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Finite jet determination of CR mappings

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Abstract

We prove the following finite jet determination result for CR mappings: Given a smooth generic submanifold $M \subset \mathbb{C}^N$, $N \geqslant 2$, that is essentially finite and of finite type at each of its points, for every point $p \in M$ there exists an integer ℓ_p, depending upper-semicontinuously on p, such that for every smooth generic submanifold $M' \subset \mathbb{C}^N$ of the same dimension as M, if $h_1, h_2 : (M, p) \to M'$ are two germs of smooth finite CR mappings with the same ℓ_p jet at p, then necessarily $j_p^k h_1 = j_p^k h_2$ for all positive integers k. In the hypersurface case, this result provides several new unique jet determination properties for holomorphic mappings at the boundary in the real-analytic case; in particular, it provides the finite jet determination of arbitrary real-analytic CR mappings between real-analytic hypersurfaces in $\mathbb{C}^N$ of D'Angelo finite type. It also yields a new boundary version of H. Cartan's uniqueness theorem: if $\Omega, \Omega' \subset \mathbb{C}^N$ are two bounded domains with smooth real-analytic boundary, then there exists an integer k, depending only on the boundary $\partial\Omega$, such that if $H_1, H_2 : \Omega \to \Omega'$ are two proper holomorphic mappings extending smoothly up to $\partial\Omega$ near some point $p \in \partial\Omega$ and agreeing up to order k at p, then necessarily $H_1 = H_2$.

Keywords: CR mapping; Finite jet determination

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1. Introduction

There exists a wide variety of results concerned with the rigidity of automorphisms of a given geometric structure. In CR geometry, one classical result of this type is given by the uniqueness result stating that every pseudo-conformal map (or equivalently local biholomorphic map) sending Levi-nondegenerate real-analytic hypersurfaces of $\mathbb{C}^N$ into each other, $N \geqslant 2$, is uniquely determined by its 2-jet at any given point; this is a consequence of the solution to the biholomorphic equivalence problem for the class of Levi-nondegenerate hypersurfaces, obtained by E. Cartan [12,13] in $\mathbb{C}^2$ and Tanaka [37] and Chern–Moser [15] in $\mathbb{C}^N$ for arbitrary $N \geqslant 2$. This result has been the source of many recent developments and generalizations in several directions, see e.g. the works [1,3,4,6,8,9,20,21,27,28,30,32–34] and also the surveys [5,26,36,38,39] for complete references on the subject. Most of the work mentioned above is concerned with establishing the unique jet determination property *for holomorphic automorphisms*. In this paper, we are concerned with understanding the same phenomenon for *finite holomorphic mappings* (or even arbitrary CR mappings) between generic manifolds that we allow to be of any codimension and to have strong Levi-degeneracies. More precisely, we prove the following theorem (see Section 2 for relevant definitions and notation).

Theorem 1. *Let $M \subset \mathbb{C}^N$ be a smooth generic submanifold that is essentially finite and of finite type at each of its points. Then for every point $p \in M$ there exists an integer ℓ_p, depending upper-semicontinuously on p, such that for every smooth generic submanifold $M' \subset \mathbb{C}^N$ of the same dimension as that of M, if $h_1, h_2 : (M, p) \to M'$ are two germs of smooth finite CR mappings with the same ℓ_p jet at p, then necessarily $j_p^k h_1 = j_p^k h_2$ for all positive integers k.*

Here and throughout the paper by smooth we mean $\mathcal{C}^\infty$-smooth. To put our main result into the proper perspective, we should mention that Theorem 1 improves the very few finite jet determination results for finite mappings in two important different directions. Under the same assumptions as that of Theorem 1, Baouendi, Ebenfelt and Rothschild proved in [4] (see also [6]) the finite jet determination of finite mappings whose k-jet at a given point, for k sufficiently large, is *the same as that of a given fixed finite map*; the integer k does actually depend on this fixed map.

Our result allows, on the one hand, to compare *arbitrary pairs of finite maps*, and cannot be derived from the mentioned result of [4]. From this point of view, Theorem 1 is more natural and satisfactory. On the other hand, our main result also provides a dependence of the jet order (required to get the determination of the maps) on the base point. This explicit control cannot be obtained by the techniques of [4,6] and is of fundamental importance in order to derive for instance Theorem 3 below.

Note that Theorem 1 is new even in the case where the manifolds and mappings are real-analytic, in which case the conclusion is that the mappings are identical. Note also that the upper-semicontinuity of the jet order with respect to the base point mentioned in Theorem 1 was already obtained by the authors in [32] in the case of local biholomorphic self-maps of real-analytic generic submanifolds of $\mathbb{C}^N$. The proof that we are giving of this fact in this paper has the advantage to extend to a more general situation and to be at the same time somewhat simpler than the proof given in [32].

Theorem 1 offers a number of remarkable new consequences. The first one is given by the following finite jet determination result for arbitrary CR mappings between D'Angelo finite type hypersurfaces (in the sense of [16]). To the authors' knowledge, this result is the first of its kind in the Levi-degenerate case. (See also Corollary 30 below for a slightly more general version.)

Corollary 2. *Let $M, M' \subset \mathbb{C}^N$ be smooth real hypersurfaces of D'Angelo finite type. Then for every point $p \in M$, there exists a positive integer $\ell = \ell(M,p)$, depending upper-semicontinuously on p, such that for any pair $h_1, h_2 : (M,p) \to M'$ of germs of smooth CR mappings, if $j_p^\ell h_1 = j_p^\ell h_2$, then necessarily $j_p^k h_1 = j_p^k h_2$ for all positive integers k. If in addition both M and M' are real-analytic, it follows that $h_1 = h_2$.*

In another direction, a further consequence of Theorem 1 is given by the following.

Theorem 3. *Let M be a compact real-analytic CR submanifold of $\mathbb{C}^N$ that is of finite type at each of its points. Then there exists a positive integer k, depending only on M, such that for every real-analytic CR submanifold $M' \subset \mathbb{C}^N$ of the same dimension as that of M and for every point $p \in M$, local smooth CR finite mappings sending a neighborhood of p in M into M' are uniquely determined by their k-jet at p.*

Theorem 3 follows from the conjunction of the upper-semicontinuity of the integer ℓ_p on p in Theorem 1, a well-known result of Diederich–Fornæss [17] stating that compact real-analytic CR submanifolds of $\mathbb{C}^N$ do necessarily not contain any analytic disc and hence are essentially finite (see e.g. [2]) and the combination of the regularity result due to Meylan [35] with the recent transversality result due to Ebenfelt–Rothschild [22]. In the case of local CR diffeomorphisms, Theorem 3 was already obtained by the authors in [32].

When both manifolds M and M' are compact hypersurfaces in Theorem 3, we have the following neater statement as an immediate consequence of Corollary 2.

Corollary 4. *Let $M, M' \subset \mathbb{C}^N$ be compact real-analytic hypersurfaces. Then there exists a positive integer k depending only on M, such that for every point $p \in M$, local smooth CR mappings sending a neighborhood of p in M into M' are uniquely determined by their k-jet at p.*

We note that the conclusion of Corollary 4 does not hold (even for automorphisms) if the compactness assumption is dropped, as the following example shows.

Example 5. [3] Let $\Phi : \mathbb{C} \to \mathbb{C}$ be a nonzero entire function satisfying

$$\frac{\partial^j \Phi}{\partial z^j}(n) = 0, \quad j \leqslant n,\ n \in \mathbb{N},$$

and consider the hypersurface $M \subset \mathbb{C}^3_{z_1,z_2,w}$ given by the equation

$$\operatorname{Im} w = \operatorname{Re}\big(z_1 \overline{\Phi(z_2)}\big).$$

Then the entire automorphism

$$H(z_1, z_2, w) = \big(z_1 + i\Phi(z_2), z_2, w\big)$$

[3] This is an adaptation of an example which appeared in [21], which grew out of a discussion at the workshop "Complexity of mappings in CR-geometry" at the American Institute of Mathematics in September 2006. The authors would like to take this opportunity to thank the Institute for its hospitality.

sends M into itself, agrees with the identity up to order n at each point $(0, n, 0)$, $n \in \mathbb{N}$, but is not equal to the identity. This example shows that despite of the fact that local holomorphic automorphisms of M are uniquely determined by a finite jet at every arbitrary fixed point of M (since M is holomorphically nondegenerate and of finite type, see [6]), a uniform bound for the jet order valid at all points of the manifold need not exist in general, unless additional assumptions (like compactness) are added. Note also that in view of the results in [21], the above phenomenon cannot happen in $\mathbb{C}^2$.

By a classical result of H. Cartan [14], given any bounded domain $\Omega \subset \mathbb{C}^N$, any holomorphic self-map of Ω agreeing with the identity mapping up to order one at any fixed point of Ω must be the identity mapping. Our last application provides a new boundary version of this uniqueness theorem for proper holomorphic mappings.

Corollary 6. *Let $\Omega \subset \mathbb{C}^N$ be a bounded domain with smooth real-analytic boundary. Then there exists an integer k, depending only on the boundary $\partial\Omega$, such that for every other bounded domain Ω' with smooth real-analytic boundary, if $H_1, H_2 : \Omega \to \Omega'$ are two proper holomorphic maps extending smoothly up to $\partial\Omega$ near some point $p \in \partial\Omega$ which satisfy $H_1(z) = H_2(z) + o(|z-p|^k)$, then necessarily $H_1 = H_2$.*

Corollary 6 follows immediately from Corollary 4. The authors do not know any other analog of H. Cartan's uniqueness theorem for arbitrary pairs of proper maps. A weaker version of Corollary 6 appears in the authors' paper [32] (namely when $\Omega = \Omega'$ and one of the map is assumed to be the identity mapping). For other related results, we refer the reader to the papers [11,23,24].

