j_0}=N$. By applying Proposition 10.1 j_0 times, we conclude that there exists an integer $K_0=K_0(H^0)>0$[3] such that if $H:(\mathbb{C}^N,0)\to(\mathbb{C}^{N'},0)$ is a formal map whose complexification $\mathcal{H}$ sends $\mathcal{M}$ into $\mathcal{M}'$ and such that

$$^{H^0}\rho'_{Z^\delta}\left(v^0,\zeta'\right)={}^{H}\rho'_{Z^\delta}\left(v^0,\zeta'\right),\quad |\delta|\le K_0\,, \tag{10.7}$$

then,

$$^{H^0}\rho'\left(v^{j_0}\left(t^{[j_0]}\right),\zeta'\right)={}^{H}\rho'\left(v^{j_0}\left(t^{[j_0]}\right),\zeta'\right)\,. \tag{10.8}$$

Recall that $v^0=0\in\mathbb{C}^N$, and hence we may rewrite (10.7) in the form

$$\partial_Z^\delta\left[\rho'\left(H(Z),\zeta'\right)\right]\big|_{Z=0}=\partial_Z^\delta\left[\rho'\left(H^0(Z),\zeta'\right)\right]\Big|_{Z=0},\quad |\delta|\le K_0\,. \tag{10.9}$$

[3]To find K_0, we proceed as follows. We define inductively a finite sequence of nonnegative integers K_q, $0\le q\le j_0$, by putting $K_{j_0}=0$ and $K_q=K(H^0,K_{q+1},q)$ where $K(H^0,l,j)$ is the integer given by Proposition 10.1.

It is, then, clear that if H is a formal map such that $\mathcal{H}$ sends $\mathcal{M}$ into $\mathcal{M}'$ with $j_0^{K_0}H = j_0^{K_0}H^0$, then (10.9) and hence (10.7) and (10.8) hold. Since Rk $v^{j_0} = N$, it follows e. g., from Proposition 5.3.5 of [6] that (10.8) implies

$$ {}^{H^0}\rho'\left(Z,\zeta'\right) = {}^{H}\rho'\left(Z,\zeta'\right) . \tag{10.10}$$

From the definition of the reflection ideal $\mathcal{I}^H$ given in (2.6), we conclude that (10.10) implies that the reflection ideals $\mathcal{I}^H$ and $\mathcal{I}^{H^0}$ are the same. The proof of Theorem 2.5 is complete. □

11. Proof of Theorem 2.6

In this section, we consider two germs $(M,0)$ and $(M',0)$ of real-analytic generic submanifolds in $\mathbb{C}^N$ and $\mathbb{C}^{N'}$, respectively. We let $\mathcal{M} \subset \mathbb{C}^N_Z \times \mathbb{C}^N_\zeta$ and $\mathcal{M}' \subset \mathbb{C}^{N'}_{Z'} \times \mathbb{C}^{N'}_{\zeta'}$ be their complexifications. For generators of $\mathcal{I}(\mathcal{M}')$, we take a convergent mapping $\rho'(Z',\zeta')$ as in (6.3). We shall also use the corresponding notation for $\tilde{\rho}'(Z',\zeta')$ given by (6.4). Moreover, we choose a convergent Segre variety mapping γ relative to $\mathcal{M}$ as defined in (7.1); hence the corresponding iterated Segre mappings v^j defined in (8.1) are also convergent. Using the notation of Section 8, we have the following proposition.

Proposition 11.1. *Let $(M,0)$ and $(M',0)$ be germs of generic real-analytic submanifolds in $\mathbb{C}^N$ and $\mathbb{C}^{N'}$ of codimension d and d', respectively. Let $H : (\mathbb{C}^N,0) \to (\mathbb{C}^{N'},0)$ be a formal map sending M into M'. Assume that H is not totally degenerate (as in Definition 2.4). Then, for every nonnegative integer j, the following holds. If*

$$ {}^{H}\rho'_{Z^\beta}\left(v^j\left(t^{[j]}\right),\zeta'\right) \in \left(\mathbb{C}\left\{t^{[j]},\zeta'\right\}\right)^{d'}, \quad \forall \beta \in \mathbb{N}^N , \tag{11.1}$$

then,

$$ {}^{H}\rho'_{Z^\beta}\left(v^{j+1}\left(t^{[j+1]}\right),\zeta'\right) \in \left(\mathbb{C}\left\{t^{[j+1]},\zeta'\right\}\right)^{d'}, \quad \forall \beta \in \mathbb{N}^N . \tag{11.2}$$

Here ${}^H\rho'(Z,\zeta')$ is the formal mapping given by (5.2) *relative to the choice of the convergent mapping $\rho'(Z',\zeta')$ given by* (6.3).

Proof. We fix a pair of nonnegative integers l, j, and we shall prove that if (11.1) holds, then,

$$ {}^{H}\rho'_{Z^\nu}\left(v^{j+1}\left(t^{[j+1]}\right),\zeta'\right) \in \left(\mathbb{C}\left\{t^{[j+1]},\zeta'\right\}\right)^{d'}, \quad \forall \nu \in \mathbb{N}^N,\ |\nu| \le l . \tag{11.3}$$

The proposition will clearly follow.

