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Holomorphic Deformations of Real-Analytic CR Maps and Analytic Regularity of CR Mappings

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Abstract Let $M \subset \mathbb{C}^N$ and $M' \subset \mathbb{C}^{N'}$ be real-analytic CR submanifolds, with M minimal. We provide a new sufficient condition, that happens to be also essentially necessary, for all sufficiently smooth CR maps $h: U \to M'$ defined on a connected open subset of M and of rank larger than a prescribed integer r to be real-analytic on a dense open subset of U. This condition corresponds to the nonexistence of nontrivial holomorphic deformations of germs of real-analytic CR mappings whose rank is larger than r. As a consequence, we obtain several new results about analyticity of CR mappings that, at the same time, generalize and unify a number of previous existing ones.

Keywords CR map · Analyticity · Holomorphic deformation

Mathematics Subject Classification 32V20 · 32V25 · 32V30 · 32V40 · 32H02 · 32H40

1 Introduction

Let M and M' be real-analytic CR submanifolds embedded in complex space $\mathbb{C}^N$ and $\mathbb{C}^{N'}$, respectively, and $h: M \to M'$ a sufficiently smooth CR map. A very much studied question over the last decades is to understand under what conditions on M and M' the mapping h is real-analytic. The topic has naturally attracted the attention of a number of mathematicians as it lies at the intersection of Partial Differential Equa-

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tions (analyticity of solutions of first-order nonlinear systems) and Several Complex Variables (analytic continuation and reflection principle).

Since the works of Lewy [25] and Pinchuk [31] in the 70s, the above question has led to numerous investigations and results in the equidimensional case $N = N'$. We cannot mention here all related works but rather refer, e.g., to the monograph by Baouendi–Ebenfelt–Rothschild [5] and the survey papers by Forstnerič [17] and Huang [20] for a more detailed account on this matter as well as to the recent work by Kossovskiy-Lamel [24] for the latest developments. On the other hand, in the more general case where N and N' are not necessarily equal, the problem seems to be more difficult and the related literature is much less abundant. In this paper, we shall tackle that situation for minimal CR submanifolds M. Let us recall that M is said to be minimal in the sense of Tumanov [34] if it does not contain any CR submanifold S of the same CR dimension as that of M with $\dim S < \dim M$ (see also the books [5,9]). One of our main results provides a new sufficient condition, that happens to be also essentially necessary, for all sufficiently smooth CR maps $h\colon U \to M'$ defined on a connected open subset of M and of rank larger than a prescribed integer r to be real-analytic on a dense open subset of U. Here, by the rank of the mapping h (over U), denoted by $\mathrm{Rk}_U\, h$, we mean the maximum rank of h over U. As a consequence, we obtain several new results about analyticity of CR mappings that, at the same time, generalize and unify a number of previous existing ones.

Let us now describe how one may construct nonanalytic CR mappings from M into M'. Firstly, it may happen that M carries only real-analytic CR functions (see, e.g., [5,10] and [1] for recent work), in which case the original question is trivially answered by the affirmative. Hence, one may assume, without loss of generality, that M admits nonanalytic CR functions. If the target manifold M' contains, say, some holomorphic curve parametrized by $\mathbb{C} \ni t \mapsto \gamma(t)$, one obvious way to construct a nonanalytic CR map from (an open piece of) M into M' is simply to consider $\gamma \circ f$, where f is a chosen nonanalytic CR function on M. However, in this way, one can only construct nonanalytic CR maps of rank two, and, therefore, such a procedure cannot be used to produce, e.g., nonanalytic CR immersions.

In order to generate nonanalytic CR maps of prescribed rank, we shall consider *nontrivial holomorphic deformations of real-analytic CR maps*. Given $k \in \mathbb{Z}_+$ and $p \in M$, a nontrivial holomorphic deformation of germs at p of real-analytic CR mappings from M into M', denoted by $(\Psi_t)_{t\in\mathbb{C}^k}$, consists of a real-analytic CR map $(t, z) \mapsto \Psi_t(z)$ from a (connected) neighborhood of $(0, p)$ in $\mathbb{C}^k \times M$ into M' satisfying $\mathrm{rk}\, \frac{\partial \Psi_t}{\partial t}(p)|_{t=0} = k$. We also define the rank of the deformation, denoted by $\mathrm{Rk}_M\, \Psi$, to be the maximum of the (generic) ranks of the maps Ψ_t for t sufficiently small. The interest of such deformations in the analyticity problem lies in the following observation: if $p \in M$ and if there exists a nontrivial holomorphic deformation $(\Psi_t)_{t\in\mathbb{C}^k}$ of germs at p of real-analytic CR maps from M into M', then for any nonanalytic CR function f defined near p, vanishing at p, and for sufficiently small generic values of $u \in \mathbb{C}$ and $t \in \mathbb{C}^k$, the mapping $h\colon z \mapsto \Psi_{(t_1+uf(z),t_2,\ldots,t_k)}(z)$ is a nonanalytic CR map from a neighborhood U of p in M into M' satisfying $\mathrm{Rk}_U\, h \geq \mathrm{Rk}_M\, \Psi$. Hence, to construct nonanalytic CR maps near a point $p \in M$ whose rank is greater than or equal to a prescribed integer r, it is sufficient that a nontrivial holomorphic deformation $(\Psi_t)_{t\in\mathbb{C}^k}$ as above satisfying $\mathrm{Rk}_M\,\Psi \geq r$ exists. Our main result in this paper establishes essentially a converse to this statement.

Theorem 1.1 *Let $M \subset \mathbb{C}^N$ and $M' \subset \mathbb{C}^{N'}$ be connected real-analytic CR submanifolds of CR dimensions n and n', respectively, and $h\colon M \to M'$ be a CR mapping of class $\mathcal{C}^{m_0}$, $n, n' \geq 1$, $m_0 \geq n'$. Assume that M is minimal and that there exists a nonempty open subset $\Omega \subset M$ where h is nowhere real-analytic. Then there exists a dense open subset ω of Ω such that for every point $p \in \omega$, there exist an integer $k \in \{1,\dots,n'\}$ and a nontrivial holomorphic deformation $(\Psi_t)_{t\in\mathbb{C}^k}$ of germs at p of real-analytic CR mappings from M into M' with the following properties:*

(i) $\mathrm{Rk}_M\,\Psi \geq \mathrm{Rk}_M\,h$;
(ii) *there exists a CR map $\varphi_p\colon M \to \mathbb{C}^k$, of class $\mathcal{C}^{m_0}$, such that $\Psi_{\varphi_p(z)}(z) = h(z)$ for all z in some neighborhood of p in M.*

As an immediate consequence of Theorem 1.1 and the above discussion, we obtain the characterization announced in the beginning of the introduction.

Corollary 1.2 *Let $M \subset \mathbb{C}^N$ and $M' \subset \mathbb{C}^{N'}$ be real-analytic CR submanifolds of CR dimensions n and n', respectively, with M minimal and $n, n' \geq 1$. Assume that M admits a nowhere real-analytic CR function of class $\mathcal{C}^{n'}$ and let r be a nonnegative integer. Then the following conditions are equivalent:*

(i) *every CR map $h\colon U \to M'$, with $U \subset M$ open and connected, of class $\mathcal{C}^{n'}$ with $\mathrm{Rk}_U\,h \geq r$ is real-analytic on some dense open subset of U;*
(ii) *there does not exist any nontrivial holomorphic deformation of germs of real-analytic CR mappings from M into M' of rank $\geq r$.*

Furthermore, simple examples show that, in general, one cannot expect in (i) the map h to be real-analytic all over U (see Remark 2.2).

Another consequence of Theorem 1.1 we would like to mention is given by the following existence result, previously established by Sunyé [33] in the special case of $\mathcal{C}^\infty$ mappings.

Corollary 1.3 *Let $M \subset \mathbb{C}^N$ and $M' \subset \mathbb{C}^{N'}$ be real-analytic CR submanifolds of CR dimensions n and n', respectively. Assume that M is minimal and let $h\colon M \to M'$ be a CR mapping of class $\mathcal{C}^{m_0}$ with $m_0 \geq n'$. Then for every point p in some dense open subset of M, there exists a germ at p of a real-analytic CR map from M into M' that agrees with h up to order m_0 at p.*

Theorem 1.1 and Corollary 1.2 reduce the analyticity problem for CR mappings (of class $\mathcal{C}^{n'}$) to the nonexistence of nontrivial holomorphic deformations of germs of real-analytic CR maps from M into M'. Note that in the special case of CR diffeomorphisms between CR submanifolds M, M' of the same CR dimension, condition (ii) in Corollary 1.2 is equivalent to saying that either M and M' are nowhere locally biholomorphically equivalent or that M is holomorphically nondegenerate (provided M is connected). Hence, Corollary 1.2 recovers in that special situation a well-known statement that can be obtained by combining several results from the existing literature ([3,5]).