The paper is organized as follows. In the next section, we recall the basic concepts concerning formal generic submanifolds and mappings which allow us to state a general finite jet determination result (Theorem 9) in such a context for so-called CR-transversal mappings, and from which Theorem 1 will be derived. In Section 4 we give the proof of Theorem 9 which involves the Segre set machinery recently developed by Baouendi, Ebenfelt and Rothschild [2–4]. In order to be able to compare arbitrary pairs of mappings, we have to derive a number of new properties of the mappings under consideration, when restricted to the first Segre set. As a byproduct of the proof, we also obtain a new sufficient condition for a CR-transversal map to be an automorphism (Corollary 16). The last part of the proof, concerned with the iteration to higher order Segre sets, is established by a careful analysis of standard reflection identities. During the course of the proof, we also have to keep track of the jet order needed to get the determination of the maps so that this order behaves upper-semicontinuously on base points when applied at varying points of smooth generic submanifolds. This is done in the formal setting by defining new numerical invariants associated to any formal generic submanifold; such invariants are used to provide an explicit jet order that behaves upper-semicontinuously on the source manifold when this latter is subject to arbitrary continuous deformations. The proofs of the results mentioned in the introduction are then derived from Theorem 9 in Section 5.

2. Formal submanifolds and mappings

2.1. Basic definitions

For $x = (x_1, \ldots, x_k) \in \mathbb{C}^k$, we denote by $\mathbb{C}[[x]]$ the ring of formal power series in x and by $\mathbb{C}\{x\}$ the subring of convergent ones. If $I \subset \mathbb{C}[[x]]$ is an ideal and $F : (\mathbb{C}^k_x, 0) \to (\mathbb{C}^{k'}_{x'}, 0)$

is a formal map, then we define the *pushforward* $F_*(I)$ of I to be the ideal in $\mathbb{C}[\![x']\!]$, $x' \in \mathbb{C}^{k'}$, $F_*(I) := \{h \in \mathbb{C}[\![x']\!]\colon\ h \circ F \in I\}$. We also call the *generic rank* of F and denote by $\mathrm{Rk}\, F$ the rank of the Jacobian matrix $\partial F/\partial x$ regarded as a $\mathbb{C}[\![x]\!]$-linear mapping $(\mathbb{C}[\![x]\!])^k \to (\mathbb{C}[\![x]\!])^{k'}$. Hence $\mathrm{Rk}\, F$ is the largest integer r such that there is an $r \times r$ minor of the matrix $\partial F/\partial x$ which is not 0 as a formal power series in x. Note that if F is convergent, then $\mathrm{Rk}\, F$ is the usual generic rank of the map F. In addition, for any complex-valued formal power series $h(x)$, we denote by $\bar{h}(x)$ the formal power series obtained from h by taking complex conjugates of the coefficients. We also denote by $\mathrm{ord}\, h \in \mathbb{N} \cup \{+\infty\}$ the order of h i.e. the smallest integer r such that $\partial^\alpha h(0) = 0$ for all $\alpha \in \mathbb{N}^k$ with $|\alpha| \leqslant r-1$ and for which $\partial^\beta h(0) \neq 0$ for some $\beta \in \mathbb{N}^k$ with $|\beta| = r$ (if $h \equiv 0$, we set $\mathrm{ord}\, h = +\infty$). Moreover, if $S = S(x, x') \in \mathbb{C}[\![x, x']\!]$, we write $\mathrm{ord}_x S$ to denote the order of S viewed as a power series in x with coefficients in the ring $\mathbb{C}[\![x']\!]$.

2.2. *Formal generic submanifolds and normal coordinates*

For $(Z, \zeta) \in \mathbb{C}^N \times \mathbb{C}^N$, we define the involution $\sigma : \mathbb{C}[\![Z, \zeta]\!] \to \mathbb{C}[\![Z, \zeta]\!]$ by $\sigma(f)(Z, \zeta) := \bar{f}(\zeta, Z)$. Let $r = (r_1, \ldots, r_d) \in (\mathbb{C}[\![Z, \zeta]\!])^d$ such that *r is invariant under the involution σ*. Such an r is said to define a *formal generic submanifold* through the origin, which we denote by M, if $r(0) = 0$ and the vectors $\partial_Z r_1(0), \ldots, \partial_Z r_d(0)$ are linearly independent over $\mathbb{C}$. In this case, the number $n := N - d$ is called the *CR dimension* of M, the number $2N - d$ the *dimension* of M and the number d the *codimension* of M. Throughout the paper, we shall freely write $M \subset \mathbb{C}^N$. The complex space of vectors of $T_0\mathbb{C}^N$ which are in the kernel of the complex linear map $\partial_Z r(0)$ will be denoted by $T_0^{1,0}M$. Furthermore, in the case $d = 1$, a formal generic submanifold will be called a *formal real hypersurface*. These definitions are justified by the fact that, on the one hand, if $r \in (\mathbb{C}\{Z, \zeta\})^d$ defines a formal generic submanifold then the set $\{Z \in \mathbb{C}^N\colon\ r(Z, \bar{Z}) = 0\}$ is a germ through the origin in $\mathbb{C}^N$ of a real-analytic generic submanifold and $T_0^{1,0}M$ is the usual space of $(1,0)$ tangent vectors of M at the origin (see e.g. [2]). On the other hand, if Σ is a germ through the origin of a smooth generic submanifold of $\mathbb{C}^N$, then the complexified Taylor series of a local smooth vector-valued defining function for Σ near 0 gives rise to a formal generic submanifold as defined above. These observations will be used to derive the results mentioned in the introduction from the corresponding results for formal generic submanifolds given in Section 3.

Given a topological space T, by a *continuous family of formal generic submanifolds* $(M_t)_{t\in T}$, we mean the data of a formal power series mapping $r(Z, \zeta; t) = (r_1(Z, \zeta; t), \ldots, r_d(Z, \zeta; t))$ in (Z, ζ) with coefficients that are continuous functions of t and such that for each $t \in T$, M_t defines a formal submanifold as described above. When T is furthermore a smooth submanifold and the coefficients depend smoothly on t, we say that $(M_t)_{t\in T}$ is a *smooth family of formal generic submanifolds*. An important example (for this paper) of such a family is given when considering a smooth generic submanifold of $\mathbb{C}^N$ near some point $p_0 \in \mathbb{C}^N$ and allowing the base point to vary. In such a case, the smooth family of formal submanifolds is just obtained by considering a smooth defining function $\rho = (\rho_1, \ldots, \rho_d)$ for M near p_0 and by setting $r(Z, \zeta; p)$ to be the complexified Taylor series mapping of ρ at the point p, for p sufficiently close to p_0.

Given a family $\mathcal{E}$ of formal generic submanifolds of $\mathbb{C}^N$, a numerical invariant ι attached to the family $\mathcal{E}$ and a submanifold $M \in \mathcal{E}$, we will further say that $\iota(M)$ *depends upper-semicontinuously on continuous deformations of M* if for every continuous family of formal generic submanifolds $(M_t)_{t\in T}$ with $M_{t_0} = M$ for some $t_0 \in T$, there exists a neighborhood ω of t_0 in T such that $M_t \in \mathcal{E}$ for all $t \in \omega$ and such that the function $\omega \ni t \mapsto \iota(M_t)$ is upper-semicontinuous.

Throughout this paper, it will be convenient to use (formal) *normal coordinates* associated to any formal generic submanifold M of $\mathbb{C}^N$ of codimension d (see e.g. [2]). They are given as follows. There exists a formal change of coordinates in $\mathbb{C}^N \times \mathbb{C}^N$ of the form $(Z,\zeta) = (Z(z,w), \bar{Z}(\chi,\tau))$, where $Z = Z(z,w)$ is a formal change of coordinates in $\mathbb{C}^N$ and where $(z,\chi) = (z_1,\ldots,z_n,\chi_1,\ldots,\chi_n) \in \mathbb{C}^n \times \mathbb{C}^n$, $(w,\tau) = (w_1,\ldots,w_d,\tau_1,\ldots,\tau_d) \in \mathbb{C}^d \times \mathbb{C}^d$ so that M is defined through the following defining equations

$$r\big((z,w),(\chi,\tau)\big) = w - Q(z,\chi,\tau), \tag{1}$$

where $Q = (Q^1,\ldots,Q^d) \in (\mathbb{C}[\![z,\chi,\tau]\!])^d$ satisfies

$$Q^j(0,\chi,\tau) = Q^j(z,0,\tau) = \tau_j, \quad j = 1,\ldots,d. \tag{2}$$

Furthermore if $(M_t)_{t\in T}$ is a continuous (respectively smooth) family of formal generic submanifolds with $M = M_{t_0}$ for some $t_0 \in T$, then one may construct normal coordinates so that the formal power series mapping $Q = Q(z,\chi,\tau;t)$ depends continuously (respectively smoothly) on t for t sufficiently close to t_0.

2.3. Formal mappings

Let $r, r' \in (\mathbb{C}[\![Z,\zeta]\!])^d \times (\mathbb{C}[\![Z,\zeta]\!])^d$ define two formal generic submanifolds M and M' respectively of the same dimension and let $\mathcal{I}(M)$ (respectively $\mathcal{I}(M')$) be the ideal generated by r (respectively by r'). Throughout the paper, given a formal power series mapping φ with components in the ring $\mathbb{C}[\![Z,\zeta]\!]$, we write $\varphi(Z,\zeta) = 0$ for $(Z,\zeta) \in \mathcal{M}$ to mean that each component of φ belongs to the ideal $\mathcal{I}(M)$. Let now $H : (\mathbb{C}^N,0) \to (\mathbb{C}^N,0)$ be a formal holomorphic map. For every integer k, the k-jet of H, denoted by $j_0^k H$, is simply the usual k-jet at 0 of H. We associate to the map H another formal map $\mathcal{H} : (\mathbb{C}^N \times \mathbb{C}^N,0) \to (\mathbb{C}^N \times \mathbb{C}^N,0)$ defined by $\mathcal{H}(Z,\zeta) = (H(Z),\bar{H}(\zeta))$. We say that H *sends M into M'* if $\mathcal{I}(M') \subset \mathcal{H}_*(\mathcal{I}(M))$ and write $H(M) \subset M'$. Note that if M, M' are germs through the origin of real-analytic generic submanifolds of $\mathbb{C}^N$ and H is convergent, then $H(M) \subset M'$ is equivalent to say that H sends a neighborhood of 0 in M into M'. On the other hand, observe that if M, M' are merely smooth generic submanifolds through the origin and $h : (M,0) \to (M',0)$ is a germ of a smooth CR mapping, then there exists a unique (see e.g. [2]) formal (holomorphic) map $H : (\mathbb{C}^N,0) \to (\mathbb{C}^N,0)$ extending the Taylor series of h at 0 (in any local coordinate system). Then the obtained formal map H sends M into M' in the sense defined above when M and M' are viewed as formal generic submanifolds.