It follows from (11.1) and (5.4) that one has

$$ \tilde{\rho}'^H_{\zeta^\nu}\left(Z',\bar{v}^j\left(t^{[j]}\right)\right) \in \left(\mathbb{C}\left\{Z',t^{[j]}\right\}\right)^{d'}, \quad \forall \nu \in \mathbb{N}^N , \tag{11.4}$$

where $\tilde{\rho}'^H(Z',\zeta)$ is the formal mapping given by (5.3). It follows from Lemma 8.3 and (11.4) that the components of the formal power series mappings $\tilde{\Theta}^{[l,j]}_{\nu,\epsilon}(H;\Lambda^1,t^{[j+1]})$, for $\nu \in \mathbb{N}^N$, $|\nu| \le l$, and $\epsilon \in \mathbb{N}^n$, [defined by (8.13)], are in the ring $\mathbb{C}\{\Lambda^1_0,t^{[j+1]}\}[\hat{\Lambda}^1]$. (We should observe at this point that the components of the formal mappings $\tilde{\psi}^{[l,j]}(H;\Lambda^1,t^{[j+2]})$, defined in (8.4), are not yet known to be convergent.) By Lemma 8.1, it follows that $\Lambda^1 = \left((\partial^\alpha H)(v^{j+1}(t^{[j+1]}))\right)_{|\alpha|\le l}$ is a formal solution of the system of equations

$$ \tilde{\psi}^{[l,j]}\left(H;\Lambda^1,t^{[j+2]}\right) = 0 , \tag{11.5}$$

and hence by Remark 8.4, it is also a formal solution of the system of equations

$$\tilde{\Theta}^{[l,j]}_{\nu,\epsilon}\left(H;\Lambda^1,t^{[j+1]}\right)=0,\quad |\nu|\le l,\ \epsilon\in\mathbb{N}^n\ . \tag{11.6}$$

Since the mappings $\tilde{\Theta}^{[l,j]}_{\nu,\epsilon}(H;\cdot)$ are convergent, it follows from Artin's approximation theorem [2], Theorem (1.2), that there exists a convergent solution of (11.6) given by $\Lambda^1=S(t^{[j+1]})=(S_0(t^{[j+1]}),\hat{S}(t^{[j+1]}))$, where

$$S:\left(\mathbb{C}^{n(j+1)},0\right)\to J^l_0\left(\mathbb{C}^N,\mathbb{C}^{N'}\right),\quad S(0)=j^l_0H\ . \tag{11.7}$$

Since the convergent mapping $S(t^{[j+1]})$ is independent of the variable t^{j+2}, it follows from Remark 8.4 that $\Lambda^1=S(t^{[j+1]})$ is also a solution of the system of equations given by (11.5). Hence, by Lemma 8.1, $\Lambda^1=S(t^{[j+1]})$ is a solution of the system of equations

$$\psi^{[l,j]}\left(H;\Lambda^1,t^{[j+2]}\right)=0\ .$$

We may now apply Lemma 9.1 for the convergent solution $S(t^{[j+1]})$ to obtain (9.2). To conclude that (11.3) holds, it suffices to observe that the right-hand side of (9.2) is a convergent map. This completes the proof of Proposition 11.1. □

Proof of Theorem 2.6. Since M is of finite type at 0, by Theorem 10.5.5 of [6] (see also [10] and [5]), there exists an integer k_0, $2\le k_0\le 2(d+1)$ (where d is the codimension of M) such that in any neighborhood U of $0\in\mathbb{C}^{nk_0}$, there exists $t_0^{[k_0]}\in U$ such that

$$\mathrm{rk}\ \frac{\partial v^{k_0}}{\partial t^{[k_0]}}\left(t_0^{[k_0]}\right)=N,\quad v^{k_0}\left(t_0^{[k_0]}\right)=0\ . \tag{11.8}$$

Since $v^0=0\in\mathbb{C}^N$, we observe that for any multiindex $\beta\in\mathbb{N}^N$,

$${}^H\rho'_{Z^\beta}\left(v^0,\zeta'\right)=\partial_Z^\beta\left[\rho'\left(H(Z),\zeta'\right)\right]\big|_{Z=0}\in\left(\mathbb{C}\left\{\zeta'\right\}\right)^{d'},\quad\forall\beta\in\mathbb{N}^N\ . \tag{11.9}$$

Applying Proposition 11.1 k_0 times, we conclude in particular that

$${}^H\rho'\left(v^{k_0}\left(t^{[k_0]}\right),\zeta'\right)\in\left(\mathbb{C}\left\{t^{[k_0]},\zeta'\right\}\right)^{d'}\ . \tag{11.10}$$

Hence there exists an open neighborhood $U\times V\subset\mathbb{C}^{nk_0}\times\mathbb{C}^{N'}$ of 0 where the mapping ${}^H\rho'(v^{k_0}(t^{[k_0]}),\zeta')$ is convergent. If we choose $t_0^{k_0}\in U$ such that (11.8) holds and apply the rank theorem, we obtain that the mapping ${}^H\rho'(Z,\zeta')$ is convergent. By the definition of the reflection ideal $\mathcal{I}^H$ given in (2.6) and Definition 2.3, it follows that $\mathcal{I}^H$ is convergent. This completes the proof of Theorem 2.6. □

Remark 11.2. As mentioned in Section 1, Theorem 2.6 was first proved in [24] for an invertible formal map H and in the case where M and M' are real-analytic hypersurfaces in $\mathbb{C}^N$. We should point out here that the techniques used in this article are somewhat different from those of [24]. For instance, the use of Cauchy estimates was a crucial tool in [24], but is not needed in our approach in this article. We should also note that Corollary 7.4 and Theorem 7.1 in [24], which are proved there in the case of invertible formal mappings between real-analytic hypersurfaces of finite type, can be extended to the case of finite formal mappings between generic real-analytic submanifolds of finite type of $\mathbb{C}^N$ by making use of Theorem 2.6. We do not give any further details.