Back to the general case, a first obvious condition that implies the nonexistence of nontrivial holomorphic deformations of germs of CR maps of any rank is the nonexistence of holomorphic curves embedded in the target manifold M'. Hence we obtain:

Corollary 1.4 *Let $M \subset \mathbb{C}^N$ and $M' \subset \mathbb{C}^{N'}$ be real-analytic CR submanifolds of CR dimensions n and n', respectively, and $h\colon M \to M'$ be a CR mapping of class $\mathcal{C}^{n'}$, $n, n' \geq 1$. Assume that M is minimal and that M' does not contain any complex curve. Then h is real-analytic on some dense open subset of M.*

To the author's knowledge, the regularity result given in Corollary 1.4 seems to be new. Under the stronger assumption that $h \in \mathcal{C}^\infty$, the result was obtained by Damour [12].

Theorem 1.1 can also be used to provide analyticity results for CR mappings even if the target manifold M' does contain holomorphic curves and, in fact, even foliated by complex curves such as, e.g., in the case of holomorphically nondegenerate real-analytic CR manifolds with everywhere degenerate Levi form (see [5]). In fact, such maps naturally appear in the theory of proper holomorphic maps between hermitian symmetric domains (see, e.g., [23,30] and the references therein). We shall illustrate this by considering the case of the tube of the light cone $\mathbb{T}^{N'} \subset \mathbb{C}^{N'}$, $N' \geq 3$. Recall that $\mathbb{T}^{N'}$ is the (everywhere Levi-degenerate) smooth real-algebraic hypersurface of $\mathbb{C}^{N'}$ given by the smooth part of the real-algebraic variety

$$X = \left\{ (z_1, \ldots, z_{N'}) \in \mathbb{C}^{N'} : (Re\, z_{N'})^2 = \sum_{j=1}^{N'-1} (Re\, z_j)^2 \right\}. \tag{1.1}$$

We have the following result.

Corollary 1.5 *Let $M \subset \mathbb{C}^N$ be a connected minimal real-analytic CR submanifold and $N' \geq 3$. Then any CR mapping $h\colon M \to \mathbb{T}^{N'}$, of class $\mathcal{C}^{N'-1}$, whose rank is ≥ 3, is real-analytic on some dense open subset of M. In particular, any CR immersion from M into $\mathbb{T}^{N'}$, of class $\mathcal{C}^{N'-1}$, is real-analytic on some dense open subset of M.*

Corollary 1.5 is proven by first determining all nontrivial holomorphic deformations $(\Psi_t)_t$ of germs of real-analytic CR mappings from M into $\mathbb{T}^{N'}$. This is done in Sect. 2.3 (see Lemma 2.3) where we show that the rank of any such deformation must necessarily not exceed 2. This in conjunction with Theorem 1.1 provides the proof of Corollary 1.5. Note that the rank assumption on the mapping h in this corollary is necessary as $\mathbb{T}^{N'}$ contains complex lines.

If, in Theorem 1.1, we assume more on the mapping and on the manifolds, our arguments can be used to yield more precise results. We therefore now consider the well-studied and important case of CR immersions with Levi-nondegenerate targets for which we also obtain new results as well as generalizations of the existing ones. We shall also address the case of CR transversal maps, i.e., CR maps $h\colon M \to M'$ between (real-analytic) CR submanifolds satisfying at every $p \in M$ the condition

$$T^{1,0}_{h(p)}M' + T^{0,1}_{h(p)}M' + dh(\mathbb{C}T_pM) = \mathbb{C}T_{h(p)}M'. \tag{1.2}$$

The CR transversality property can be seen as a Hopf lemma property for CR maps and has been the subject of many recent works (see, e.g., [6,14,22]) as it appears to be an important geometric tool in the study of the mapping problems. Recall also that the intrinsic complexification of M at p, denoted by $\mathcal{V}_p^M$, is the germ at p of the complex submanifold in $\mathbb{C}^N$ of smallest dimension containing the germ of M at p (and similarly for $\mathcal{V}_{h(p)}^{M'}$, see [5]). Our arguments for CR immersions provide the following.

Theorem 1.6 *Let $M \subset \mathbb{C}^N$ and $M' \subset \mathbb{C}^{N'}$ be connected real-analytic CR submanifolds of CR dimensions n and n', respectively, and $h\colon M \to M'$ be a CR immersion of class $\mathcal{C}^{m_0}$, $n' > n \geq 1$, $m_0 \geq n' - n + 1$. Assume that M is minimal, that M' is Levi-nondegenerate, and that there exists a nonempty open subset $\Omega \subset M$ where h is nowhere real-analytic. Then there exists a dense open subset ω of Ω such that the following holds:*

(a) *For every point $p \in \omega$, there exist an integer $k \in \{1, \dots, n' - n\}$ and a nontrivial holomorphic deformation $(\Psi_t)_{t\in\mathbb{C}^k}$ of germs at p of real-analytic CR immersions from M into M' satisfying (ii) of Theorem 1.1.*
(b) *If, furthermore, M and M' are of the same CR codimension, h is CR transversal, and $n' = n + 1$, then we may choose $\omega = \Omega$ and, for every $p \in \omega$, the CR map $(t, z) \mapsto \Psi_t(z)$ to extend near $(0, p)$ as a biholomorphic map from $\mathbb{C} \times \mathcal{V}_p^M$ to $\mathcal{V}_{h(p)}^{M'}$ sending a neighborhood of $(0, p)$ in $\mathbb{C} \times M$ into M'.*

Theorem 1.6 yields the following stronger version of Corollary 1.2 for immersive maps with Levi-nondegenerate targets.

Corollary 1.7 *Let $M \subset \mathbb{C}^N$ and $M' \subset \mathbb{C}^{N'}$ be real-analytic CR submanifolds of CR dimensions n and n', respectively, with M minimal, M' Levi-nondegenerate, and $n' > n \geq 1$. Assume that M admits a nowhere real-analytic CR function of class $\mathcal{C}^{n'-n+1}$. Then the following conditions are equivalent:*

(i) *every CR immersion $h\colon U \to M'$, with $U \subset M$ open, of class $\mathcal{C}^{n'-n+1}$, is real-analytic on some dense open subset of U;*
(ii) *there does not exist any nontrivial holomorphic deformation of germs of real-analytic CR immersions from M into M'.*

Part (a) of Theorem 1.6 (or Corollary 1.7) immediately implies the following result that seems to be new even for the case of real hypersurfaces.

Corollary 1.8 *Let $M \subset \mathbb{C}^N$ and $M' \subset \mathbb{C}^{N'}$ be real-analytic CR submanifolds of CR dimensions n and n', respectively, $n' \geq n \geq 1$. Assume that M is minimal and M' is Levi-nondegenerate and does not contain any complex curve. Then any CR immersion $h\colon M \to M'$, of class $\mathcal{C}^{n'-n+1}$, is real-analytic on some dense open subset of M.*

Corollary 1.8 contains several previously studied situations. In the case where M and M' are strongly pseudoconvex hypersurfaces, the result goes back to Huang [18] (see [7] for another recent proof of this and [15] for the case where $h \in \mathcal{C}^\infty$). When M, M' are manifolds of higher codimension and M' is a strongly pseudoconvex quadric, Corollary 1.8 was proved by Forstnerič [16, Theorem 1.6]. Note that for $n' = n$ the statement is well known and is contained in the work of Baouendi–Jacobowitz–Treves [3].

On the other hand, using Part (b) of Theorem 1.6, one gets the following noteworthy statement :

Corollary 1.9 *Let $M \subset \mathbb{C}^N$ and $M' \subset \mathbb{C}^{N+1}$ be real-analytic generic submanifolds of CR dimension, n and n', respectively, $n' \geq n \geq 1$. Assume that M is minimal and that M' is Levi-nondegenerate. Then any CR transversal immersion $h\colon M \to M'$, of class $\mathcal{C}^2$, is real-analytic on some dense open subset of M.*

In the case where M and M' are hypersurfaces, Corollary 1.9 has recently been proved by Berhanu–Xiao [8] (see also [13] when $h \in \mathcal{C}^\infty$ and [28] for an analogous result in the setting of formal maps).