A formal map $H : (\mathbb{C}^N,0) \to (\mathbb{C}^N,0)$ sending M into M' where M, M' are formal generic submanifolds of $\mathbb{C}^N$ is called *CR-transversal* if

$$T_0^{1,0}M' + dH\big(T_0\mathbb{C}^N\big) = T_0\mathbb{C}^N, \tag{3}$$

where dH denotes the differential of H (at 0). We say that H is a *finite* map if the ideal generated by the components of the map H is of finite codimension in the ring $\mathbb{C}[\![Z]\!]$. If M, M' are merely smooth generic submanifolds through the origin and $h : (M,0) \to (M',0)$ is a germ of a smooth CR mapping, we say that h is CR-transversal (respectively finite) if its unique associated formal (holomorphic) power series mapping extension is CR-transversal (respectively finite).

Finally, given M, M' two real-analytic CR submanifolds of $\mathbb{C}^N$, $h: M \to M'$ a smooth CR mapping, k a positive integer and p a point in M, we will denote by $j_p^k h$ the usual k-jet of h at p. Note that there exists a (not necessarily unique) formal holomorphic map $(\mathbb{C}^N, p) \to (\mathbb{C}^N, h(p))$ extending the power series of h at p whose restriction to the intrinsic complexification of M at p is unique (see e.g. [2]). We then say that h is a finite CR mapping if the above restricted map is a finite formal holomorphic map.

2.4. *Nondegeneracy conditions for formal submanifolds and numerical invariants*

A formal vector field V in $\mathbb{C}^N \times \mathbb{C}^N$ is a $\mathbb{C}$-linear derivation of the ring $\mathbb{C}[[Z, \zeta]]$. If M is a formal generic submanifold of $\mathbb{C}^N$, we say that V is tangent to M if $V(f) \in \mathcal{I}(M)$ for every $f \in \mathcal{I}(M)$.

A formal $(1,0)$-vector field X in $\mathbb{C}_Z^N \times \mathbb{C}_\zeta^N$ is of the form

$$X = \sum_{j=1}^N a_j(Z, \zeta) \frac{\partial}{\partial Z_j}, \quad a_j(Z, \zeta) \in \mathbb{C}[[Z, \zeta]],\ j = 1, \ldots, N. \tag{4}$$

Similarly, a $(0,1)$-vector field Y in $\mathbb{C}_Z^N \times \mathbb{C}_\zeta^N$ is given by

$$Y = \sum_{j=1}^N b_j(Z, \zeta) \frac{\partial}{\partial \zeta_j}, \quad b_j(Z, \zeta) \in \mathbb{C}[[Z, \zeta]],\ j = 1, \ldots, N. \tag{5}$$

For a formal generic submanifold M of $\mathbb{C}^N$ of codimension d, we denote by $\mathfrak{g}_M$ the Lie algebra generated by the formal $(1,0)$ and $(0,1)$ vector fields tangent to M. The formal generic submanifold M is said to be *of finite type* if the dimension of $\mathfrak{g}_M(0)$ over $\mathbb{C}$ is $2N - d$, where $\mathfrak{g}_M(0)$ is the vector space obtained by evaluating the vector fields in $\mathfrak{g}_M$ at the origin of $\mathbb{C}^{2N}$. Note that if $M \subset \mathbb{C}^N$ is a smooth generic submanifold through the origin, then the above definition coincides with the usual finite type condition due to Kohn [29] and Bloom–Graham [10].

We now need to introduce a nondegeneracy condition for formal generic submanifolds, which in the real-analytic case was already defined by the authors in [32]. Let therefore M be a formal generic submanifold of $\mathbb{C}^N$ of codimension d and choose normal coordinates as in Section 2.2. For every $\alpha \in \mathbb{N}^n$, we set $\Theta_\alpha(\chi) = (\Theta_\alpha^1(\chi), \ldots, \Theta_\alpha^d(\chi)) := (Q_{z^\alpha}^1(0, \chi, 0), \ldots, Q_{z^\alpha}^d(0, \chi, 0))$.

Definition 7. We say that a formal submanifold M defined in normal coordinates as above is in the class $\mathcal{C}$ if for k large enough the generic rank of the formal (holomorphic) map $\chi \mapsto (\Theta_\alpha(\chi))_{|\alpha| \leqslant k}$ is equal to n. If this is the case, we denote by κ_M the smallest integer k for which the rank condition holds.

If the formal submanifold $M \notin \mathcal{C}$, we set $\kappa_M = +\infty$. In Section 4, we will show that for a formal submanifold M, being in the class $\mathcal{C}$ is independent of the choice of normal coordinates. Further, it will also be shown that $\kappa_M \in \mathbb{N} \cup \{+\infty\}$ is invariantly attached to M (see Corollary 15). Note that if $(M_t)_{t \in T}$ is a continuous family of formal generic submanifolds (parametrized by some topological space T) such that $M_{t_0} = M$ for some $t_0 \in T$ and $M \in \mathcal{C}$, then there exists a neighborhood ω of t_0 in T such that $M_t \in \mathcal{C}$ for all $t \in \omega$ and furthermore the map $\omega \ni t \mapsto \kappa_{M_t}$

is clearly upper-semicontinuous. This remark is useful to keep in mind during the proof of Theorem 9 below. Note also that the definition of the class $\mathcal{C}$ given here coincides with that given in [32] in the real-analytic case. We therefore refer the reader to the latter paper for further details on that class in the real-analytic case. We only note here that several comparison results between the class $\mathcal{C}$ and other classes of generic submanifolds still hold in the formal category. For instance, recall that a formal manifold is said to be *essentially finite* if the formal holomorphic map $\chi \mapsto (\Theta_\alpha(\chi))_{|\alpha|\leqslant k}$ is finite for k large enough. It is therefore clear that if M is essentially finite, then $M \in \mathcal{C}$. As in the real-analytic case, there are also other classes of formal submanifolds that are not essentially finite and that still belong to the class $\mathcal{C}$. We leave the interested reader to mimic in the formal setting what has been done in the real-analytic case in [32].

If M is a smooth generic submanifold of $\mathbb{C}^N$ and $p \in M$, we say that (M, p) is in the class $\mathcal{C}$ (respectively essentially finite) if the formal generic submanifold associated to (M, p) (as explained in Section 2.2) is in the class $\mathcal{C}$ (respectively is essentially finite).

For every formal submanifold $M \subset \mathbb{C}^N$, we need to define another numerical quantity that will be used to give an explicit bound on the number of jets needed in Theorem 9. Given a choice of normal coordinates $Z = (z, w)$ for M, we set for any n-tuple of multiindices $\underline{\alpha} := (\alpha^{(1)}, \ldots, \alpha^{(n)})$, $\alpha^{(j)} \in \mathbb{N}^n$, and any n-tuple of integers $\underline{s} := (s_1, \ldots, s_n) \in \{1, \ldots, d\}^n$

$$D_M^Z(\underline{\alpha}, \underline{s}) = \det \begin{pmatrix} \frac{\partial \Theta^{s_1}_{\alpha^{(1)}}}{\partial \chi_1} & \cdots & \frac{\partial \Theta^{s_1}_{\alpha^{(1)}}}{\partial \chi_n} \\ \vdots & & \vdots \\ \frac{\partial \Theta^{s_n}_{\alpha^{(n)}}}{\partial \chi_1} & \cdots & \frac{\partial \Theta^{s_n}_{\alpha^{(n)}}}{\partial \chi_n} \end{pmatrix}. \tag{6}$$

Let us write $|\underline{\alpha}| := \max\{|\alpha^{(j)}|\colon 1 \leqslant j \leqslant n\}$. We now define for every integer $k \geqslant 1$,

$$\nu_M^Z(k) := \inf\bigl\{\operatorname{ord} D_M^Z(\underline{\alpha}, \underline{s})\colon |\underline{\alpha}| \leqslant k\bigr\} \in \mathbb{N} \cup \{+\infty\}. \tag{7}$$

Note that for a general formal submanifold M, the numerical quantity $\nu_M^Z(k)$ depends *a priori* on a choice of normal coordinates for M; it will be shown in Section 4.1 that $\nu_M^Z(k)$ is in fact independent of such a choice, and thus is a biholomorphic invariant of M. In view of this result, we will simply write $\nu_M(k)$ for $\nu_M^Z(k)$ for every k. Observe also that if $M \in \mathcal{C}$ then for all $k \geqslant \kappa_M$, $\nu_M(k) < +\infty$.

We also define the following quantity

$$\nu_M(\infty) := \lim_{k\to\infty} \nu_M(k) = \inf_{k\in\mathbb{N}} \nu_M(k) \in \mathbb{N} \cup \{+\infty\}, \tag{8}$$

and notice that $\nu_M(\infty) = 0$ if and only if for some k, the map $\chi \mapsto (\Theta_\alpha(\chi))_{|\alpha|\leqslant k}$ is immersive; this is equivalent to M being *finitely nondegenerate* (for other possible ways of expressing this condition, see e.g. [2]).

Given the invariance of $\nu_M(k)$ for each k, it is also easy to see that if $(M_t)_{t\in T}$ is a continuous family of generic submanifolds, then for every $k \in \mathbb{N}^* \cup \{\infty\}$, the mappings $T \ni t \mapsto \kappa_{M_t}$ and $T \ni t \mapsto \nu_{M_t}(k)$ are clearly upper-semicontinuous. Hence, the numerical quantities κ_M and $\nu_M(k)$ for $k \in \mathbb{N}^* \cup \{\infty\}$ depend upper-semicontinuously on continuous deformations of M. This fact has also to be kept in mind during the proof of Theorem 9 below.

2.5. Finite type and Segre sets mappings

We here briefly recall the definition of the Segre sets mappings associated to any formal generic submanifold as well as the finite type criterion in terms of these mappings due to Baouendi, Ebenfelt and Rothschild [3].