12. Proof of Theorem 2.7

In this section, we consider two germs $(M, 0)$ and $(M', 0)$ of real-algebraic generic submanifolds in $\mathbb{C}^N$ and $\mathbb{C}^{N'}$, respectively. We let $\mathcal{M} \subset \mathbb{C}^N_Z \times \mathbb{C}^N_\zeta$ and $\mathcal{M}' \subset \mathbb{C}^{N'}_{Z'} \times \mathbb{C}^{N'}_{\zeta'}$ be their complexifications. For generators of $\mathcal{I}(\mathcal{M}')$, we take the components of an algebraic mapping $\rho'(Z', \zeta')$ as in (6.3). We also use the corresponding notation for $\tilde{\rho}'(Z', \zeta')$ given by (6.4). Moreover, we choose an algebraic Segre variety mapping γ relative to $\mathcal{M}$ as defined in (7.1); hence the corresponding iterated Segre mappings v^j defined in (8.1) are also algebraic. We have the following analog of Theorem 2.6 for generic real-algebraic submanifolds.

Theorem 12.1. *Let $(M, 0)$ and $(M', 0)$ be germs of real-algebraic generic submanifolds in $\mathbb{C}^N$ and $\mathbb{C}^{N'}$ respectively and $H : (\mathbb{C}^N, 0) \to (\mathbb{C}^{N'}, 0)$ a formal map sending M into M'. Assume that M is of finite type at 0 and H is not totally degenerate. Then, the reflection ideal $\mathcal{I}^H$, as defined by (2.6), is algebraic.*

This theorem will be used in the proof of Theorem 2.7 in the case where H is a convergent mapping. The proof of Theorem 12.1 follows the same lines as that of Theorem 2.6, by making use of the following analog of Proposition 11.1 in the algebraic setting.

Proposition 12.2. *Let $(M, 0)$ and $(M', 0)$ be germs of generic real-algebraic submanifolds in $\mathbb{C}^N$ and $\mathbb{C}^{N'}$ of codimension d and d' respectively. Let $H : (\mathbb{C}^N, 0) \to (\mathbb{C}^{N'}, 0)$ be a formal map sending M into M'. Assume that H is not totally degenerate (as in Definition 2.4). Then, for every nonnegative integer j, the following holds. If*

$$ {}^H\rho'_{Z^\beta}\left(v^j\left(t^{[j]}\right), \zeta'\right) \in \left(\mathcal{A}\left\{t^{[j]}, \zeta'\right\}\right)^{d'}, \quad \forall \beta \in \mathbb{N}^N , \tag{12.1} $$

then,

$$ {}^H\rho'_{Z^\beta}\left(v^{j+1}\left(t^{[j+1]}\right), \zeta'\right) \in \left(\mathcal{A}\left\{t^{[j+1]}, \zeta'\right\}\right)^{d'}, \quad \forall \beta \in \mathbb{N}^N . \tag{12.2} $$

Here ${}^H\rho'(Z, \zeta')$ is the formal mapping given by (5.2) relative to the choice of the algebraic mapping $\rho'(Z', \zeta')$ given by (6.3).

Proof. The proof of this proposition follows very closely that of Proposition 11.1. One has to note that all the convergent mappings involved in the latter are also algebraic in the present case. Also, the convergent solution $S(t^{[j+1]})$ of the system (11.6), given in (11.7) and obtained by making use of Artin's approximation theorem, can be chosen to be algebraic. Indeed, in the present case, the mappings involved in (11.6) are algebraic and another version of Artin's approximation theorem [1] yields a solution which is also algebraic. We omit further details. □

Proof of Theorem 2.7. Choose $U, U' \subset \mathbb{C}^N$, two open polydiscs centered at the origin such that H is holomorphic in U and $H(U \cap M) \subset U' \cap M'$. We may assume that the real-algebraic generic submanifold $M' \subset \mathbb{C}^N_{Z'}$ is given by $\tilde{\rho}'(Z', \bar{Z}') = 0$ where

$$ \tilde{\rho}'\left(Z', \bar{Z}'\right) := w' - Q'\left(z', \bar{Z}'\right), \quad Z' = \left(z', w'\right) \in \mathbb{C}^n \times \mathbb{C}^d , \tag{12.3} $$

with $\tilde{\rho}'(Z', \zeta')$ a $\mathbb{C}^d$ valued algebraic map defined in $U' \times U'$. Here we recall that d is the codimension of M (and of M') and $n = N - d$. Equivalently, M' is also given by $\rho'(Z', \bar{Z}') = 0$ where

$$ \rho'\left(Z', \bar{Z}'\right) := \bar{w}' - \bar{Q}'\left(\bar{z}', Z'\right) . \tag{12.4} $$

To prove Theorem 2.7, by Proposition 6.1 (iii), it suffices to show that the convergent generators ${}^H\rho'(Z,\zeta')$ of the reflection ideal $\mathcal{I}^H$ are algebraic, where we have used the notation given by (2.6) and (5.2). Since the Jacobian of H is not identically zero and there is no germ at 0 of a nonconstant holomorphic function $h:(\mathbb{C}^N,0)\to\mathbb{C}$ with $h(M)\subset\mathbb{R}$, it follows that there exists $p_0\in U\cap M$ such that M is of finite type at p_0 and the Jacobian of H at p_0 is not zero (see e. g., Lemma 13.3.2 of [6]). Put $p_0':=H(p_0)\in U'\cap M'$. We define the translation maps $\varphi_{p_0}(Z):=Z-p_0$ and $\varphi_{p_0'}(Z'):=Z'-p_0'$. We put $M_{p_0}:=\varphi_{p_0}(M)$ and $M'_{p_0'}:=\varphi_{p_0'}(M')$. Observe that M_{p_0} and $M'_{p_0'}$ are real-algebraic generic submanifolds through the origin in $\mathbb{C}^N$ with M_{p_0} of finite type at 0. We also define

$$\check{H}\left(\check{Z}\right):=\left(\varphi_{p_0'}\circ H\circ\varphi_{p_0}^{-1}\right)\left(\check{Z}\right) \tag{12.5}$$