Apart from the condition of nonexistence of holomorphic curves in the target manifold M' and the codimension one assumption appearing in Corollary 1.9, it is desirable to find some other sufficient conditions implying the nonexistence of nontrivial holomorphic deformations of germs of real-analytic CR (transversal) immersions from M into M'. In the case where M and M' are hypersurfaces with M strongly pseudoconvex and M' Levi-nondegenerate, such a sufficient condition involving the signature of M' can be provided (see Proposition 3.1). Combining this with Theorem 1.6 yields another proof of the following recent result due to Berhanu–Xiao [8].

Corollary 1.10 *Let $M \subset \mathbb{C}^{n+1}$ and $M' \subset \mathbb{C}^{n'+1}$ be real-analytic hypersurfaces with M strongly pseudoconvex and M' Levi-nondegenerate, $n' \geq n \geq 1$. Denote by ℓ'_+ (resp. ℓ'_-) the number of positive (resp. negative) eigenvalues of the Levi form of M'. If $\max(\ell'_+, \ell'_-) \leq n$, then any CR transversal immersion $h\colon M \to M'$, of class $\mathcal{C}^{n'-n+1}$, is real-analytic on some dense open subset of M.*

As explained in Sect. 3.4, the above condition on the signature of M' (i.e., $\max(\ell'_+, \ell'_-) \leq n$) is easily seen not to be necessary for the nonexistence of nontrivial holomorphic deformations of germs of CR transversal immersions from M into M'. Nevertheless, in the case where M' is the nondegenerate hyperquadric $\mathcal{H}^{n'}_{\ell'}$, i.e.,

$$\mathcal{H}^{n'}_{\ell'} = \left\{ (z'_1, \ldots, z'_{n'}, w') \in \mathbb{C}^{n'+1} : \operatorname{Im} w' = \sum_{j=1}^{\ell'} |z'_j|^2 - \sum_{j=1+\ell'}^{n'} |z'_j|^2 \right\}, \; 0 \leq \ell' \leq n', \tag{1.3}$$

we provide in Sect. 3.4 a necessary and sufficient criterion for the nonexistence of such deformations that almost coincides with the above-mentioned signature condition (see Proposition 3.2). Such a characterization implies the next result that also follows from the recent work by Berhanu–Xiao [8].

Corollary 1.11 *Let $\mathcal{H}^{n'}_{\ell'}$ be the nondegenerate hyperquadric in $\mathbb{C}^{n'+1}$ as in (1.3) and let $n \in \mathbb{Z}_+$ with $n' \geq n \geq 1$. If $\max(\ell', n' - \ell') \leq n$ or $\max(\ell', n' - \ell') = n'$, then given any strongly pseudoconvex real-analytic hypersurface $M \subset \mathbb{C}^{n+1}$, every CR transversal immersion $h\colon M \to \mathcal{H}^{n'}_{\ell'}$, of class $\mathcal{C}^{n'-n+1}$, is real-analytic on some dense open subset of M. Conversely, if $n < \max(\ell', n' - \ell') < n'$ and if $M = \mathcal{H}^n_0$, then there exists a CR transversal immersion $h_0\colon M \to \mathcal{H}^{n'}_{\ell'}$, of class $\mathcal{C}^{n'-n+1}$, that is nowhere real-analytic on M.*

We conclude this introduction by mentioning some open problems. Firstly, it would be interesting to know if the smoothness assumption on the mapping h can be weakened in Theorems 1.1 and 1.6. To the author's knowledge, weaker differentiability assumptions on the map h implying analyticity are known only in the model cases of spheres under some additional codimension restrictions (see, e.g., [19,21] and the references therein). Secondly, in view of Remark 2.2, it would be desirable to know if instead of assuming that the mapping h is nowhere analytic on some open subset of M in Theorems 1.1 and 1.6, one may assume that $h \in \mathcal{C}^\infty$ and not real-analytic at some point p_0 and arrive at the same conclusion for some sufficiently small open neighborhood Ω of p_0. If such a result were true, one could conclude in several of the subsequent corollaries that the mapping h is everywhere real-analytic. We should point out that, if in Theorems 1.1 and 1.6 we assume that the mapping $h \in \mathcal{C}^\infty$ and the target M' is real-algebraic, then the arguments of the present paper can be refined to answer by the affirmative the question raised just above. Hence, if, in Corollaries 1.5 and 1.11, one assumes the map h to be $\mathcal{C}^\infty$ to start with, then the conclusion is that h is everywhere real-analytic. For other related results in that direction, we refer the reader to [27,29,32].

The paper is organized as follows. In Sect. 2, we collect the proofs of Theorem 1.1 and Corollaries 1.3 and 1.5. Section 3 contains the proofs of all results for mappings with Levi-nondegenerate targets.

2 Proofs of Theorem 1.1 and Corollaries 1.3 and 1.5

2.1 Proof of Theorem 1.1

Let M, M', Ω, and $h\colon M \to M'$ be as given in Theorem 1.1 with h nowhere real-analytic on Ω. Without loss of generality, we may assume that both M and M' are generic and we write $N' = n' + d'$. For a point $p \in M$, we say that p satisfies condition ($\sharp$) if there exist an integer $k \in \{1, \ldots, n'\}$ and a nontrivial holomorphic deformation $(\Psi_t)_{t\in\mathbb{C}^k}$ of germs at p of real-analytic CR mappings from M into M' satisfying conditions (i) and (ii) of Theorem 1.1. Set

$$W := \{p \in M : p \text{ satisfies } (\sharp)\}.$$

In order to prove Theorem 1.1, we need to show that $W \cap \Omega$ is dense in Ω, or, equivalently, that $\Omega \setminus W$ is of empty interior in Ω. We are going to prove that if a nonempty open subset $M_0 \subset \Omega$ satisfies $M_0 \subset \Omega \setminus W$, then h is real-analytic on some nonempty open subset of M_0.

So let $M_0 \subset \Omega$ be a nonempty open subset and suppose that $M_0 \subset \Omega \setminus W$. Shrinking M_0 if necessary, we may assume that M_0 has a basis of CR vector fields $\bar{L}_1, \ldots, \bar{L}_n$ with real-analytic coefficients defined all over M_0. For every multiindex $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, denote by $\bar{L}^\alpha = \bar{L}_{\alpha_1} \ldots \bar{L}_{\alpha_n}$.

The main tool in the proof of Theorem 1.1 lies in the following lemma, which allows us to "solve" every component of the mapping successively. In what follows, all neighborhoods are assumed to be open and connected.

Lemma 2.1 *Let h, Ω, M_0 be as above with $M_0 \subset \Omega \setminus W$ and $\ell \le n'-1$. Assume that in a fixed choice of holomorphic coordinates in $\mathbb{C}^{N'}$, the map h splits as $h = (\widetilde{h}, \widehat{h}) \in \mathbb{C}^r \times \mathbb{C}^{N'-r}$ for some $r \in \{1, \dots, n'\}$ and satisfies in a neighborhood M_1 of some point $p \in M_0$ an identity of the form*

$$\widehat{h}(z) = \Theta(z, \bar{z}, ((\bar{L}^\alpha \bar{h})(z))_{|\alpha|\le \ell}, \widetilde{h}(z)), \tag{2.1}$$

for some $\mathbb{C}^{N'-r}$-valued holomorphic map Θ defined in a neighborhood of $(p, \bar{p}, ((\bar{L}^\alpha \bar{h})(p))_{|\alpha|\le \ell}, \widetilde{h}(p))$. Then we can select one component of the map $\widetilde{h}$, denoted by $\widetilde{h}_1$, such that, writing $\widetilde{h} = (\widetilde{h}_1, \widetilde{h}_2) \in \mathbb{C} \times \mathbb{C}^{r-1}$, the following identity is satisfied in a neighborhood M_2 of some point $q \in M_1 \subset M_0$

$$(\widehat{h}(z), \widetilde{h}_1(z)) = \Delta(z, \bar{z}, ((\bar{L}^\alpha \bar{h})(z))_{|\alpha|\le \ell+1}, \widetilde{h}_2(z)), \tag{2.2}$$

for some $\mathbb{C}^{N'-r+1}$-valued holomorphic map Δ defined in a neighborhood of $(q, \bar{q}, ((\bar{L}^\alpha \bar{h})(q))_{|\alpha|\le \ell+1}, \widetilde{h}_2(q))$.