Let M be a formal submanifold of codimension d in $\mathbb{C}^N$ given for simplicity in normal coordinates as in Section 2.2. Then for every integer $j \geqslant 1$, we define a formal mapping $v^j : (\mathbb{C}^{nj}, 0) \to (\mathbb{C}^N, 0)$ called the Segre set mapping of order j as follows. We first set $v^1(t^1) = (t^1, 0)$ and define inductively the v^j by the formula

$$v^{j+1}(t^1, \dots, t^{j+1}) = \big(t^{j+1}, Q\big(t^{j+1}, \bar{v}^j\big(t^1, \dots, t^j\big)\big)\big). \tag{9}$$

Here and throughout the paper, each $t^k \in \mathbb{C}^n$ and we shall also use the notation $t^{[j]} = (t^1, \dots, t^j)$ for brevity. Note that for every formal power series mapping $h \in \mathbb{C}[[Z, \zeta]]$ such that $h(Z, \zeta) = 0$ for $(Z, \zeta) \in \mathcal{M}$, one has the identities $h(v^{j+1}, \bar{v}^j) \equiv 0$ in the ring $\mathbb{C}[[t^1, \dots, t^{j+1}]]$ and $h(v^1(t^1), 0) \equiv 0$ in $\mathbb{C}[[t^1]]$.

The following well-known characterization of finite type for a formal generic submanifold in terms of its Segre sets mappings will be useful in the conclusion of the proof of Theorem 9.

Theorem 8. *(See [3].) Let M be a formal generic submanifold of $\mathbb{C}^N$. Then M is of finite type if and only if there exists an integer $1 \leqslant m \leqslant (d+1)$ such that $\operatorname{Rk} v^k = N$ for all $k \geqslant m$.*

3. Statement of the main result for formal submanifolds

We will derive in Section 5 the results mentioned in the introduction from the following finite jet determination result for formal mappings between formal submanifolds.

Theorem 9. *Let $M \subset \mathbb{C}^N$ be a formal generic submanifold of finite type which is in the class $\mathcal{C}$. Then there exists an integer K depending only on M satisfying the following properties:*

(i) *For every formal generic manifold M' of $\mathbb{C}^N$ with the same dimension as M, and for any pair $H_1, H_2 : (\mathbb{C}^N, 0) \to (\mathbb{C}^N, 0)$ of formal CR-transversal holomorphic mappings sending M into M' it holds that if the K-jets of H_1 and H_2 agree, then necessarily $H_1 = H_2$.*
(ii) *The integer K depends upper-semicontinuously on continuous deformations of M.*

The upper-semicontinuity of the jet order K on continuous perturbations of M in the above theorem is of fundamental importance in order to provide the upper-semicontinuity of the integer ℓ_p on p in Theorem 1 (see Section 5 for details). We also mention here the following consequence of Theorem 9 which, under additional assumptions on the manifolds, provides a finite jet determination result valid for pairs of *arbitrary* maps. In what follows, we say that a formal manifold M of $\mathbb{C}^N$ contains a formal curve if there exists a nonconstant formal map $\gamma : (\mathbb{C}_t, 0) \to (\mathbb{C}^N, 0)$ such that for every $h \in \mathcal{I}(M)$, $h(\gamma(t), \overline{\gamma(t)}) \equiv 0$.

Corollary 10. *Let $M, M' \subset \mathbb{C}^N$ be a formal real hypersurfaces. Assume that $M \in \mathcal{C}$ and that M' does not contain any formal curve. Then there exists an integer K, depending only on M, such that for any pair of formal holomorphic maps $H_1, H_2 : (\mathbb{C}^N, 0) \to (\mathbb{C}^N, 0)$ sending M into*

M' it holds that if the K-jets of H_1 and H_2 agree, then necessarily $H_1 = H_2$. Furthermore, the integer K can be chosen to depend upper-semicontinuously on continuous deformations of M.

Proof. The corollary is an immediate consequence of Theorem 9 by noticing that any formal real hypersurface that belongs to the class $\mathcal{C}$ is necessarily of finite type and by using [31, Corollary 2.4] that in the setting of Corollary 10, any formal holomorphic mapping $H : (\mathbb{C}^N, 0) \to (\mathbb{C}^N, 0)$ sending M into M' is either constant or CR-transversal. $\square$

The proof of Theorem 9 is given the next section. In order to prove this theorem, we need to establish several new properties of CR-transversal maps *along the Segre variety* (which is done through Sections 4.1–4.2). Since the maps we consider will turn out to be *not totally degenerate*, that is, their restriction to the Segre variety is of generic full rank, a careful analysis of the usual reflection identities will suffice to iterate the determination property along higher order Segre sets (this is carried out in Section 4.3). The well-known finite type criterion (given in Theorem 8) is finally used to conclude the proof of the theorem.

4. Proof of Theorem 9

In this section, we use the notation and terminology introduced in Section 2. We let M, M' be two formal generic submanifolds of $\mathbb{C}^N$ with the same codimension d and *fix* a choice of normal coordinates $Z = (z, w)$ (respectively $Z' = (z', w')$) so that M (respectively M') is defined through the power series mapping $Q = Q(z, \chi, \tau)$ (respectively $Q' = Q'(z', \chi', \tau')$) given in (1). Recall that we write

$$\Theta_\alpha(\chi) = Q_{z^\alpha}(0, \chi, 0), \quad \alpha \in \mathbb{N}^n. \tag{10}$$

In what follows, we use analogous notations for M' by just adding a "prime" to the corresponding objects.

For every formal map $H : (\mathbb{C}^N, 0) \to (\mathbb{C}^N, 0)$, we split the map

$$H = (F, G) = \left(F^1, \dots, F^n, G^1, \dots, G^d\right) \in \mathbb{C}^n \times \mathbb{C}^d$$

according to the above choice of normal coordinates for M'. If H sends M into M', we have the following fundamental $\mathbb{C}^d$-valued identity

$$G\big(z, Q(z, \chi, \tau)\big) = Q'\big(F\big(z, Q(z, \chi, \tau)\big), \bar{F}(\chi, \tau), \bar{G}(\chi, \tau)\big), \tag{11}$$

which holds in the ring $\mathbb{C}[[z, \chi, \tau]]$. Note that H is CR-transversal if and only the $d \times d$ matrix $G_w(0)$ is invertible (see e.g. [22]). Recall also that H is *not totally degenerate* if $\operatorname{Rk} F_z(z, 0) = n$.

For every positive integer k, we denote by $J^k_{0,0}(\mathbb{C}^N, \mathbb{C}^N)$ the jet space of order k of formal holomorphic maps $(\mathbb{C}^N, 0) \to (\mathbb{C}^N, 0)$ and by j^k_0 be the k-jet mapping. (After identifying the jet space with polynomials of degree k, this is just the map which truncates the Taylor series at degree k.) As done before, we equip the source space $\mathbb{C}^N$ with normal coordinates Z for M and the target space $\mathbb{C}^N$ with normal coordinates Z' for M'. This choice being fixed, we denote by Λ^k the corresponding coordinates on $J^k_{0,0}(\mathbb{C}^N, \mathbb{C}^N)$ and by $\mathcal{T}^k_0(\mathbb{C}^N)$ the open subset of $J^k_{0,0}(\mathbb{C}^N, \mathbb{C}^N)$ consisting of k-jets of holomorphic maps $H = (F, G)$ for which $G_w(0)$ is invertible. Hence, for every formal CR-transversal mapping H sending M into M', we have $j^k_0 H \in \mathcal{T}^k_0(\mathbb{C}^N)$.

4.1. Properties of CR-transversal maps on the first Segre set

We start by establishing here a few facts concerning CR-transversal formal holomorphic mappings sending formal generic submanifolds into each other. We will in particular derive the following list of important properties:

(1) We provide the invariance of the condition to be in the class $\mathcal{C}$ for a formal submanifold M as well as the invariance of the associated numerical quantities κ_M and $\nu_M(k)$ for $k \in \mathbb{N}^*$ (Corollary 15).
(2) We obtain some rigidity properties of CR-transversal mappings between submanifolds in the class $\mathcal{C}$, e.g. the fact that they are necessarily not totally degenerate with a certain uniform bound on the degeneracy considered (see Corollary 12 and Eq. (20)) as well as their determination on the first Segre set by a finite jet (Corollary 19).
(3) As a byproduct of the proofs, we obtain a new sufficient condition on M that force any CR transversal formal map sending M into another formal submanifold M' of the same dimension to be a formal biholomorphism (Corollary 16).

All the above mentioned properties will be obtained as consequences of the following result, which can be seen as a generalization in higher codimension of an analogous version obtained for the case of hypersurfaces in [18].

Proposition 11. *Let M, M' be formal generic submanifolds of $\mathbb{C}^N$ of the same dimension. Then for every $\alpha \in \mathbb{N}^n$, there exists a universal $\mathbb{C}^d$-valued holomorphic map Φ_α defined in a neighborhood of $\{0\} \times \mathcal{T}_0^{|\alpha|}(\mathbb{C}^N) \subset \mathbb{C}^{dr_{|\alpha|}} \times \mathcal{T}_0^{|\alpha|}(\mathbb{C}^N)$, where $r_{|\alpha|} := \operatorname{card}\{\beta \in \mathbb{N}^n\colon 1 \leqslant |\beta| \leqslant |\alpha|\}$, such that for every CR-transversal formal map $H : (\mathbb{C}^N, 0) \to (\mathbb{C}^N, 0)$ sending M into M', we have*

$$\Theta_\alpha(\chi) = \Phi_\alpha\big(\big(\Theta'_\beta\big(\bar F(\chi,0)\big)\big)_{|\beta| \leqslant |\alpha|}, j_0^{|\alpha|}H\big). \tag{12}$$

Proof. We proceed by induction on the length of α. For every $j = 1, \ldots, n$, we denote by e_j the multiindex of $\mathbb{N}^n$ having 1 at the jth digit and zero elsewhere. Let H be as in the statement of the proposition. Differentiating (11) with respect to z_j, evaluating at $z = \tau = 0$ and using the fact that $G(z, 0) \equiv 0$ (which follows directly from (11)) yields

$$G_w(0) \cdot \Theta_{e_j}(\chi) = \sum_{k=1}^{n} \Theta'_{e_k}\big(\bar F(\chi,0)\big)\big(F^k_{z_j}(0) + F^k_w(0) \cdot \Theta_{e_j}(\chi)\big), \tag{13}$$

where Θ_{e_j} is considered as a column vector and $F^k_w(0)$ as a row vector. We thus define polynomial maps

$$A : \mathbb{C}^{dn} \times J^1_{0,0}\big(\mathbb{C}^N, \mathbb{C}^N\big) \to \mathbb{M}_d(\mathbb{C}), \qquad B_j : \mathbb{C}^{dn} \times J^1_{0,0}\big(\mathbb{C}^N, \mathbb{C}^N\big) \to \mathbb{C}^d, \quad j = 1, \ldots, n,$$

where $\mathbb{M}_d$ denotes the space of $d \times d$ complex-valued matrices, so that for each $j = 1, \ldots, n$, so that for every map H as above

$$A\big(\big(\Theta'_\beta\big(\bar F(\chi,0)\big)\big)_{|\beta|=1}, j_0^1 H\big) = \frac{\partial G}{\partial w}(0) - \sum_{k=1}^n \Theta'_{e_k}\big(\bar F(\chi,0)\big)\cdot F_w^k(0),$$

$$B_j\big(\big(\Theta'_\beta\big(\bar F(\chi,0)\big)\big)_{|\beta|=1}, j_0^1 H\big) = \sum_{k=1}^n \Theta'_{e_k}\big(\bar F(\chi,0)\big) F_{z_j}^k(0). \tag{14}$$

Note also that for all $\Lambda^1 \in \mathcal{T}_0^1(\mathbb{C}^N)$, $\det A(0,\Lambda^1) \neq 0$. Therefore, $\Phi_{e_j} := A^{-1}\cdot B_j$ is holomorphic in a neighborhood of $\{0\}\times \mathcal{T}_0^1(\mathbb{C}^N) \subset \mathbb{C}^{dn}\times \mathcal{T}_0^1(\mathbb{C}^N)$ and satisfies the desired property in view of (13).