for $\check{Z}$ close enough to the origin in $\mathbb{C}^N$. We can regard $\check{H}$ as a germ at the origin of a biholomorphism sending the germ $(M_{p_0},0)$ onto $(M'_{p_0'},0)$. Note also that the germ $(M'_{p_0'},0)$ is defined by $\check{\rho}'(\check{Z}',\overline{\check{Z}'})=0$ where

$$\check{\rho}'\left(\check{Z}',\check{\zeta}'\right):=\check{\tau}'+\bar{w}'_{p_0'}-\bar{Q}'\left(\check{\chi}'+\bar{z}'_{p_0'},\check{Z}'+p_0'\right),\quad \check{\zeta}'=\left(\check{\chi}',\check{\tau}'\right)\in\mathbb{C}^n\times\mathbb{C}^d\ , \tag{12.6}$$

with $p_0'=(z'_{p_0'},w'_{p_0'})\in\mathbb{C}^n\times\mathbb{C}^d$. It follows from Theorem 12.1 and Proposition 6.1 (iii) that the convergent mapping ${}^{\check{H}}\check{\rho}'(\check{Z},\check{\zeta}')=\check{\rho}'(\check{H}(\check{Z}),\check{\zeta}')$ is in $(\mathcal{A}\{\check{Z},\check{\zeta}\})^d$, i. e., that the components of the map

$$\left(\mathbb{C}^N\times\mathbb{C}^n,0\right)\ni\left(\check{Z},\check{\chi}'\right)\mapsto\bar{Q}'\left(\check{\chi}'+\bar{z}'_{p_0'},\check{H}\left(\check{Z}\right)+p_0'\right)\in\mathbb{C}^d$$

are in $\mathcal{A}\{\check{Z},\check{\zeta}\}$. In view of (12.5), we conclude that the map

$$\left(\mathbb{C}^N\times\mathbb{C}^n,\left(p_0,\bar{z}'_{p_0'}\right)\right)\ni\left(Z,\chi'\right)\mapsto\bar{Q}'\left(\chi',H(Z)\right)\in\mathbb{C}^d$$

is algebraic i. e., each component of this map satisfies a non-trivial polynomial equation with polynomial coefficients for Z near p_0 and χ' near $\bar{z}'_{p_0'}$. By unique continuation, the same equations hold for (Z,χ') close to $0\in\mathbb{C}^{N+n}$. This shows that the components of ${}^H\rho'(Z,\zeta')$ are in $\mathcal{A}\{Z,\zeta'\}$ which gives the desired conclusion of Theorem 2.7. □

13. Proofs of Propositions 2.9, 2.10, and 2.12 and Theorems 3.1 and 3.2

In this section, we consider a formal generic manifold $\mathcal{M}'\subset\mathbb{C}^{N'}_{Z'}\times\mathbb{C}^{N'}_{\zeta'}$ of codimension d' and we assume that the ideal $\mathcal{I}(\mathcal{M}')$ is generated by the components of the formal map $\rho'(Z',\zeta')$ given by (6.3). We write

$$\rho'(Z',\zeta')=\tau'-\bar{Q}'\left(\chi',Z'\right)=\tau'-\sum_{\alpha\in\mathbb{N}^{n'}}q_\alpha\left(Z'\right)\chi'^{\alpha}\ , \tag{13.1}$$

where the $q_\alpha(Z')=\left(q_{1,\alpha}(Z'),\ldots,q_{d',\alpha}(Z')\right)$ are in $(\mathbb{C}[\![Z']\!])^{d'}$ and $n'=N'-d'$.

The proof of the following criterion for holomorphic nondegeneracy of formal generic manifolds is left to the reader (see e. g., [28] and [6], Chapter 11, for the case where $\mathcal{M}'$ is the complexification of a real-analytic generic submanifold).

Lemma 13.1. *The formal generic manifold $\mathcal{M}'$ as above is holomorphically nondegenerate if and only if there exist $\alpha^1, \ldots, \alpha^{N'} \in \mathbb{N}^{n'}$ and $j_1, \ldots, j_{N'} \in \{1, \ldots, d'\}$ such that*

$$\det\left(\frac{\partial q_{j_l,\alpha^l}}{\partial Z'_m}\left(Z'\right)\right)_{1\le l,m\le N'} \neq 0, \quad \text{in } \mathbb{C}[\![Z']\!], \tag{13.2}$$

where the formal power series $q_\alpha(Z')$ are given by (13.1).

We also need the following lemma for the proof of Proposition 2.10.

Lemma 13.2. *Let $R(x, y) = (R_1(x, y), \ldots, R_r(x, y)) \in (\mathbb{C}[\![x, y]\!])^r$, $x \in \mathbb{C}^q$, $y \in \mathbb{C}^r$, and $h^0 : (\mathbb{C}^q, 0) \to (\mathbb{C}^r, 0)$ be a formal map such that*

(i) $R(x, h^0(x)) = 0$,

(ii) $\det\left(\dfrac{\partial R_i}{\partial y_j}(x, h^0(x))\right)_{1\le i,j\le r} \neq 0.$

Then, there exists a positive integer $k = k(h^0)$ such that the following holds. If $h : (\mathbb{C}^q, 0) \to (\mathbb{C}^r, 0)$ is a formal map such that $R(x, h(x)) = 0$ and $j_0^k h = j_0^k h^0$, then necessarily $h(x) = h^0(x)$.