Proof of Lemma 2.1 We write $z' = (\widetilde{z}', \widehat{z}')$ for the coordinates in $\mathbb{C}^{N'}$ associated with the splitting of the map $h = (\widetilde{h}, \widehat{h})$. Differentiating (2.1), we obtain on M_1

$$\begin{aligned} 0 &= \bar{L}_j \left(\Theta(z, \bar{z}, (\bar{L}^\alpha \bar{h}(z))_{|\alpha|\le \ell}, \widetilde{h}(z))\right) \\ &=: \Phi^j(z, \bar{z}, (\bar{L}^\alpha \bar{h}(z))_{|\alpha|\le \ell+1}, \widetilde{h}(z)), \quad j = 1, \dots, n, \end{aligned} \tag{2.3}$$

for some $\mathbb{C}^{N'-r}$-valued holomorphic map $\Phi^j = \Phi^j(z, \zeta, (\Lambda_\alpha)_{|\alpha|\le \ell+1}, \widetilde{z}')$ defined in a neighborhood of $(p, \bar{p}, ((\bar{L}^\alpha \bar{h})(p))_{|\alpha|\le \ell+1}, \widetilde{h}(p))$. Set $\Phi := (\Phi^1, \dots, \Phi^n)$. There are two cases to consider.

First case : $\Phi(z, \bar{z}, ((\bar{L}^\alpha \bar{h}(z))_{|\alpha|\le \ell+1}, \widetilde{z}')) \not\equiv 0$ in some neighborhood $M_1^* \times V$ of $(p, \widetilde{h}(p))$ in $M_1 \times \mathbb{C}^r$. (Note that this implies, by the minimality of M and Tumanov's extension theorem [34] applied to the map h, that the above-mentioned mapping cannot vanish on any open subset of $M_1^* \times V$, see, e.g., [26, Sect. 6].) It then follows from (2.3) that

$$0 < \delta_0 := \inf \{|\gamma| : \gamma \in \mathbb{N}^r,\ \Phi_{\widetilde{z}'^\gamma}(z, \bar{z}, ((\bar{L}^\alpha \bar{h}(z))_{|\alpha|\le \ell+1}, \widetilde{h}(z))|_{M_1^*} \not\equiv 0\} < \infty.$$

Since $\Phi_{\widetilde{z}'^\gamma}(z, \bar{z}, ((\bar{L}^\alpha \bar{h}(z))_{|\alpha|\le \ell+1}, \widetilde{h}(z)) = 0$ for $z \in M_1^*$ and $|\gamma| < \delta_0$, the conclusion of the lemma follows from the choice of δ_0 and the implicit function theorem.

Second case : $\Phi(z, \bar{z}, ((\bar{L}^\alpha \bar{h}(z))_{|\alpha|\le \ell+1}, \widetilde{z}')) = 0$ for $(z, \widetilde{z}')$ in some neighborhood $M_1^* \times V$ of $(p, \widetilde{h}(p))$ in $M_1 \times \mathbb{C}^r$. From the definition of Φ and (2.3), we therefore get on $M_1^* \times V$

$$0 = \bar{L}_j \left(\Theta(z, \bar{z}, (\bar{L}^\alpha \bar{h}(z))_{|\alpha|\le \ell}, \widetilde{z}')\right), \ j = 1, \dots, n, \tag{2.4}$$

i.e., that the map $M_1^* \times V \ni (z, \widetilde{z}') \mapsto \Theta(z, \bar{z}, ((\bar{L}^\alpha \bar{h})(z))_{|\alpha|\le \ell}, \widetilde{z}')$ is CR. Since M is minimal, $M_1^* \times V$ is minimal too, and, therefore, by the standard reflection principle (see, e.g., [26]), the latter map extends holomorphically to a neighborhood $U \times V$ of

$M_1^* \times V$ in $\mathbb{C}^N \times \mathbb{C}^r$. We denote the extended mapping by $A = A(z, \widetilde{z}')$ for $z \in U$ and $\widetilde{z}' \in V$. Note that by (2.1), we have

$$\widehat{h}(z) = A(z, \widetilde{h}(z)), \quad z \in M_1^*. \tag{2.5}$$

Let $\rho'(z', \bar{z}') = (\rho'_1, \dots, \rho'_{d'})$ be a real-analytic vector-valued defining function of M' near $h(p)$. We claim that

$$\rho'(\widetilde{z}', A(z, \widetilde{z}'), \overline{\widetilde{z}'}, \overline{A(z, \widetilde{z}')}) \not\equiv 0, \ (z, \widetilde{z}') \in M_1^* \times V. \tag{2.6}$$

Indeed, by contradiction, suppose that (2.6) does not hold. We now show that for a generic point $z_0 \in M_1^*$, we necessarily have $z_0 \in W$, reaching a contradiction, since $M_1^* \subset M_0 \subset \Omega \setminus W$. Consider the set $E \subset M_1^*$ of points z at which the rank of h is maximal, i.e., equals $\mathrm{Rk}_M\, h$. Since M is minimal, E is dense in M_1^*. Pick $z_0 \in E$. Since $\widetilde{h}$ is CR of class $\mathcal{C}^{m_0}$, there exists a $\mathbb{C}^r$-valued holomorphic polynomial $P(z)$ of degree at most m_0 such that the m_0-jet at z_0 of $\widetilde{h}$ coincides with that of $P|_M$ (see, e.g., [5]). Set $k := r$ and define for $t \in \mathbb{C}^k$ sufficiently close to 0 and $z \in M$ sufficiently close to z_0

$$\Psi_t(z) := (t + P(z), A(z, t + P(z))). \tag{2.7}$$

It is easy to check that $(\Psi_t)_{t \in \mathbb{C}^k}$ is a nontrivial deformation of germs at z_0 of real-analytic CR mappings from M to M'. Furthermore, the rank of Ψ_0 at z_0 is equal to $\mathrm{Rk}_M\, h$ since, in view of our choice of P and (2.5), Φ_0 agrees with h at z_0 up to order $m_0 \geq n' \geq 1$. Finally, it follows from (2.5) that for $z \in M$ sufficiently close to z_0, we have $\Psi_{\varphi(z)}(z) = h(z)$ for $\varphi(z) := \widetilde{h}(z) - P(z)$ that is CR all over M and of class $\mathcal{C}^{m_0}$. We therefore have shown that $z_0 \in W$, which yields the desired contradiction and hence proves that (2.6) holds. Set, for $(z, \widetilde{z}') \in M_1^* \times V$,

$$\rho^*(z, \bar{z}, \widetilde{z}', \overline{\widetilde{z}'}) := \rho'(\widetilde{z}', A(z, \widetilde{z}'), \overline{\widetilde{z}'}, \overline{A(z, \widetilde{z}')}). \tag{2.8}$$

Subcase 1 : $\eta_0 := \inf \{|\beta| : \beta \in \mathbb{N}^r, \ \rho^*_{\widetilde{z}'^\beta}(z, \bar{z}, \widetilde{h}(z), \overline{\widetilde{h}}(z)) \not\equiv 0, \text{ on } M_1^*\} < \infty$.

Note that it follows from (2.5) that $\eta_0 > 0$. Since $\rho^*_{\widetilde{z}'^\beta}(z, \bar{z}, \widetilde{h}(z), \overline{\widetilde{h}}(z)) = 0$ on M_1^* for $|\beta| < \eta_0$, the conclusion of the lemma follows from the choice of η_0 and the implicit function theorem.

Subcase 2 : $\forall \beta \in \mathbb{N}^r$, $\rho^*_{\widetilde{z}'^\beta}(z, \bar{z}, \widetilde{h}(z), \overline{\widetilde{h}}(z)) = 0$ on M_1^*.

This implies that $\rho^*(z, \bar{z}, \widetilde{z}', \overline{\widetilde{h}}(z)) = 0$ for $z \in M_1^*$ and $\widetilde{z}' \in V$. Now observe that (2.6) and (2.8) imply that there exists $\nu \in \mathbb{N}^r$ such that $\frac{\partial \rho^*}{\partial \overline{\widetilde{z}'}^\nu}(z, \bar{z}, \widetilde{z}', \overline{\widetilde{h}}(z)) \not\equiv 0$ on $M_1^* \times V$. As before, the implicit function theorem leads to the desired conclusion. □

With Lemma 2.1 now at hand, we can proceed with the proof of Theorem 1.1.