To prove (12) for $|\alpha|>1$, one differentiates (11) with respect to z^α and evaluates at $z=\tau=0$. Using the induction to express every term Θ_β with $|\beta|<|\alpha|$ by Φ_β, we obtain for every formal map $H:(\mathbb{C}^N,0)\to(\mathbb{C}^N,0)$ sending M into M' an expression of the form

$$A\big(\big(\Theta'_\beta\big(\bar F(\chi,0)\big)\big)_{|\beta|=1}, j_0^1 H\big)\cdot \Theta_\alpha(\chi) = B_\alpha\big(\big(\Theta'_\beta\big(\bar F(\chi,0)\big)\big)_{|\beta|\leqslant|\alpha|}, j_0^{|\alpha|} H\big),$$

where $B_\alpha : \mathbb{C}^{dr_{|\alpha|}} \times J_{0,0}^1(\mathbb{C}^N,\mathbb{C}^N) \to \mathbb{C}^d$ is a universal polynomial map and A is given by (14). As in the case of multiindices of length one, we conclude by setting $\Phi_\alpha := A^{-1}\cdot B_\alpha$. The proof of Proposition 11 is complete. □

A number of interesting consequences may be derived from Proposition 11. For instance, it immediately yields the following corollary; we note that we have not yet proved the independence of the quantities κ_M and $\nu_M(k)$ for $k\in\mathbb{N}^*\cup\{\infty\}$ on the choice of coordinates; however, this invariance, stated in Corollary 15 below, is an immediate consequence of Corollary 12, so we already state this latter in the invariant way.

Corollary 12. *Let $M, M' \subset \mathbb{C}^N$ be a formal generic submanifolds of the same dimension. Suppose that M belongs to the class $\mathcal{C}$ (as defined in Section 2.4) and that there exists a formal CR-transversal map $H:(\mathbb{C}^N,0)\to(\mathbb{C}^N,0)$ sending M into M'. Then necessarily H is not totally degenerate, $M'\in\mathcal{C}$, $\kappa_{M'}\leqslant\kappa_M$, and for every integer $k\geqslant 1$, $\nu_M(k)\geqslant \nu_{M'}(k)+\operatorname{ord}\det \bar F_\chi(\chi,0)$.*

Proof. We start the proof by introducing some notation which will be used consistently from now on. For any n-tuple of multiindices of $\mathbb{N}^n$ $\underline{\alpha}=(\alpha^{(1)},\dots,\alpha^{(n)})$ and integers $\underline{s}=(s_1,\dots,s_n)\in\{1,\dots,d\}^n$, we write

$$\Theta_{\underline{\alpha},\underline{s}} = \big(\Theta^{s_1}_{\alpha^{(1)}},\dots,\Theta^{s_n}_{\alpha^{(n)}}\big), \tag{15}$$

and

$$\Phi_{\underline{\alpha},\underline{s}} := \big(\Phi^{s_1}_{\alpha^{(1)}},\dots,\Phi^{s_n}_{\alpha^{(n)}}\big) \tag{16}$$

for the corresponding map given by Proposition 11. We thus have from the same proposition that

$$\Theta_{\underline{\alpha},\underline{s}}(\chi) = \Phi_{\underline{\alpha},\underline{s}}\big(\big(\Theta'_\beta\big(\bar F(\chi,0)\big)\big)_{|\beta|\leqslant|\underline{\alpha}|}, j_0^{|\underline{\alpha}|}H\big) = \Phi_{\underline{\alpha},\underline{s}}\big(\Theta'_{|\underline{\alpha}|}\big(\bar F(\chi,0)\big), j_0^{|\underline{\alpha}|}H\big), \tag{17}$$

where we use the notation $\Theta'_k = (\Theta'_\beta)_{|\beta|\leqslant k}$ for every integer k. We also write for any $\underline{\alpha}$, $\underline{s}$

$$\Upsilon^H_{\underline{\alpha},\underline{s}}(\chi') := \Phi_{\underline{\alpha},\underline{s}}\big(\Theta'_{|\underline{\alpha}|}(\chi'), j_0^{|\underline{\alpha}|}H\big), \tag{18}$$

where we recall that $\Phi_{\underline{\alpha},\underline{s}} = \Phi_{\underline{\alpha},\underline{s}}(X, \Lambda^{|\underline{\alpha}|})$ is holomorphic in a neighborhood of $\{0\} \times \mathcal{T}_0^{|\underline{\alpha}|}(\mathbb{C}^N) \subset \mathbb{C}^{dr_{|\underline{\alpha}|}} \times \mathcal{T}_0^{|\underline{\alpha}|}(\mathbb{C}^N)$. Since $M \in \mathcal{C}$, we can choose n-tuples of multiindices $\underline{\alpha}$ and integers $\underline{s}$ with $|\underline{\alpha}| = \kappa_M$ such that the formal map $\chi \mapsto \Theta_{\underline{\alpha},\underline{s}}(\chi)$ is of generic rank n. Differentiating (17) with respect to χ yields

$$\frac{\partial \Theta_{\underline{\alpha},\underline{s}}}{\partial \chi}(\chi) = \frac{\partial \Upsilon^H_{\underline{\alpha},\underline{s}}}{\partial \chi'}\big(\bar{F}(\chi,0)\big) \cdot \bar{F}_\chi(\chi,0). \tag{19}$$

From (19), we immediately get that $\operatorname{Rk} \bar{F}_\chi(\chi,0) = n$ i.e. that H is not totally degenerate. We also immediately get that

$$\operatorname{Rk} \frac{\partial \Upsilon^H_{\underline{\alpha},\underline{s}}}{\partial \chi'}(\chi') = n,$$

which implies in view of (18) that the generic rank of the map $\chi' \mapsto \Theta'_{\kappa_M}(\chi')$ is also n, which shows that $M' \in \mathcal{C}$ and that $\kappa_{M'} \leqslant \kappa_M$.

Let us now prove the inequality for ν_M. To this end, for every integer $k \geqslant 1$ and for every choice of $\underline{\alpha} = (\alpha^{(1)}, \ldots, \alpha^{(n)}) \in \mathbb{N}^n \times \cdots \times \mathbb{N}^n$ with $|\underline{\alpha}| \leqslant k$ and $\underline{s} = (s_1, \ldots, s_n) \in \{1, \ldots, d\}^n$, we consider the resulting equation (17). Differentiating (17) with respect to χ yields that the determinant considered in (6) is expressed as the product of $\det \bar{F}_\chi(\chi,0)$ with the determinant of

$$\frac{\partial \Phi_{\underline{\alpha},\underline{s}}}{\partial X}\big(\Theta'_{|\underline{\alpha}|}(\chi'), j_0^{|\underline{\alpha}|} H\big)\big|_{\chi'=\bar{F}(\chi,0)} \cdot \frac{\partial \Theta'_{|\underline{\alpha}|}}{\partial \chi'}\big(\bar{F}(\chi,0)\big).$$

Applying the Cauchy–Binet formula (allowing to express the determinant of this matrix product as the sum of the product of corresponding minors of the factors), we get the equation

$$D^Z_M(\underline{\alpha},\underline{s}) = \Bigg(\sum_{\substack{|\underline{\beta}| \leqslant k \\ \underline{t} \in \{1,\ldots,d\}^n}} a_{\underline{\beta},\underline{t}}(\chi) D^{Z'}_{M'}(\underline{\beta},\underline{t})(\chi')|_{\chi'=\bar{F}(\chi,0)}\Bigg) \det \bar{F}_\chi(\chi,0).$$

From this we see that the order of the right-hand side is at least $\nu_{M'}(k) + \operatorname{ord} \det \bar{F}_\chi(\chi,0)$, and since this holds for any choice of $\underline{\alpha}$ and $\underline{s}$ as above, we obtain the inequality $\nu_M(k) \geqslant \nu_{M'}(k) + \operatorname{ord} \det \bar{F}_\chi(\chi,0)$. $\square$

Remark 13. Under the assumptions and notation of the proof of Corollary 12, it also follows from (19) that the order of the power series

$$\chi \mapsto \det\bigg(\frac{\partial \Upsilon^H_{\underline{\alpha},\underline{s}}}{\partial \chi'}\big(\bar{F}(\chi,0)\big)\bigg)$$

is *uniformly bounded* by $\nu_M(k)$ for any choice of n-tuple of multiindices $\underline{\alpha}$ with $|\underline{\alpha}| \leqslant k$ and of integers $\underline{s} = (s_1, \ldots, s_n)$ for which $\operatorname{ord} D^Z_M(\underline{\alpha},\underline{s}) = \nu_M(k)$. This fact will be useful in the proof of Corollary 19 and Proposition 24 below.