Proof. We may write

$$R(x, y) - R(x, t) = P(x, y, t) \cdot (y - t) \tag{13.3}$$

where P is an $r \times r$ matrix with entries in $\mathbb{C}[\![x, y, t]\!]$ satisfying $P(x, y, y) = \dfrac{\partial R}{\partial y}(x, y)$. By assumption, we know that $\det\ P(x, h^0(x), h^0(x)) \neq 0$. This implies that one can find an integer k such that if $h : (\mathbb{C}^q, 0) \to (\mathbb{C}^r, 0)$ is a formal mapping which agrees up to order k with h^0, then $\det\ P(x, h^0(x), h(x)) \neq 0$. If, in addition, h satisfies $R(x, h(x)) = 0$, it follows from (13.3) that $P(x, h^0(x), h(x))\cdot(h^0(x)-h(x)) = 0$ in $\mathbb{C}[[x]]$. Since $\det\ P(x, h^0(x), h(x)) \neq 0$, we conclude that $h(x) = h^0(x)$ and hence the lemma follows. □

Proof of Proposition 2.10. First observe that if $H, H^0 : (\mathbb{C}^N, 0) \to (\mathbb{C}^{N'}, 0)$ are two formal mappings with $\mathcal{I}^H = \mathcal{I}^{H^0}$, then by Proposition 6.1 (i) and in view of (13.1), necessarily for any $\alpha \in \mathbb{N}^{n'}$, $q_\alpha \circ H = q_\alpha \circ H^0$. Since $\mathcal{M}'$ is holomorphically nondegenerate, we may choose $\alpha^1, \ldots, \alpha^{N'} \in \mathbb{N}^{N'}$ and $j_1, \ldots, j_{N'} \in \{1, \ldots, d'\}$ as in Lemma 13.1. For any $l = 1, \ldots, N'$, we define a formal map $R_l : (\mathbb{C}^N \times \mathbb{C}^{N'}, 0) \to (\mathbb{C}, 0)$ as follows

$$R_l\left(Z, Z'\right) := q_{j_l,\alpha^l}\left(Z'\right) - q_{j_l,\alpha^l}\left(H^0(Z)\right). \tag{13.4}$$

Observe that $R_l(Z, H^0(Z)) = 0$, for $l = 1, \ldots, N'$, and moreover, since Rk $H^0 = N'$, by (13.2) and e. g., Proposition 5.3.5 in [6], we have

$$\det\left(\frac{\partial q_{j_l,\alpha^l}}{\partial Z'_m}\left(H^0(Z)\right)\right)_{1\le l,m\le N'} \neq 0\,, \tag{13.5}$$

or equivalently,

$$\det\left(\frac{\partial R_l}{\partial Z'_m}\left(Z, H^0(Z)\right)\right)_{1\le l,m\le N'} \neq 0\,.$$

By Lemma 13.2, there exists a positive integer $k = k(H^0)$ such that if $H : (\mathbb{C}^N, 0) \to (\mathbb{C}^{N'}, 0)$ is a formal map satisfying $R_l(Z, H(Z)) = 0$, for $l = 1, \ldots, N'$, and $j_0^k H = j_0^k H^0$, then

$H = H^0$. On the other hand, as mentioned in the beginning of the proof, if $\mathcal{I}^H = \mathcal{I}^{H^0}$, then $R_l(Z, H(Z)) = R_l(Z, H^0(Z)) = 0$, for $l = 1, \ldots, N'$. This completes the proof of Proposition 2.10. □

Proof of Proposition 2.9. Since M' is real-analytic, we may assume that the corresponding formal mappings $\rho'(Z', \zeta')$ and $q_\alpha(Z')$ given in (13.1) are convergent. First, note if there exists a convergent map $\check{H} : (\mathbb{C}^N, 0) \to (\mathbb{C}^{N'}, 0)$ such that $\mathcal{I}^H = \mathcal{I}^{\check{H}}$, then, since $\mathcal{I}^{\check{H}}$ is convergent, so is $\mathcal{I}^H$. Now, assume that $\mathcal{I}^H$ is convergent. By Proposition 6.1 (ii) and in view of (13.1),

$$r_\alpha(Z) := q_\alpha(H(Z)) \tag{13.6}$$

is a convergent mapping for all $\alpha \in \mathbb{N}^{n'}$. By Artin's approximation theorem [2], Theorem (1.2), for any positive integer κ, there exists a convergent map $H^\kappa : (\mathbb{C}^N, 0) \to (\mathbb{C}^{N'}, 0)$ which agrees with H up to order κ and such that $q_\alpha(H^\kappa(Z)) = r_\alpha(Z)$, for all $\alpha \in \mathbb{N}^{n'}$. It follows from (13.1) and (13.6) that $\rho'(H^\kappa(Z), \zeta') = \rho'(H(Z), \zeta')$ and hence

$$\mathcal{I}^H = \left(\rho'\left(H(Z), \zeta'\right)\right) = \left(\rho'\left(H^\kappa(Z), \zeta'\right)\right) = \mathcal{I}^{H^\kappa} .$$

This completes the proof of Proposition 2.9 in the convergent case. In the case when M' is real-algebraic, the $q_\alpha(Z')$ given by (13.1) are algebraic. As before, if there exists an algebraic map $\check{H} : (\mathbb{C}^N, 0) \to (\mathbb{C}^{N'}, 0)$ such that $\mathcal{I}^H = \mathcal{I}^{\check{H}}$, then, since $\mathcal{I}^{\check{H}}$ is algebraic, so is $\mathcal{I}^H$. Moreover, it follows from the algebraic version of Artin's theorem [1] that, in this case, one can choose H^κ as above to be algebraic so that $\mathcal{I}^H = \mathcal{I}^{H^\kappa}$. The proof of the proposition is now complete. □

For the proof of Proposition 2.12, we need the following lemma whose proof is in the spirit of that of Lemma 13.2 but also makes use of Artin's approximation theorem [2]. We refer the reader to Proposition 4.2 of [24] for the proof of this lemma.