Let $M_0 \subset \Omega \setminus W$ be as above. Without loss of generality, we may assume that $0 \in M_0$ and $h(0) = 0$. We may also assume that there exists an open neighborhood $U' \subset \mathbb{C}^{N'}$ of 0 such that $M' \cap U' = \{(z', w') \in U' : w' = Q'(z', \bar{z}', \bar{w}')\}$, in some fixed choice of coordinates $(z', w') \in \mathbb{C}^{n'} \times \mathbb{C}^{d'}$, with $Q' = Q'(z', \chi', \tau')$ being a $\mathbb{C}^{d'}$-valued holomorphic map defined in a neighborhood of 0 in $\mathbb{C}^{2n'+d'}$. Write $h = (f, g) \in \mathbb{C}^{n'} \times \mathbb{C}^{d'}$. Shrinking M_0 if necessary, we get the following basic identity on M_0:

$$g = Q'(f, \bar{h}). \tag{2.9}$$

By Lemma 2.1, there exists (at least) one component of f, say f_1, a neighborhood M_1 of some point $p_1 \in M_0$ and a $\mathbb{C}^{N'-n'+1}$-valued holomorphic map Δ_1 defined in a neighborhood of $(p_1, \bar{p}_1, ((\bar{L}^\alpha \bar{h})(p_1))_{|\alpha|\le 1}, \widetilde{f}(p_1))$ where $f = (f_1, \widetilde{f})$ such that

$$(f_1(z), g(z)) = \Delta_1(z, \bar{z}, ((\bar{L}^\alpha \bar{h})(z))_{|\alpha|\le 1}, \widetilde{f}(z)), \quad z \in M_1. \tag{2.10}$$

Applying inductively Lemma 2.1 $n' - 1$ more times, we obtain that there exists a point $p^* \in M_0$ and a neighborhood M^* of p^* in M_0 and a $\mathbb{C}^{N'}$-valued holomorphic map Δ^* defined in a neighborhood of $(p^*, \bar{p}^*, ((\bar{L}^\alpha \bar{h})(p^*))_{|\alpha|\le n'})$ such that

$$h(z) = (f(z), g(z)) = \Delta^*(z, \bar{z}, ((\bar{L}^\alpha \bar{h})(z))_{|\alpha|\le n'}), \quad z \in M^*. \tag{2.11}$$

By the standard reflection principle (see, e.g., [26]), h is real-analytic all over M^*. This concludes the proof of Theorem 1.1.

2.2 Proof of Corollary 1.3

Assume first that $n' \ge 1$. If h is real-analytic on some dense open subset of M, there is nothing to prove. Otherwise, let Ω be the interior of the set of points of M where h is not real-analytic. It is enough to check the conclusion of the corollary for a generic point in Ω. Since h is nowhere real-analytic on Ω, it follows from Theorem 1.1 that for a generic point $p \in \Omega$, there exist a nontrivial holomorphic deformation of germs at p of real-analytic CR mappings from M into M' and a CR function φ_p on M of class $\mathcal{C}^{m_0}$ such that $\Psi_{\varphi_p(z)}(z) = h(z)$ for all $z \in M$ near p. Let $R(z)$ be a holomorphic polynomial of degree at most m_0 such that $R|_M$ and φ_p agree up to order m_0 at p (see [5]). Then the real-analytic CR map defined near p given by $z \mapsto \Psi_{R(z)}(z)$ sends a neighborhood of p in M into M' and agrees with h up to order m_0 at p.

It remains to deal with the case $n' = 0$, i.e., M' is totally real. For this, the reader can check that the required conclusion is easy to derive using the fact that real-valued CR functions on a minimal connected real-analytic CR submanifold are necessarily constant (see [4]). The proof is complete.

Remark 2.2 The following example shows that one cannot expect, in general, the map h to be real-analytic all over U in Corollary 1.2. Consider the real-algebraic hypersurfaces $M \subset \mathbb{C}^2$ and $M' \subset \mathbb{C}^3$:

$$M: \quad \operatorname{Im} w = |z|^2 + |w|^{10},$$
$$M': \quad \operatorname{Im} w' = |z'_1|^2 + |z'_2|^4.$$

Fix a relatively compact open subset U of 0 in M. Then U is strongly pseudoconvex and by [8, Theorem 2.7] there exists a nowhere real-analytic CR function on U of class $\mathcal{C}^2$. Furthermore, M' does not contain any complex curve and therefore condition (ii) in Corollary 1.2 is satisfied. For an appropriate branch of the square root function, the CR map $(z, w) \mapsto (z, w^2\sqrt{w}, w)$ is a CR immersion from U into M' of class $\mathcal{C}^2$ that is real-analytic on $U \setminus \{0\}$ but not at 0.

2.3 Mapping to the Tube over the Light Cone

In order to apply Theorem 1.1 to the case of the tube over the light cone $\mathbb{T}^{N'}$ and prove Corollary 1.5, we need the following result that describes all nontrivial holomorphic deformations of real-analytic CR mappings valued in $\mathbb{T}^{N'}$. Note that since $\mathbb{T}^{N'}$ does not contain any complex submanifold of dimension ≥ 2, we only need to consider one-dimensional holomorphic deformations.

Lemma 2.3 *Let $M \subset \mathbb{C}^N$ be a real-analytic CR submanifold that is everywhere minimal. Let $p \in M$ and assume that there exists a nontrivial holomorphic deformation $(\Psi_t)_{t\in\mathbb{C}}$ of germs at p of real-analytic CR mappings from M into $\mathbb{T}^{N'}$, $N' = n' + 1$. Then there exists a real-analytic CR function $\mu = \mu(z, t)$ defined in a neighborhood of $(p, 0)$ in $M \times \mathbb{C}$ such that for $(z, t) \in M \times \mathbb{C}$ near $(p, 0)$*

$$\Psi_t(z) = (\alpha_1\mu(z, t) + i\delta_1, \dots, \alpha_{n'}\mu(z, t) + i\delta_{n'}, \mu(z, t)),$$

where $\alpha_1, \dots, \alpha_{n'}, \delta_1, \dots, \delta_{n'}$ are real numbers satisfying $\sum_{j=1}^{n'} \alpha_j^2 = 1$. In particular, $\mathrm{Rk}_M\, \Psi \leq 2$ *necessarily holds.*

Proof of Lemma 2.3 Without loss of generality, we may assume that M is generic. Write

$$\Psi_t(z) = (\psi_1(z, t), \dots, \psi_{n'}(z, t), \mu(z, t)).$$

In a neighborhood Ω of $(p, 0)$ in $M \times \mathbb{C}$, we have

$$(Re\, \mu)^2 = \sum_{j=1}^{n'} (Re\, \psi_j)^2. \tag{2.12}$$

In what follows, we write μ' for the holomorphic derivative of μ with respect to t (as well as for the ψ_j's). Differentiating (2.12) with respect to t, we get on Ω

$$(Re\, \mu)\, \mu' = \sum_{j=1}^{n'} (Re\, \psi_j)\, \psi_j'. \tag{2.13}$$

Differentiating (2.13) with respect to $\bar{t}$, we get on Ω

$$|\mu'|^2 = \sum_{j=1}^{n'} |\psi_j'|^2. \tag{2.14}$$

Since $\frac{\partial \Psi_t}{\partial t}\Big|_{\substack{t=0\\ z=p}} \neq 0$, we may assume, using (2.14) and shrinking Ω if necessary, that $\mu' \neq 0$ on Ω. Note also that since $\mathbb{T}^{n'+1}$ is the smooth part of the real-algebraic variety given by (1.1), we have $Re\,\mu \neq 0$ on Ω. From (2.12), (2.13), and (2.14), we see that the vectors in $\mathbb{C}^{n'}$ given by

$$\left(\frac{Re\,\psi_1}{Re\,\mu}, \ldots, \frac{Re\,\psi_{n'}}{Re\,\mu}\right) \text{ and } \left(\frac{\psi_1'}{\mu'}, \ldots, \frac{\psi_{n'}'}{\mu'}\right)$$

are unit vectors and their scalar product is equal to one. Hence, for $j = 1, \ldots, n'$, we have

$$\frac{\psi_j'}{\mu'} = \frac{Re\,\psi_j}{Re\,\mu}, \text{ on } \Omega.$$

The previous equality shows that each real-analytic CR function $\frac{\psi_j'}{\mu'}$ is real valued on Ω. Since M is minimal, Ω is minimal too, and therefore by [4], each CR function $\frac{\psi_j'}{\mu'}$ is a real-valued constant that we denote by α_j. From (2.14), we see that $\sum_{j=1}^{n'} \alpha_j^2 = 1$. From $\frac{\psi_j'}{\mu'} = \frac{Re\,\psi_j}{Re\,\mu} = \alpha_j$, we deduce that $\psi_j = \alpha_j \mu + \eta_j$ for some real-analytic CR function η_j defined near p on M, whose real part is identically zero. Hence, since M is minimal, η_j is a purely imaginary constant. This completes the proof of Lemma 2.3. □

Proof of Corollary 1.5 When dim $M \geq 3$, Corollary 1.5 follows directly from combining Theorem 1.1 and Lemma 2.3. When dim $M = 2$, the second part of Corollary 1.5 follows from the fact that CR maps on (one)-dimensional complex manifolds are merely holomorphic maps. □

3 Mapping to a Levi-Nondegenerate Target

Theorem 1.6 follows from either slightly modifying or inspecting the proof of Theorem 1.1. For sake of completeness, we include the details here.