Remark 14. It is easy to see that the inequality $\nu_{M'}(k) + \operatorname{ord}\det \bar{F}_\chi(\chi, 0) \leqslant \nu_M(k)$ may be strict; consider for example $M = \{(z,w) \in \mathbb{C}^2\colon \operatorname{Im} w = |z|^8\}$, $M' = \{(z,w) \in \mathbb{C}^2\colon \operatorname{Im} w = |z|^4\}$, and $H(z,w) = (z^2, w)$. Our proof also gives the somewhat better inequality

$$\nu_{M'}(k) \cdot \operatorname{ord} \bar{F}(\chi, 0) + \operatorname{ord}\det \bar{F}_\chi(\chi, 0) \leqslant \nu_M(k)$$

(in which equality holds in the above example in $\mathbb{C}^2$, but not in general). The inequality given in Corollary 12 is strong enough in order to derive the invariance in Corollary 15 below, so we will not dwell on this matter any longer.

From Corollary 12, the invariance of κ_M and $\nu_M(k)$ immediately follows.

Corollary 15. *Let M be a formal generic submanifold of $\mathbb{C}^N$. Then the condition for M to be in the class $\mathcal{C}$ is independent of the choice of (formal) normal coordinates. Moreover, for M arbitrary, the integers κ_M and $\nu_M(k)$ for $k \in \mathbb{N}^* \cup \{\infty\}$, defined in Section* 2.4, *are also independent of a choice of such coordinates and hence invariantly attached to the formal submanifold M.*

Another consequence that is noteworthy to point out is given by the following criterion for a CR-transversal map to be an automorphism. Note that the inequality for the numerical invariant ν_M given in Corollary 12 implies that for any CR-transversal map H sending the formal generic submanifold M of $\mathbb{C}^N$, where $M \in \mathcal{C}$, into another formal generic submanifold M' of $\mathbb{C}^N$ with the same dimension, it follows that

$$\operatorname{ord}\det \bar{F}_\chi(\chi, 0) \leqslant \nu_M(\infty). \tag{20}$$

Recalling that $\nu_M(\infty) = 0$ if and only if M is finitely nondegenerate, we therefore get:

Corollary 16. *Let $M, M' \subset \mathbb{C}^N$ be formal generic submanifolds of the same dimension, and assume that $M \in \mathcal{C}$. Then a formal CR-transversal holomorphic map sending M into M' is an automorphism if and only if for some $k \geqslant \kappa_M$, $\nu_M(k) = \nu_{M'}(k)$. Furthermore, if M is finitely nondegenerate, every formal CR-transversal map is a formal biholomorphism.*

Remark 17. (i) A criterion analogous to the second part of Corollary 16 for a formal *finite* holomorphic mapping to be a biholomorphism was obtained in [22, Theorem 6.5] under the additional assumption that M is of finite type. In fact, this latter result can also be seen as a consequence of Corollary 16 in conjunction with the transversality result [22, Theorem 3.1]. Note also that the second part of Corollary 16 does not hold for finite maps as can be seen by considering $M = M' = \{(z, w_1, w_2) \in \mathbb{C}^3\colon \operatorname{Im} w_1 = |z|^2,\ \operatorname{Im} w_2 = 0\}$ and $H(z, w_1, w_2) = (z, w_1, w_2^2)$.

(ii) A nice application of the preceding corollary is also a "one-glance" proof of the fact that (for example) the hypersurfaces

$$\begin{aligned} M_1\colon &\quad \operatorname{Im} w = |z_1|^2 + \operatorname{Re} z_1^2 \bar{z}_2^3 + \operatorname{Re} z_1^4 \bar{z}_2 + O(6), \\ M_2\colon &\quad \operatorname{Im} w = |z_1|^2 + \operatorname{Re} z_1^2 \bar{z}_2^2 + \operatorname{Re} z_1^4 \bar{z}_2 + O(6), \end{aligned}$$

are not biholomorphically equivalent; indeed, both are finitely nondegenerate, and we have

$$\kappa_{M_1} = \kappa_{M_2} = 2, \quad \nu_{M_1}(k) = \nu_{M_2}(k), \quad \text{for } k \neq 2, \text{ but } 2 = \nu_{M_1}(2) \neq \nu_{M_2}(2) = 1.$$

As a consequence of (20) and [31, Corollary 2.4], we also get following property that under some additional assumptions on the manifolds, tangential flatness up to a certain order of a given map implies that it is necessarily constant.

Corollary 18. *Let $M \subset \mathbb{C}^N$ be a formal real hypersurface given in normal coordinates as above, and assume that $M \in \mathcal{C}$. Then there exists an integer k such that for every formal real hypersurface $M' \subset \mathbb{C}^N$ not containing any formal curve and every formal holomorphic map $H : (\mathbb{C}^N, 0) \to (\mathbb{C}^n, 0)$ sending M into M', $H = (F, G)$ is constant if and only if*

$$F_{z^\alpha}(0) = 0, \quad 1 \leqslant |\alpha| \leqslant k.$$

For the purposes of this paper, the most important consequence of Proposition 11 lies in the following finite jet determination property.

Corollary 19. *Let $M, M' \subset \mathbb{C}^N$ be formal generic submanifolds of the same dimension, given in normal coordinates as above. Assume that M belongs to the class $\mathcal{C}$. Then the integer*

$$k_0 := \min_{k \geqslant \kappa_M} \max\{k, \nu_M(k)\}$$

satisfies the following property: For any pair $H_1, H_2 : (\mathbb{C}^N, 0) \to (\mathbb{C}^N, 0)$ of formal CR-transversal holomorphic mappings sending M into M', if the k_0-jets of H_1 and H_2 agree, then necessarily $H_1(z, 0) = H_2(z, 0)$. Furthermore, k_0 depends upper-semicontinuously on continuous deformations of M.

Proof. Let $\tilde{k}$ be an integer with $\max\{\tilde{k}, \nu_M(\tilde{k})\} = k_0$. We choose $\underline{\alpha} = (\alpha^{(1)}, \ldots, \alpha^{(n)})$ with $|\underline{\alpha}| \leqslant \tilde{k}$ and $\underline{s} = (s_1, \ldots, s_n)$ such that $\operatorname{ord} D_M^Z(\underline{\alpha}, \underline{s}) = \nu_M(\tilde{k})$. We use the notation of the proof of Corollary 12, in particular, we consider the function $\Upsilon_{\underline{\alpha},\underline{s}}^{H_j}$ defined there, with this choice of $\underline{\alpha}$ and $\underline{s}$ and for a given pair H_1, H_2 of formal CR-transversal maps satisfying $j_0^{k_0} H_1 = j_0^{k_0} H_2$. In view of (18), we have

$$\Upsilon_{\underline{\alpha},\underline{s}}^{H_1}(\chi') = \Upsilon_{\underline{\alpha},\underline{s}}^{H_2}(\chi') =: \Upsilon_{\underline{\alpha},\underline{s}}(\chi'). \tag{21}$$

We write $H_j = (F_j, G_j) \in \mathbb{C}^n \times \mathbb{C}^d$, $j = 1, 2$. We now claim that $\bar{F}_1(\chi, 0) = \bar{F}_2(\chi, 0)$ which yields the desired result. Indeed first note that the identity

$$\Upsilon_{\underline{\alpha},\underline{s}}(y) - \Upsilon_{\underline{\alpha},\underline{s}}(x) = (y - x) \cdot \int_0^1 \frac{\partial \Upsilon_{\underline{\alpha},\underline{s}}}{\partial \chi'}\big(ty + (1-t)x\big)\, dt,$$

gives in view of (17) and (18) that

$$0 = \big(\bar{F}_2(\chi, 0) - \bar{F}_1(\chi, 0)\big) \cdot \int_0^1 \frac{\partial \Upsilon_{\underline{\alpha},\underline{s}}}{\partial \chi'}\big(t\bar{F}_2(\chi, 0) + (1-t)\bar{F}_1(\chi, 0)\big)\, dt. \tag{22}$$

To prove the claim, it is therefore enough to show that

$$\det\left(\int_0^1 \frac{\partial \Upsilon_{\underline{\alpha},\underline{s}}}{\partial \chi'}\big(t\bar{F}_2(\chi,0)+(1-t)\bar{F}_1(\chi,0)\big)\,dt\right)\not\equiv 0. \tag{23}$$

By Remark 13, the order of the power series $\chi \mapsto \det(\frac{\partial \Upsilon_{\underline{\alpha},\underline{s}}}{\partial \chi'}(\bar{F}_2(\chi,0)))$ is at most $\nu_M(\tilde{k})$ and since $\bar{F}_1(\chi,0)$ agrees with $\bar{F}_2(\chi,0)$ up to order $k_0 \geqslant \nu_M(\tilde{k})$, it follows that (23) automatically holds. The proof of Corollary 19 is complete, up to the upper-semicontinuity of the integer k_0, which is a direct consequence of the upper-semicontinuity on continuous deformations of M of the numerical invariants κ_M and $\nu_M(k)$ for all $k \in \mathbb{N}^* \cup \{\infty\}$. □

4.2. Finite jet determination of the derivatives on the first Segre set

Our next goal is to establish a finite jet determination property similar to that obtained in Corollary 19, but this time for the derivatives of the maps. For this, we will need a number of small technical lemmas. In what follows, for every positive integer ℓ, we write $\hat{j}_\zeta^\ell \bar{H}$ for $(\partial_\zeta^\alpha \bar{H}(\zeta))_{1\leqslant|\alpha|\leqslant\ell}$ and similarly for $\hat{j}_Z^\ell H$ to mean $(\partial_Z^\alpha H(Z))_{1\leqslant|\alpha|\leqslant\ell}$. We also keep the notation introduced in previous sections. We start with the following.