Lemma 13.3. *Let $R(x, y) = (R_1(x, y), \ldots, R_r(x, y)) \in (\mathbb{C}\{x, y\})^r$, $x \in \mathbb{C}^q$, $y \in \mathbb{C}^r$, and $h : (\mathbb{C}^q, 0) \to (\mathbb{C}^r, 0)$ a formal map satisfying $R(x, h(x)) = 0$. If $\det\left(\frac{\partial R}{\partial y}(x, h(x))\right) \neq 0$ in $\mathbb{C}[[x]]$, then $h(x)$ is convergent.*

Proof of Proposition 2.12. By Proposition 6.1 (ii), if $\mathcal{I}^H$ is convergent, then, in view of (13.1), it follows that for any $\alpha \in \mathbb{N}^{n'}$ and $j = 1, \ldots, d'$, the formal power series $r_{j,\alpha}(Z) := q_{j,\alpha}(H(Z))$ is convergent. Since M' is holomorphically nondegenerate, we may choose $\alpha^1, \ldots, \alpha^{N'} \in \mathbb{N}^{N'}$ and $j_1, \ldots, j_{N'} \in \{1, \ldots, d'\}$ as in Lemma 13.1. For any $l = 1, \ldots, N'$, we define a convergent map $R_l : (\mathbb{C}^N \times \mathbb{C}^{N'}, 0) \to (\mathbb{C}, 0)$ as follows

$$R_l\left(Z, Z'\right) := q_{j_l,\alpha^l}\left(Z'\right) - r_{j_l,\alpha^l}(Z) . \tag{13.7}$$

Observe that $R_l(Z, H(Z)) = 0$, $l = 1, \ldots, N'$, and moreover, since Rk $H = N'$, by (13.2) and e. g., Proposition 5.3.5 in [6], we have

$$\det\left(\frac{\partial q_{j_l,\alpha^l}}{\partial Z'_m}(H(Z))\right)_{1\le l,m\le N'} \neq 0 , \tag{13.8}$$

or equivalently,

$$\det\left(\frac{\partial R_l}{\partial Z'_m}(Z, H(Z))\right)_{1\le l,m\le N'} \neq 0 .$$

We may now apply Lemma 13.3 to conclude that H is convergent. The proof of Proposition 2.12 is complete. □

Proof of Theorems 3.1 and 3.2. Theorem 3.1 is a consequence of Theorem 2.5 and Proposition 2.10, while Theorem 3.2 follows from Theorem 2.6 and Proposition 2.12. □

14. Proofs of Theorems 1.1, 1.2, 1.3, 1.5, and 3.4 and Corollaries 1.7 and 1.8

We begin with the following lemma, which will be used in the proofs in this section.

Lemma 14.1. *Let $\mathcal{M}, \mathcal{M}' \subset \mathbb{C}^N \times \mathbb{C}^N$ be two formal generic manifolds of the same codimension d and $H : (\mathbb{C}^N, 0) \to (\mathbb{C}^N, 0)$ a formal finite map. Then,* Rk $H = N$. *Moreover, if the complexification $\mathcal{H}$ of H maps $\mathcal{M}$ into $\mathcal{M}'$, then H is not totally degenerate.*

Proof. The proof that Rk $H = N$ is standard (see e. g., Theorem 5.1.37 of [6]). To prove the second part of the lemma, it suffices to show that if $\gamma(\zeta, t)$ is a Segre variety mapping as defined in (7.1) relative to $\mathcal{M}$, then Rk $(H \circ v^1) = n$, where $n = N - d$ and $v^1(t) = \gamma(0, t)$ as in (8.1). We claim that the formal map $H \circ v^1$ is finite. Indeed, it is a composition of the finite map H and of the formal map v^1 whose rank at 0 is n and hence is finite. The claim follows from the fact that the composition of two formal finite mappings is again finite. (This could be seen by e. g., making use of Proposition 5.1.5 of [6].) As before, the fact that $H \circ v^1$ is finite implies that Rk $(H \circ v^1) = n$, which completes the proof of the lemma. □

Proof of Theorem 1.1. Without loss of generality, we may assume that $p = p' = 0$. Since $M, M' \subset \mathbb{C}^N$ are smooth generic submanifolds through the origin, we can consider the associated formal generic manifolds $\mathcal{M}, \mathcal{M}' \subset \mathbb{C}^N \times \mathbb{C}^N$ as described in Section 2. In this case, the complexification $\mathcal{H}$ of any formal map $H : (\mathbb{C}^N, 0) \to (\mathbb{C}^N, 0)$ sending M into M' sends $\mathcal{M}$ into $\mathcal{M}'$. Since the given formal map H^0 is finite, it follows from Lemma 14.1 that H^0 is not totally degenerate and Rk $H^0 = N$. Theorem 1.1 is then a consequence of Theorem 3.1. □

Proof of Theorem 1.2. Without loss of generality, we may assume that $p = p' = 0$. Since the given formal map H is finite, it follows from Lemma 14.1 that H is not totally degenerate and Rk $H = N$. Theorem 1.2 is then a consequence of Theorem 3.2. □

For the proofs of Theorems 1.3 and 1.5, we need the following lemma.

Lemma 14.2. *Let $I \subset \mathbb{C}[\![Z, \zeta]\!]$ and $J \subset \mathbb{C}[\![Z', \zeta']\!]$ be two ideals and $H, \check{H} : (\mathbb{C}^N_Z, 0) \to (\mathbb{C}^{N'}_{Z'}, 0)$ be two formal mappings. Let $\mathcal{H}, \check{\mathcal{H}}$ be the complexifications of H and $\check{H}$, respectively as defined in* (2.2)*. Assume that:*

(i) *J is a real ideal;*

(ii) *$J \subset \mathcal{H}_*(I)$, where $\mathcal{H}_*(I)$, the pushforward of I by $\mathcal{H}$ as defined by* (2.1)*;*

(iii) *$J^{\check{H}} \subset J^H$, where the ideals $J^{\check{H}}, J^H \subset \mathbb{C}[\![Z, \zeta']\!]$ are defined by* (2.5).