3.1 Proof of Theorem 1.6(a)

Let M and M' be as in Theorem 1.6 with, without loss of generality, M, M' being generic, and let h be a CR immersion of class $\mathcal{C}^{m_0}$, $m_0 \geq n'-n+1$, that is nowhere real-analytic on a nonempty open subset $\Omega \subset M$. Following the proof of Theorem 1.1, we say that a point $p \in M$ satisfies condition $\widehat{(\sharp)}$ if there exist an integer $k \in \{1, \dots, n'-n\}$ and a nontrivial holomorphic deformation $(\Psi_t)_{t\in\mathbb{C}^k}$ of germs at p of real-analytic CR immersions from M into M' satisfying (ii) of Theorem 1.1. Set $\widehat{W} := \{p \in M : p \text{ satisfies } \widehat{(\sharp)}\}$. In order to prove Theorem 1.6, we only need to show that if a nonempty open subset $M_0 \subset \Omega$ satisfies $M_0 \subset \Omega \setminus \widehat{W}$, then h is real-analytic on some nonempty subset of M_0. We pick such a subset M_0 with a basis of CR vector fields $\bar{L}_1, \dots, \bar{L}_n$ with real-analytic coefficients on M_0. We first note that a slightly modified version of Lemma 2.1 still holds with $\ell \leq n' - n + 1$ (instead of n'), $r \leq n' - n$, and $\widehat{W}$ (instead of W).

We may assume that $0 \in M_0$, $h(0) = 0$ and that there exists an open neighborhood $U' \subset \mathbb{C}^{N'}$ of 0 such that $M' \cap U' = \{(z', w') \in U' : w' = Q'(z', \bar{z}', \bar{w}')\}$, where $(z', w') \in \mathbb{C}^{n'} \times \mathbb{C}^{d'}$ and $Q' = Q'(z', \chi', \tau')$ is a $\mathbb{C}^{d'}$-valued holomorphic map defined in a neighborhood of 0 in $\mathbb{C}^{2n'+d'}$. We may also assume that $Q'(z', \bar{z}', \bar{w}') - \bar{w}'$ vanishes at the origin up to order 1. We write $h = (f, g) \in \mathbb{C}^{n'} \times \mathbb{C}^{d'}$. Shrinking M_0 if necessary, we get the following basic identity on M_0:

$$g = Q'(f, \bar{h}). \tag{3.1}$$

Let $\bar{Q}'$ be the holomorphic map obtained by taking complex conjugates of the coefficients of Q'. Applying each CR vector field $\bar{L}_j$ to the identity $\bar{g} = \bar{Q}'(\bar{f}, h)$ on M_0, one gets

$$\bar{L}_j \bar{g} = \bar{L}_j \bar{f} \cdot \bar{Q}'_{\bar{z}'}(\bar{f}, h), \quad j = 1, \dots, n. \tag{3.2}$$

Since h is CR immersive, we can select n components $\widehat{f}$ among those of f such that, writing $f = (\widehat{f}, \widetilde{f})$ and $\widehat{f} = (\widehat{f}_1, \dots, \widehat{f}_n)$, the $n \times n$ matrix $(\bar{L}_j \overline{\widehat{f}_k})$ has full rank n over M_0. Since M' is Levi-nondegenerate, we can use the implicit function theorem to the set of equations (3.1) and (3.2) to conclude that, shrinking M_0 near 0 if necessary, we have

$$(\widehat{f}(z), g(z)) = \Theta(((\bar{L}^\alpha \bar{h})(z))_{|\alpha| \leq 1}, \widetilde{f}(z)), \tag{3.3}$$

for some $\mathbb{C}^{N'-n'+n}$-valued holomorphic map Θ defined in a neighborhood of $(((\bar{L}^\alpha \bar{h})(0))_{|\alpha| \leq 1}, 0)$. Applying successively the above-mentioned modified version of Lemma 2.1 $n' - n$ times, we get that there exist a point $p^* \in M_0$, a neighborhood M^* of p^* in M_0, and a $\mathbb{C}^{N'}$-valued holomorphic map Δ^* defined in a neighborhood of $(p^*, \bar{p}^*, ((\bar{L}^\alpha \bar{h})(p^*))_{|\alpha| \leq n'-n+1})$ such that

$$h(z) = \Delta^*(z, \bar{z}, ((\bar{L}^\alpha \bar{h})(z))_{|\alpha| \leq n'-n+1}), \quad z \in M^*. \tag{3.4}$$

As in the proof of Theorem 1.1, (3.4) shows that h is real-analytic all over M^*. This proves Theorem 1.6(a).

3.2 Proof of Theorem 1.6(b)

Let M and M' be as in Theorem 1.6(b) with, without loss of generality, M, M' being generic, $n' = n + 1$, and let h be a CR transversal immersion of class $\mathcal{C}^{m_0}$, $m_0 \geq 2$. Let also d be the CR codimension of M (that coincides with that of M'). Assume that h is nowhere real-analytic on a nonempty open subset $\Omega \subset M$. Let $p \in \Omega$ and let Ω_0 be an open neighborhood of p admitting a basis of CR vector fields $\bar{L}_1, \ldots, \bar{L}_n$ with real-analytic coefficients on Ω_0. As in the proof of Theorem 1.6(a), we may assume that $0 \in \Omega_0$, $h(0) = 0$ and that there exists an open neighborhood $U' \subset \mathbb{C}^{N+1}$ of 0 such that

$$M' \cap U' = \left\{(z', w') \in U' : w' = Q'(z', \bar{z}', \bar{w}')\right\}, \tag{3.5}$$

where $(z', w') \in \mathbb{C}^{n+1} \times \mathbb{C}^d$ and $Q' = Q'(z', \chi', \tau')$ is a $\mathbb{C}^d$-valued holomorphic map defined in a neighborhood of 0 in $\mathbb{C}^{2n+2+d}$. Without loss of generality, we may also assume that $Q'(z', \bar{z}', \bar{w}') - \bar{w}'$ vanishes at the origin up to order 1. We write $h = (f, g) \in \mathbb{C}^{n+1} \times \mathbb{C}^d$. Since h is of class $\mathcal{C}^2$, we may consider $H = (F, G)$ the unique holomorphic polynomial map of degree at most 2 such that the 2-jet at 0 of $H|_M$ coincides with that of h. As h is immersive, interchanging the z'-coordinates if necessary, we may write $z' = (\widetilde{z}', \widehat{z}') \in \mathbb{C} \times \mathbb{C}^n$ so that the $n \times n$ matrix $(\bar{L}_j \widehat{f}_k)$ has full rank n over Ω_0. Furthermore, since h is CR transversal, the polynomial map $(\widehat{F}, G)$ is a local biholomorphism of $\mathbb{C}^N$ near the origin. Performing a complex linear change of coordinates in $\mathbb{C}^{N+1}$ leaving $(\widehat{z}', w')$ fixed, we may assume that

$$\mathrm{d}\widetilde{F}(0) = 0. \tag{3.6}$$

(The change of coordinates might affect the chosen defining function for M', but using the implicit function theorem if necessary, one can keep a defining function of the form (3.5).) Shrinking Ω_0 if necessary, we obtain the identities (3.1) and (3.2). As in the proof of Theorem 1.6(a), since M' is Levi-nondegenerate, the implicit function theorem applied to the set of equations (3.1) and (3.2) implies, shrinking Ω_0 near 0 if necessary, that