Lemma 20. *Let $M, M' \subset \mathbb{C}^N$ be formal generic submanifolds of codimension d given in normal coordinates as above. Then for every multiindex $\mu \in \mathbb{N}^d \setminus \{0\}$, there exists a universal $\mathbb{C}^d$-valued power series mapping $\mathcal{S}_\mu = \mathcal{S}_\mu(Z,\zeta,Z',\zeta';\cdot)$ polynomial in its last argument with coefficients in the ring $\mathbb{C}[[Z,\zeta,Z',\zeta']]$ such that for every formal holomorphic map $H:(\mathbb{C}^N,0)\to(\mathbb{C}^N,0)$ sending M into M' with $H=(F,G)\in\mathbb{C}^n\times\mathbb{C}^d$, the following identity holds for $(Z,\zeta)\in\mathcal{M}$:*

$$\begin{aligned}&\bar{F}_{\tau^\mu}(\zeta)\cdot Q'_{\chi'}\big(f(Z),\bar{H}(\zeta)\big)\\&\quad=\mathcal{S}_\mu\big(Z,\zeta,H(Z),\bar{H}(\zeta);\hat{j}_Z^{|\mu|}H,\big(\bar{F}_{\tau^\gamma}(\zeta)\big)_{|\gamma|\leqslant|\mu|-1},\big(\bar{G}_{\tau^\eta}(\zeta)\big)_{|\gamma|\leqslant|\mu|}\big).\end{aligned} \tag{24}$$

Proof. The proof follows easily by induction and differentiating (11) with respect to τ. We leave the details of this to the reader. □

The following lemma is stated in [32, Lemma 9.3] for the case of biholomorphic self-maps of real-analytic generic submanifolds but it (along with the proof) also applies to the case of arbitrary formal holomorphic maps between formal generic submanifolds.

Lemma 21. *Let $M, M' \subset \mathbb{C}^N$ be formal generic submanifolds of codimension d given in normal coordinates as above. Then for every multiindex $\mu \in \mathbb{N}^d \setminus \{0\}$, there exists a universal $\mathbb{C}^d$-valued power series mapping $W_\mu(Z,\zeta,Z',\zeta';\cdot)$ polynomial in its last argument with coefficients in the ring $\mathbb{C}[[Z,\zeta,Z',\zeta']]$ such that for every formal holomorphic map $H:(\mathbb{C}^N,0)\to(\mathbb{C}^N,0)$ sending M into M' with $H=(F,G)\in\mathbb{C}^n\times\mathbb{C}^d$ the following identity holds*

$$\bar{G}_{\tau^\mu}(\zeta)=\bar{F}_{\tau^\mu}(\zeta)\cdot\bar{Q}'_{\chi'}\big(\bar{F}(\zeta),H(Z)\big)+W_\mu\big(Z,\zeta,H(Z),\bar{H}(\zeta);\hat{j}_Z^{|\mu|}H,\hat{j}_\zeta^{|\mu|-1}\bar{H}\big). \tag{25}$$

In particular, there exists a universal $\mathbb{C}^d$-valued polynomial map $\mathcal{R}_\mu = \mathcal{R}_\mu(\chi, \chi'; \cdot)$ of its arguments with coefficients in the ring $\mathbb{C}[\![\chi, \chi']\!]$ such that for every map H as above, the following holds:

$$\bar{G}_{\tau^\mu}(\chi, 0) = \mathcal{R}_\mu\big(\chi, \bar{F}(\chi, 0); \big(\partial^\beta \bar{H}(\chi, 0)\big)_{1\leqslant|\beta|\leqslant|\mu|-1}, j_0^{|\mu|} H\big). \tag{26}$$

Combining Lemma 20 and Lemma 21 together, we get the following.

Lemma 22. *In the situation of Lemma* 20, *there exists, for every multiindex $\mu \in \mathbb{N}^d \setminus \{0\}$, a universal $\mathbb{C}^d$-valued power series mapping $\mathcal{A}_\mu = \mathcal{A}_\mu(z, \chi, Z', \zeta'; \cdot)$ polynomial in its last argument with coefficients in the ring $\mathbb{C}[\![z, \chi, Z', \zeta']\!]$ such that for every formal holomorphic map $H : (\mathbb{C}^N, 0) \to (\mathbb{C}^N, 0)$ sending M into M' with $H = (F, G) \in \mathbb{C}^n \times \mathbb{C}^d$ the following identity holds*

$$\begin{aligned} &\bar{F}_{\tau^\mu}(\chi, 0) \cdot Q'_{\chi'}\big(F\big(z, Q(z, \chi, 0)\big), \bar{H}(\chi, 0)\big) \\ &\quad = \mathcal{A}_\mu\big(z, \chi, H\big(z, Q(z, \chi, 0)\big), \bar{H}(\chi, 0); \big(\big(\partial^\beta H\big)\big(z, Q(z, \chi, 0)\big)\big)_{1\leqslant|\beta|\leqslant|\mu|}, \\ &\qquad \big(\partial^\beta \bar{H}(\chi, 0)\big)_{1\leqslant|\beta|\leqslant|\mu|-1}, j_0^{|\mu|} H\big). \end{aligned} \tag{27}$$

Proof. Setting $Z = (z, Q(z, \chi, 0))$ and $\zeta = (\chi, 0)$ in (24) and substituting $\bar{G}_{\tau^\eta}(\chi, 0)$ by its expression given by (26) yields the required conclusion of the lemma. □

We need a last independent lemma before proceeding with the proof of the main proposition of this section.

Lemma 23. *Let $A = A(u, v)$ be a $\mathbb{C}^k$-valued formal power series mapping, $u, v \in \mathbb{C}^k$, satisfying $\det A_u(u, v) \not\equiv 0$ and $A(0, v) \equiv 0$. Assume that $\operatorname{ord}_u(\det A_u(u, v)) \leqslant \nu$ for some nonnegative integer ν. Then for every nonnegative integer r and for every formal power series $\psi(t, v) \in \mathbb{C}[\![t, v]\!]$, $t \in \mathbb{C}^k$, if $\operatorname{ord}_u(\psi(A(u, v), v)) > r(\nu + 1)$, then necessarily $\operatorname{ord}_t \psi(t, v) > r$.*

Proof. We prove the lemma by induction on r and notice that the statement automatically holds for $r = 0$. Suppose that ψ is as in the lemma and satisfies $\operatorname{ord}_u(\psi(A(u, v), v)) > r(\nu + 1)$ for some $r \geqslant 1$. Differentiating $\psi(A(u, v), v)$ with respect to u, we get that the order (in u) of each component of $\psi_t(A(u, v), v) \cdot A_u(u, v)$ is strictly greater than $r\nu + r - 1$. Multiplying $\psi_t(A(u, v), v) \cdot A_u(u, v)$ by the classical inverse of $A_u(u, v)$, we get the same conclusion for each component of the power series mapping $(\det A_u(u, v))\psi_t(A(u, v), v)$. By assumption, $\operatorname{ord}_u(\det A_u(u, v)) \leqslant \nu$ and therefore the order (in u) of each component of $\psi_t(A(u, v), v)$ is strictly greater than $r\nu + r - 1 - \nu = (r - 1)(\nu + 1)$. From the induction assumption, we conclude that the order in t of each component of $\psi_t(t, v)$ (strictly) exceeds $r - 1$. To conclude that $\operatorname{ord}_t \psi(t, v) > r$ from the latter fact, it is enough to notice that $\psi(0, v) \equiv 0$ since $\operatorname{ord}_u(\psi(A(u, v), v)) > r(\nu + 1) \geqslant 1$ and $A(0, v) \equiv 0$. The proof of Lemma 23 is complete. □

We are now completely ready to prove the following main goal of this section.

Proposition 24. *Let $M, M' \subset \mathbb{C}^N$ be formal generic submanifolds of the same dimension, given in normal coordinates as above. Assume that M belongs to the class $\mathcal{C}$ et let k_0 be the integer given in Corollary* 19. *Then the integer $k_1 := \max\{k_0, \kappa_M(\nu_M(\infty) + 1)\}$ has the following* *property*: *for any pair $H_1, H_2 : (\mathbb{C}^N, 0) \to (\mathbb{C}^N, 0)$ of formal CR-transversal holomorphic mappings sending M into M' and any nonnegative integer ℓ, if $j_0^{k_1+\ell} H_1 = j_0^{k_1+\ell} H_2$, then necessarily $(\partial^\alpha H_1)(z,0) = (\partial^\alpha H_2)(z,0)$ for all $\alpha \in \mathbb{N}^N$ with $|\alpha| \leqslant \ell$. Furthermore, k_1 depends upper-semicontinuously on continuous deformations of M.*

Proof. The proposition is proved by induction on ℓ. For $\ell = 0$, the proposition follows immediately from Corollary 19. Consider now a pair of maps H_1, H_2 as in the statement of the proposition with the same $k_1 + \ell$ jet at 0, where $\ell \geqslant 1$. Then from the induction assumption, we know that $(\partial^\alpha H_1)(z,0) = (\partial^\alpha H_2)(z,0)$ for all $\alpha \in \mathbb{N}^N$ with $|\alpha| \leqslant \ell - 1$. Hence it is enough to show that for all multiindices $\mu \in \mathbb{N}^d$ with $|\mu| = \ell$,

$$\frac{\partial^\mu \bar{H}_1}{\partial \tau^\mu}(\chi, 0) = \frac{\partial^\mu \bar{H}_2}{\partial \tau^\mu}(\chi, 0). \tag{28}$$

This is further simplified by noticing that Lemma 21 (more precisely (25) applied with $Z = 0$ and $\zeta = (\chi, 0)$) implies that it is enough to prove that for all $\mu \in \mathbb{N}^d$ as above,

$$\frac{\partial^\mu \bar{F}_1}{\partial \tau^\mu}(\chi, 0) = \frac{\partial^\mu \bar{F}_2}{\partial \tau^\mu}(\chi, 0). \tag{29}$$

Next, applying (27) to both H_1 and H_2, we get the order in z of each component of the power series mapping given by

$$\begin{aligned} &\frac{\partial^\mu \bar{F}_1}{\partial \tau^\mu}(\chi, 0) \cdot Q'_{\chi'}\big(F_1\big(z, Q(z, \chi, 0)\big), \bar{H}_1(\chi, 0)\big) \\ &\quad - \frac{\partial^\mu \bar{F}_2}{\partial \tau^\mu}(\chi, 0) \cdot Q'_{\chi'}\big(F_2\big(z, Q(z, \chi, 0)\big), \bar{H}_2(\chi, 0)\big) \end{aligned} \tag{30}$$

is at least $k_1 + 1$. Consider the power series mapping

$$\psi(z', \chi) := \frac{\partial^\mu \bar{F}_1}{\partial \tau^\mu}(\chi, 0) \cdot Q'_{\chi'}\big(z', \bar{H}_1(\chi, 0)\big) - \frac{\partial^\mu \bar{F}_2}{\partial \tau^\mu}(\chi, 0) \cdot Q'_{\chi'}\big(z', \bar{H}_2(\chi, 0)\big), \tag{31}$$

and let $\widehat{F}(z, \chi) \in \mathbb{C}[\![\chi]\!][z]$ be the Taylor polynomial (in z) of order k_1 of $F_1(z, Q(z, \chi, 0))$ viewed as a power series in the ring $\mathbb{C}[\![\chi]\!][\![z]\!]$. Note that it follows from our assumptions that $\widehat{F}(z, \chi)$ coincides also with the Taylor polynomial (in z) of order k_1 of $F_2(z, Q(z, \chi, 0))$ (also viewed as a power series in the ring $\mathbb{C}[\![\chi]\!][\![z]\!]$). Hence since the order in z of each component of the power series mapping given by (30) is at least $k_1 + 1$, this also holds for the power series mapping $\psi(\widehat{F}(z, \chi), \chi)$. Furthermore, we claim that

$$\operatorname{ord}_z\big(\det \widehat{F}_z(z, \chi)\big) \leqslant \nu_M(\infty). \tag{32}$$