Then, $J \subset \check{\mathcal{H}}_*(I)$.

Proof. Let $s_1(Z', \zeta'), \ldots, s_m(Z', \zeta')$ be generators of J in $\mathbb{C}[\![Z', \zeta']\!]$. As usual, we write $s(Z', \zeta') = (s_1(Z', \zeta'), \ldots, s_m(Z', \zeta'))$ and $J = (s(Z', \zeta'))$. We set

$$\tilde{s}\left(Z', \zeta'\right) := \bar{s}\left(\zeta', Z'\right) . \tag{14.1}$$

By the reality of J, it follows that we also have $J = (\tilde{s}(Z', \zeta'))$. Hence there exists an $m \times m$

matrix with entries in $\mathbb{C}[\![Z', \zeta']\!]$ such that

$$s\left(Z', \zeta'\right) = u\left(Z', \zeta'\right) \tilde{s}\left(Z', \zeta'\right) . \tag{14.2}$$

Note that in view of (2.5), the ideals J^H and $J^{\check{H}}$ are generated by the components of $s(H(Z), \zeta')$ and $s(\check{H}(Z), \zeta')$ respectively in $\mathbb{C}[\![Z, \zeta']\!]$. Hence, by the inclusion (iii), we have

$$s\left(\check{H}(Z), \zeta'\right) = a\left(Z, \zeta'\right) s\left(H(Z), \zeta'\right) , \tag{14.3}$$

where $a(Z, \zeta')$ is an $m \times m$ matrix with entries in $\mathbb{C}[\![Z, \zeta']\!]$. By taking complex conjugates, it follows from (14.3) that we also have

$$\bar{s}\left(\overline{\check{H}}(\zeta), Z'\right) = \bar{a}\left(\zeta, Z'\right) \bar{s}\left(\bar{H}(\zeta), Z'\right) . \tag{14.4}$$

To prove the lemma, we must show that the components of $s(\check{H}(Z), \overline{\check{H}}(\zeta))$ are in I. For this, using (14.3), (14.2), (14.1), and (14.4), we have

$$\begin{aligned} s\left(\check{H}(Z), \overline{\check{H}}(\zeta)\right) &= a\left(Z, \overline{\check{H}}(\zeta)\right) s\left(H(Z), \overline{\check{H}}(\zeta)\right) \\ &= a\left(Z, \overline{\check{H}}(\zeta)\right) u\left(H(Z), \overline{\check{H}}(\zeta)\right) \tilde{s}\left(H(Z), \overline{\check{H}}(\zeta)\right) \\ &= a\left(Z, \overline{\check{H}}(\zeta)\right) u\left(H(Z), \overline{\check{H}}(\zeta)\right) \bar{s}\left(\overline{\check{H}}(\zeta), H(Z)\right) \\ &= a\left(Z, \overline{\check{H}}(\zeta)\right) u\left(H(Z), \overline{\check{H}}(\zeta)\right) \bar{a}\left(\zeta, H(Z)\right) \bar{s}\left(\overline{H}(\zeta), H(Z)\right) \\ &= a\left(Z, \overline{\check{H}}(\zeta)\right) u\left(H(Z), \overline{\check{H}}(\zeta)\right) \bar{a}\left(\zeta, H(Z)\right) \tilde{s}\left(H(Z), \overline{H}(\zeta)\right) . \end{aligned} \tag{14.5}$$

By (ii), the components of $\tilde{s}(H(Z), \overline{H}(\zeta))$ are in I and hence, by (14.5), so are the components of $s(\check{H}(Z), \overline{\check{H}}(\zeta))$. The proof of the lemma is complete. □

The following lemma is an immediate consequence of Lemma 14.2.

Lemma 14.3. *Let $\mathcal{M} \subset \mathbb{C}^N \times \mathbb{C}^N$ and $\mathcal{M}' \subset \mathbb{C}^{N'} \times \mathbb{C}^{N'}$ be two formal generic manifolds and $H, \check{H} : (\mathbb{C}^N, 0) \to (\mathbb{C}^{N'}, 0)$ be two formal mappings whose complexifications are denoted by $\mathcal{H}$ and $\check{\mathcal{H}}$, respectively. Assume that $\mathcal{H}$ sends $\mathcal{M}$ into $\mathcal{M}'$ and that the reflection ideals $\mathcal{I}^H$ and $\mathcal{I}^{\check{H}}$ are the same. Then, $\check{\mathcal{H}}$ also sends $\mathcal{M}$ into $\mathcal{M}'$.*

Proof of Theorem 3.4. Since M is of finite type at 0 and the formal map H is not totally degenerate, by Theorem 2.6, the reflection ideal $\mathcal{I}^H$ is convergent. By Proposition 2.9, for any positive integer κ, there exists a convergent map $H^\kappa : (\mathbb{C}^N, 0) \to (\mathbb{C}^N, 0)$ which agrees with H up to order κ such that $\mathcal{I}^H = \mathcal{I}^{H^\kappa}$. By Lemma 14.3, it follows that $\mathcal{H}^\kappa$ maps $\mathcal{M}$ into $\mathcal{M}'$ and hence H^κ maps M into M'. The proof of Theorem 3.4 is complete. □

Proof of Theorem 1.3. Without loss of generality, we may assume that $p = p' = 0$. Since the given formal map H is finite, it follows from Lemma 14.1 that H is not totally degenerate. Theorem 1.3 is, then, a consequence of Theorem 3.4. □