$$(\widehat{f}(z), g(z)) = \Phi(((\bar{L}^\alpha \bar{h})(z))_{|\alpha| \leq 1}, \widetilde{f}(z)), \tag{3.7}$$

for some $\mathbb{C}^N$-valued holomorphic map Φ defined in a neighborhood of $(((\bar{L}^\alpha \bar{h})(0))_{|\alpha| \leq 1}, 0)$. We claim that for $j = 1, \ldots, n$, $\bar{L}_j \left(\Phi(((\bar{L}^\alpha \bar{h})(z))_{|\alpha| \leq 1}, \widetilde{z}')\right) \equiv 0$ for $z \in \Omega_0$ and $\widetilde{z}'$ in some sufficiently small neighborhood of the origin in $\mathbb{C}$. Indeed, if it were not the case, one could show, following the lines of the very beginning of the proof of Lemma 2.1 and the standard reflection principle, that $\widetilde{f}$ and therefore h is real-analytic on some open subset of Ω_0, reaching a contradiction. As in the proof of Theorem 1.1, the claim implies that the map $\Phi(((\bar{L}^\alpha \bar{h})(z))_{|\alpha| \leq 1}, \widetilde{z}')$ extends holomorphically to a neighborhood $U \times V$ of $\Omega_0 \times \{0\}$ in $\mathbb{C}^N \times \mathbb{C}$. We denote the extended mapping by $A = A(z, \widetilde{z}')$ for $z \in U$ and $\widetilde{z}' \in V$. From (3.7), we get

$$(\widehat{f}(z), g(z)) = A(z, \widetilde{f}(z)), \quad z \in \Omega_0. \tag{3.8}$$

Now consider

$$\rho^*(z,\bar z,\widetilde{z}',\overline{\widetilde{z}'}) := \rho'(\widetilde{z}', A(z,\widetilde{z}'), \overline{\widetilde{z}'}, \overline{A(z,\widetilde{z}')}), \tag{3.9}$$

where $\rho'(z',w',\bar z',\bar w') = w' - Q'(z',\bar z',\bar w')$, $z' = (\widetilde{z}', \widehat{z}')$. As in the proof of Theorem 1.1 (e.g., following the arguments in subcases 1 and 2), one may show that if $\rho^*(z,\bar z,\widetilde{z}',\overline{\widetilde{z}'})\not\equiv 0$ on $\Omega_0\times V$, then $\widetilde{f}$ is real-analytic on some open subset of Ω_0 and so is h, reaching a contradiction. Hence $\rho^*(z,\bar z,\widetilde{z}',\overline{\widetilde{z}'})\equiv 0$ on $\Omega_0\times V$. Recall now that $\widetilde{F}$ (resp. $\widehat{F}$ and G) denotes the holomorphic polynomial map of degree at most 2 such that the 2-jet of $\widetilde{F}|_M$ (resp. $\widehat{F}|_M$ and $G|_M$) at 0 agrees with that of $\widetilde{f}$ (resp. $\widehat{f}$ and g). Setting for $(z,t)\in M\times\mathbb{C}$ sufficiently close to $(0,0)$, $\Psi_t(z) = (t+\widetilde{F}(z), A(z,t+\widetilde{F}(z)))$, we see that $(\Psi_t)_{t\in\mathbb{C}}$ is a nontrivial deformation of germs at 0 of real-analytic CR immersions from M to M'. We only need to check that $(z,t)\mapsto\Psi_t(z)$ extends to a biholomorphism from $\mathbb{C}^{N+1}$ into itself near the origin. For this, we shall follow one argument from [28]. Recall first that since h is immersive and CR transversal, the holomorphic map $(\widehat{F},G)$ is a local biholomorphic map from $\mathbb{C}^N$ to $\mathbb{C}^N$ near the origin. Furthermore, it follows from (3.8) that $(\widehat{F}(z),G(z))$ and $A(z,\widetilde{F}(z))$ agree at 0 up to order 2. Hence, it follows from (3.8) and (3.6) that the matrix $\frac{\partial A}{\partial z}(0)$ is of rank N. This implies that $(z,t)\mapsto\Psi_t(z)$ extends to a holomorphic map of rank $N+1$ at the origin. This finishes the proof of Theorem 1.6(b).

3.3 Proof of Corollary 1.9

Let M, M', h as in Corollary 1.9. There are two cases to consider: either $n' = n+1$ or $n' = n$. In the case $n' = n$, it is well known that h is real-analytic all over M by the standard arguments on the equidimensional reflection principle (assuming even only $\mathcal{C}^1$ smoothness on the mapping h, see, e.g., [3]). For the case $n' = n+1$, assume, by contradiction, that h is nowhere real-analytic on some open subset Ω of M. Then Theorem 1.6(b) implies that, near any point of Ω, there exists a local biholomorphism of $\mathbb{C}^{N+1}$ sending an open piece of $M\times\mathbb{C}$ onto an open piece of M'. But this is impossible as M' is Levi-nondegenerate. Hence, h must be real-analytic on a dense open subset of M.

3.4 Nontrivial Holomorphic Deformations of CR Immersions and Signature

The following result provides a necessary condition for the existence of nontrivial holomorphic deformations of CR transversal immersions from a strongly pseudoconvex hypersurface into a Levi-nondegenerate one.

Proposition 3.1 *Let $M\subset\mathbb{C}^{n+1}$ and $M'\subset\mathbb{C}^{n'+1}$ be real-analytic hypersurfaces with M strongly pseudoconvex and M' Levi-nondegenerate, $n'>n\geq 1$. Denote by ℓ'_+ (resp. ℓ'_-) the number of positive (resp. negative) eigenvalues of the Levi form of M'. If for some point $p\in M$ there exists a nontrivial holomorphic deformation $(\Psi_t)_{t\in\mathbb{C}}$ of germs at p of real-analytic CR transversal immersions from M into M', then $n<\max(\ell'_+,\ell'_-)<n'$.*

Proof of Proposition 3.1 Without loss of generality, we may assume that $p = 0$, $\Psi_0(0) = 0$ and choose holomorphic coordinates $(z', w') = (z'_1, \ldots, z'_n, w') \in \mathbb{C}^{n'} \times \mathbb{C}$ near 0 in $\mathbb{C}^{n'+1}$ so that M' is locally given by the equation

$$\operatorname{Im} w' = \sum_{j=1}^{\ell_+} |z'_j|^2 - \sum_{j=1+\ell_+}^{n'} |z'_j|^2 + R'(z', \bar{z}', Re\, w'), \tag{3.10}$$

where R' is a real-analytic function near $0 \in \mathbb{R}^{2n'+1}$ of order at least 3. Similarly, we may choose local holomorphic coordinates $(z, w) \in \mathbb{C}^n \times \mathbb{C}$ near 0 so that that M is locally given by the equation

$$\operatorname{Im} w = \sum_{j=1}^{n} |z_j|^2 + R(z, \bar{z}, Re\, w), \tag{3.11}$$

where R is a real-analytic function near $0 \in \mathbb{R}^{2n+1}$ of order at least 3. As it is well known, comparing the Levi form of M and M' and using the CR transversality of the mapping Ψ_0 yields that either $\ell'_+ \geq n$ or $\ell'_- \geq n$ (see, e.g., [2,8]). We shall treat only the case where $\ell'_+ \geq n$ since the case $\ell'_- \geq n$ can be reduced to the previous one by considering the mapping $T \circ \Phi_0$ where $T(z', w') = (z', -w')$ (that sends M to the Levi-nondegenerate hypersurface $T(M')$ that has ℓ'_- positive eigenvalues). Let us assume therefore that $\ell'_+ \geq n$. By [2], the coordinates (z, w) and (z', w') can be chosen in such a way that the mapping Ψ_0 is normalized as follows: $\Psi_0 = (f_0, \varphi_0, g_0) \in \mathbb{C}^n \times \mathbb{C}^{n'-n} \times \mathbb{C}$ with $d\varphi_0(0) = 0$ and the matrix $(\bar{L}_j \bar{f}_0(0))$ is invertible, $\bar{L}_1, \ldots, \bar{L}_n$ being a basis of real-analytic CR vector fields of M near 0. We shall also use the above splitting to write $\Psi_t = (f_t, \varphi_t, g_t)$. Since $\ell'_+ \geq n$, we shall write

$$\varphi_t = (\varphi_{t,n+1}, \ldots, \varphi_{t,\ell'_+}, \varphi_{t,1+\ell'_+}, \ldots, \varphi_{t,n'}).$$