Proof of Theorem 1.5. Without loss of generality, we may assume that $p = p' = 0$. Since M is connected and of finite type at some point, by Lemma 13.3.2 of [6], there is no germ of a nonconstant holomorphic function $h : (\mathbb{C}^N, 0) \to \mathbb{C}$ with $h(M) \subset \mathbb{R}$. It follows from

Theorem 2.7 that the reflection ideal $\mathcal{I}^H$ of the given local holomorphic map H is algebraic. By Proposition 2.9, for any positive integer κ, there exists an algebraic map $H^\kappa : (\mathbb{C}^N, 0) \to (\mathbb{C}^N, 0)$ which agrees with H up to order κ such that $\mathcal{I}^H = \mathcal{I}^{H^\kappa}$. By Lemma 14.3, it follows that $\mathcal{H}^\kappa$ maps $\mathcal{M}$ into $\mathcal{M}'$ and hence H^κ maps M into M'. The proof of Theorem 1.5 is complete. □

Proof of Corollary 1.7. Let (M, p) and (M', p') be two germs of biholomorphically equivalent real-algebraic hypersurfaces in $\mathbb{C}^N$. If there is no point of finite type in M arbitrarily close to p, then (M, p) is Levi-flat and so is (M', p'). Hence both (M, p) and (M', p') are algebraically equivalent to a real hyperplane in $\mathbb{C}^N$. If M contains points of finite type arbitrarily close to p, then we may apply Corollary 1.6 to conclude that (M, p) and (M', p') are algebraically equivalent. The proof of Corollary 1.7 is complete. □

Proof of Corollary 1.8. By Theorem 1.1 with $(M, p) = (M', p')$ and $H^0 = \mathrm{Id}$, the identity map of $(\mathbb{C}^N, p)$, there exists a positive integer K such that if $H : (\mathbb{C}^N, p) \to (\mathbb{C}^N, p)$ is a formal map sending M into itself with $j_p^K H = j_p^K \mathrm{Id}$, then $H = \mathrm{Id}$. Let $H^1, H^2 : (\mathbb{C}^N, p) \to (\mathbb{C}^N, p)$ be two invertible formal mappings sending M into itself and such that $j_p^K H^1 = j_p^K H^2$. If $H = H^1 \circ (H^2)^{-1}$, then H is formal map sending (M, p) into itself such that $j_p^K H = j_p^K \mathrm{Id}$. Hence $H = H^1 \circ (H^2)^{-1} = \mathrm{Id}$, i. e., $H^1 = H^2$. The second part of Corollary 1.8 is an immediate application of Theorem 1.2. □

15. Remarks and open problems

As mentioned in Section 1, holomorphic nondegeneracy is necessary for the conclusions of Theorems 1.1 and 1.2 to hold. Indeed, if (M, p) is a germ of a smooth generic submanifold in $\mathbb{C}^N$ which is holomorphically degenerate at p, then for any positive integer K, there exist a formal invertible mapping $H : (\mathbb{C}^N, p) \to (\mathbb{C}^N, p)$ sending M into itself and agreeing with the identity map Id up to order K at p but such that $H \not\equiv \mathrm{Id}$ (see [8] Theorem 3 and [7] Theorem 2.2.1). Similarly, if (M, p) is a germ of a real-analytic generic submanifold in $\mathbb{C}^N$ which is holomorphically degenerate at p, then there exist (infinitely many) nonconvergent formal invertible self-mappings of (M, p) (see [9]).

In constrast to holomorphic nondegeneracy, the finite type condition in Theorems 1.1 and 1.2 does not seem to be necessary. More precisely, we conjecture the following. If $M \subset \mathbb{C}^N$ is a connected holomorphically nondegenerate real-analytic generic submanifold of finite type at some point, then for any $p \in M$, $\mathrm{Aut}\,(M, p) = \mathcal{F}(M, p)$. Here, we recall that $\mathrm{Aut}\,(M, p)$ is the stability group of (M, p) and $\mathcal{F}(M, p)$ is the group of formal invertible self-mappings of (M, p). This question is open even for Levi-nonflat real analytic hypersurfaces in $\mathbb{C}^2$. We also conjecture that if M is as above, then for every $p \in M$, there exists a positive integer $K = K(p)$ such that the jet mapping $j_p^K : \mathrm{Aut}\,(M, p) \to G^K(\mathbb{C}^N, p)$ is injective, where $G^K(\mathbb{C}^N, p)$ is the jet group of order K at p. It follows from Corollary 1.8 that the above conjectures hold for all points p in a Zariski open subset of M.

Another question concerning the structure of $\mathcal{F}(M, p)$ is the following. Under the assumptions of Corollary 1.8, is the image of the group homomorphism $j_p^K : \mathcal{F}(M, p) \to G^K(\mathbb{C}^N, p)$ a closed Lie subgroup of the jet group $G^K(\mathbb{C}^N, p)$, for some suitable integer K? The question is open even when M is real-analytic, in which case $\mathcal{F}(M, p) = \mathrm{Aut}\,(M, p)$, by Corollary 1.8. It is known that the answer is positive if M is finitely nondegenerate and of finite type at p (see [9] for the hypersurface case and [32] for higher codimension). In fact it is shown in [7] that in this case the image is actually a totally real algebraic Lie subgroup of $G^K(\mathbb{C}^N, p)$ for a precise value of K.

Finally, concerning algebraic equivalence, in view of Corollary 1.7, one is led to conjecture that biholomorphic equivalence implies algebraic equivalence for germs of real-algebraic submanifolds in $\mathbb{C}^N$. To the knowledge of the authors, the question is still open even for germs of generic real-algebraic submanifolds of codimension higher than one.

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Received June 6, 2001
Revision received January 13, 2002

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Communicated by Steven Bell