For sufficiently small $t \in \mathbb{C}$, Ψ_t maps M into M' and therefore we have the following identity near 0 on M:

$$\operatorname{Im} g_t = \sum_{j=1}^{n} |f_j|^2 + \sum_{j=n+1}^{\ell'_+} |\varphi_{t,j}|^2 - \sum_{j=1+\ell'_+}^{n'} |\varphi_{t,j}|^2 + R'(f_t, \varphi_t, \bar{f}_t, \bar{\varphi}_t, Re\, g_t). \tag{3.12}$$

We also write

$$f_j = \alpha_j + t\beta_j + O(t^2), \quad \varphi_{t,j} = \mu_j + t\delta_j + O(t^2).$$

In what follows, when differentiating f_t, φ_t, g_t with respect to t, we write f'_t, φ'_t, g'_t. Differentiating (3.12) with respect to t gives the following identity near 0 in M:

$$\frac{1}{2i} g'_t = \sum_{j=1}^{n} f'_j \bar{f}_j + \sum_{j=n+1}^{\ell'_+} \varphi'_{t,j} \bar{\varphi}_{t,j} - \sum_{j=1+\ell'_+}^{n'} \varphi'_{t,j} \bar{\varphi}_{t,j} + \frac{\partial}{\partial t}\left[R'(f_t, \varphi_t, \bar{f}_t, \bar{\varphi}_t, Re\, g_t)\right]. \tag{3.13}$$

Evaluating (3.13) at $t = 0$ yields

$$\frac{1}{2i} g_t'|_{t=0} = \sum_{j=1}^{n} \beta_j \bar{\alpha}_j + \sum_{j=n}^{\ell_+'} \delta_j \bar{\mu}_j - \sum_{j=1+\ell_+'}^{n'} \delta_j \bar{\mu}_j + \frac{\partial}{\partial t}\left[R'(f_t, \varphi_t, \bar{f}_t, \bar{\varphi}_t, Re\, g_t)\right]_{t=0}. \tag{3.14}$$

Note that (3.14) implies in particular that

$$g_t'(0)|_{t=0} = 0. \tag{3.15}$$

Applying the CR vector fields $\bar{L}_1, \ldots, \bar{L}_n$ to (3.13) and evaluating at $t = 0$ yields that for $k = 1, \ldots, n$

$$\begin{aligned} 0 = & \sum_{j=1}^{n} \beta_j \bar{L}_k \bar{\alpha}_j + \sum_{j=n}^{\ell_+'} \delta_j \bar{L}_k \bar{\mu}_j - \sum_{j=1+\ell_+'}^{n'} \delta_j \bar{L}_k \bar{\mu}_j \\ & + \bar{L}_k \left(\frac{\partial}{\partial t}\left[R'(f_t, \varphi_t, \bar{f}_t, \bar{\varphi}_t, Re\, g_t)\right]\right)\Big|_{t=0}. \end{aligned} \tag{3.16}$$

Evaluating (3.16) at the origin and using the facts that the matrix $(\bar{L}_k \bar{\alpha}_j)$ is invertible and that $\bar{L}_k \bar{\mu}_j(0) = 0$ shows that

$$\beta_j(0) = 0, \quad j = 1, \ldots, n. \tag{3.17}$$

Now differentiating (3.13) with respect to $\bar{t}$ and evaluating at the origin yields

$$\begin{aligned} 0 &= \sum_{j=1}^{n} |\beta_j(0)|^2 + \sum_{j=n}^{\ell_+'} |\delta_j(0)|^2 - \sum_{j=1+\ell_+'}^{n'} |\delta_j(0)|^2 \\ 0 &= \sum_{j=n}^{\ell_+'} |\delta_j(0)|^2 - \sum_{j=1+\ell_+'}^{n'} |\delta_j(0)|^2. \end{aligned} \tag{3.18}$$

On the other, since (Ψ_t) is a nontrivial deformation, we must have $\Psi_t'(0)|_{t=0} \neq 0$. In view of (3.15) and (3.17), there must exist at least one j such that $\delta_j(0) \neq 0$. This implies together with (3.18) that necessarily $n < \ell_+' < n'$, which yields the desired conclusion. The proof of Proposition 3.1 is complete.

□

Proof of Corollary 1.10 If $n = n'$, then M' is necessarily strongly pseudoconvex (see, e.g., [2,8]). In this case, the desired conclusion follows directly from Corollary 1.8. If $n' > n$, the conclusion follows by combining Theorem 1.6(a) and Proposition 3.1. □

Note that Proposition 3.1 in codimension one, i.e., for $n' = n + 1$ shows the nonexistence of nontrivial holomorphic deformations of CR transversal immersions. Furthermore, one should observe that the necessary condition given by that proposition is in general not a sufficient one. Indeed, for any pair of integers n, n' with $n' \geq n+2$, for any strongly pseudoconvex real-analytic hypersurface $M \subset \mathbb{C}^{n+1}$ and for any Lorentzian real-analytic Levi-nondegenerate hypersurface $M' \subset \mathbb{C}^{n'+1}$ (i.e., having $n' - 1$ positive eigenvalues and 1 negative eigenvalue) containing no holomorphic curves, there does not exist any nontrivial holomorphic deformation of germs of real-analytic CR maps from M into M'. Such hypersurfaces M' are "generic" among all Lorentzian hypersurfaces (see, e.g., [11]). However, there is one particular and important instance for which the necessary condition given in Proposition 3.1 seems to be "almost" sufficient: the hyperquadric. The exact statement is given by the following result.

Proposition 3.2 *Let $\mathcal{H}^{n'}_{\ell'} \subset \mathbb{C}^{n'+1}$ be the nondegenerate hyperquadric given by (1.3) and let $1 \leq n < n'$. If $\max(n' - \ell', \ell') \leq n$ or $\max(n' - \ell', \ell') = n'$, then for every real-analytic strongly pseudoconvex $M \subset \mathbb{C}^{n+1}$, there does not exist any nontrivial holomorphic deformation of germs of real-analytic CR transversal immersions from a neighborhood of a point of M into $\mathcal{H}^{n'}_{\ell'}$. Conversely, if $n < \max(n'-\ell', \ell') < n'$ and M is the Heisenberg hypersurface $\mathcal{H}^n_0$, there exists a nontrivial holomorphic deformation of CR transversal immersions from M into $\mathcal{H}^{n'}_{\ell'}$.*

Proof of Proposition 3.2 The first part of the proposition is an immediate consequence of Proposition 3.1. For the last part, we note that $\mathcal{H}^{n'}_{\ell'}$ and $\mathcal{H}^n_0$ may be defined, respectively, by the equations

$$\operatorname{Im} w' = \sum_{j=1}^{\ell'} |z'_j|^2 - \sum_{j=\ell'+1}^{n'} |z'_j|^2, \quad \operatorname{Im} w = \sum_{j=1}^{n} |z_j|^2$$

with $(z'_1, \ldots, z'_{n'}, w') \in \mathbb{C}^{n'+1}$ and $(z_1, \ldots, z_n, w) \in \mathbb{C}^{n+1}$. Without loss of generality, we may assume that $n' > \ell' > n$. Then considering

$$\Psi_t(z_1, , \ldots, z_n, w) = (z_1, \ldots, z_n, t, 0, \ldots, 0, t, w)$$

provides the required nontrivial (one-dimensional) holomorphic deformation of CR transversal immersions from $\mathcal{H}^n_0$ into $\mathcal{H}^{n'}_{\ell'}$. This completes the proof. □

Proof of Corollary 1.11 The case $n = n'$ corresponds to the situation where necessarily $\ell' = n'$ or $\ell' = 0$, i.e., $\mathcal{H}^{n'}_{\ell'} = \mathcal{H}^n_0$. Hence the conclusion in that case follows from Corollary 1.8. In the case $n' > n$, the conclusion follows from the conjunction of Proposition 3.2, Theorem 1.6, and the existence of a nowhere real-analytic CR function on $\mathcal{H}^n_0$ of class $\mathcal{C}^{n'-n+1}$ (established in [8, Theorem 2.7]). □

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