Indeed, suppose not. Since

$$\operatorname{ord}_z\big(\widehat{F}(z, \chi) - F_1\big(z, Q(z, \chi, 0)\big)\big) \geqslant k_1 + 1,$$

we have

$$\operatorname{ord}_z\left(\widehat{F}_z(z,\chi)-\frac{\partial}{\partial z}\big[F_1\big(z,Q(z,\chi,0)\big)\big]\right)\geqslant k_1\geqslant \nu_M(\infty)+1. \tag{33}$$

Therefore (33) yields $\operatorname{ord}_z(\det\frac{\partial}{\partial z}[F_1(z,Q(z,\chi,0))])\geqslant \nu_M(\infty)+1$ and hence in particular that

$$\operatorname{ord}\left(\det\frac{\partial F_1}{\partial z}(z,0)\right)\geqslant \nu_M(\infty)+1,$$

which contradicts (20) and proves the claim. Since $\operatorname{ord}_z(\psi(\widehat{F}(z,\chi),\chi))\geqslant k_1+1>\kappa_M(\nu_M+1)$ and since $\widehat{F}(0,\chi)\equiv 0$, from (32) and Lemma 23 we conclude that $\operatorname{ord}_{z'}\psi(z',\chi)>\kappa_M$, which is equivalent to say that

$$\frac{\partial^\mu \bar{F}_1}{\partial\tau^\mu}(\chi,0)\cdot\frac{\partial\Theta'_\alpha}{\partial\chi'}\big(\bar{F}_1(\chi,0)\big)=\frac{\partial^\mu \bar{F}_2}{\partial\tau^\mu}(\chi,0)\cdot\frac{\partial\Theta'_\alpha}{\partial\chi'}\big(\bar{F}_2(\chi,0)\big), \tag{34}$$

for all $\alpha\in\mathbb{N}^n$ with $|\alpha|\leqslant\kappa_M$. By Corollary 12, the formal submanifold $M'\in\mathcal{C}$ and $\kappa_{M'}\leqslant\kappa_M$. Therefore since the formal map $\chi'\mapsto\Theta'_{\kappa_M}(\chi')$ is of generic rank n, and by assumption $\bar{F}_1(\chi,0)=\bar{F}_2(\chi,0)$, and since this map is not totally degenerate by virtue of Corollary 12, it follows from (34) that (29) holds which completes the proof of Proposition 24. □

4.3. *Iteration and proof of Theorem 9*

We now want to iterate the jet determination property along higher order Segre sets by using the reflection identities from [32] established for holomorphic self-automorphisms. Such identities could not be used to establish Corollary 19 and Proposition 24, since for CR-transversal mappings $H=(F,G)$, the matrix $F_z(0)$ need not be invertible. On the other hand, they will be good enough for the iteration process, since $F_z(z,0)$ has generic full rank in view of Corollary 12. We therefore first collect from [32] the necessary reflection identities. Even though, as mentioned above, such identities were considered in [32] only for holomorphic self-automorphisms of a given real-analytic generic submanifold of $\mathbb{C}^N$, we note here that their proof also yields the same identities for merely not totally degenerate formal holomorphic maps between formal generic submanifolds. We start with the following version of [32, Proposition 9.1].

Proposition 25. *In the situation of Lemma* 20*, there exists a universal power series* $\mathcal{D}=\mathcal{D}(Z,\zeta;\cdot)$ *polynomial in its last argument with coefficients in the ring* $\mathbb{C}[[Z,\zeta]]$ *and, for every* $\alpha\in\mathbb{N}^n\setminus\{0\}$, *another universal* $\mathbb{C}^d$*-valued power series mapping* $\mathcal{P}_\alpha=\mathcal{P}_\alpha(Z,\zeta;\cdot)$ (*whose components belong to the same ring as that of* $\mathcal{D}$), *such that for every not totally degenerate formal holomorphic map* $H:(\mathbb{C}^N,0)\to(\mathbb{C}^N,0)$ *sending* M *into* M' *with* $H=(F,G)\in\mathbb{C}^n\times\mathbb{C}^d$ *the following holds*:

(i) $\mathcal{D}(Z,\zeta;\hat{j}^1_\zeta\bar{H})|_{(Z,\zeta)=(0,(\chi,0))}=\det(\bar{F}_\chi(\chi,0))\not\equiv 0$;
(ii) $(\mathcal{D}(Z,\zeta;\hat{j}^1_\zeta\bar{H}))^{2|\alpha|-1}\bar{Q}'_{\chi'^\alpha}(\bar{F}(\zeta),H(Z))=\mathcal{P}_\alpha(Z,\zeta;\hat{j}^{|\alpha|}_\zeta\bar{H})$, *for* $(Z,\zeta)\in\mathcal{M}$.

We also need the following version of [32, Proposition 9.4].

Proposition 26. *In the situation of Lemma 20, for any $\mu \in \mathbb{N}^d \setminus \{0\}$ and $\alpha \in \mathbb{N}^n \setminus \{0\}$, there exist universal $\mathbb{C}^d$-valued power series mappings $\mathcal{B}_{\mu,\alpha}(Z,\zeta,Z',\zeta';\cdot)$ and $\mathcal{Q}_{\mu,\alpha}(Z,\zeta;\cdot)$ polynomial in their last argument with coefficients in the ring $\mathbb{C}[\![Z,\zeta,Z',\zeta']\!]$ and $\mathbb{C}[\![Z,\zeta]\!]$ respectively such that for every not totally degenerate formal holomorphic map $H:(\mathbb{C}^N,0)\to(\mathbb{C}^N,0)$ sending M into M' with $H=(F,G)\in\mathbb{C}^n\times\mathbb{C}^d$ the following holds*:

$$F_{w^\mu}(Z)\cdot\big(\bar{Q}'_{\chi'^\alpha,z'}\big(\bar{F}(\zeta),H(Z)\big)+Q'_{z'}\big(F(Z),\bar{H}(\zeta)\big)\cdot\bar{Q}_{\chi'^\alpha,w'}\big(\bar{F}(\zeta),H(Z)\big)\big)=(*)_1+(*)_2, \tag{35}$$

where $(*)_1$ is given by

$$(*)_1:=\mathcal{B}_{\mu,\alpha}\big(Z,\zeta,H(Z),\bar{H}(\zeta);\hat{j}_Z^{|\mu|-1}H,\hat{j}_\zeta^{|\mu|}\bar{H}\big), \tag{36}$$

and $(*)_2$ is given by

$$(*)_2:=\frac{\mathcal{Q}_{\mu,\alpha}(Z,\zeta,\hat{j}_\zeta^{|\alpha|+|\mu|}\bar{H})}{(\mathcal{D}(Z,\zeta,\hat{j}_\zeta^1\bar{H}))^{2|\alpha|+|\mu|-1}}, \tag{37}$$

and where $\mathcal{D}$ is given by Proposition 25.

In what follows, we use the notation introduced for the Segre mappings given in Section 2.5 (associated to a fixed choice of normal coordinates for M). We are now ready to prove the following.

Proposition 27. *Let M, M' be formal generic submanifolds of $\mathbb{C}^N$ of the same dimension given in normal coordinates as above. Assume that M' belongs to the class $\mathcal{C}$ and let j be a positive integer. Then for every nonnegative integer ℓ and for every pair $H_1,H_2:(\mathbb{C}^N,0)\to(\mathbb{C}^N,0)$ of not totally degenerate formal holomorphic mappings sending M into M', if $(\partial^\alpha H_1)\circ v^j=(\partial^\alpha H_2)\circ v^j$ for all $\alpha\in\mathbb{N}^N$ with $|\alpha|\leqslant\kappa_{M'}+\ell$ then necessarily for all $\beta\in\mathbb{N}^N$ with $|\beta|\leqslant\ell$, one has*

$$\big(\partial^\beta H_1\big)\circ v^{j+1}=\big(\partial^\beta H_2\big)\circ v^{j+1}.$$

Proof. We prove the proposition by induction on ℓ.

For $\ell=0$, suppose that $H_1,H_2:(\mathbb{C}^N,0)\to(\mathbb{C}^N,0)$ is a pair of not totally degenerate formal holomorphic mappings sending M into M' satisfying

$$\big(\partial^\alpha H_1\big)\circ v^j=\big(\partial^\alpha H_2\big)\circ v^j,\quad |\alpha|\leqslant\kappa_{M'}. \tag{38}$$

Then setting $Z=v^{j+1}(t^{[j+1]})$ and $\zeta=\bar{v}^j(t^{[j]})$ in Proposition 25(ii) and using the above assumption, one obtains that for all $\alpha\in\mathbb{N}^N$ with $|\alpha|\leqslant\kappa_{M'}$

$$\bar{Q}'_{\chi'^\alpha}\big(\bar{F}_1\circ\bar{v}^j,H_1\circ v^{j+1}\big)=\bar{Q}'_{\chi'^\alpha}\big(\bar{F}_2\circ\bar{v}^j,H_2\circ v^{j+1}\big). \tag{39}$$

In what follows, to avoid some unreadable notation, we denote by $V^j = V^j(T^1, \ldots, T^{j+1})$ the Segre mapping of order j associated to M' and also write $T^{[j]} = (T^1, \ldots, T^j) \in \mathbb{C}^n \times \cdots \times \mathbb{C}^n$. Next we note that we also have

$$H_\nu \circ v^{j+1} = V^{j+1}\big(F_\nu \circ v^{j+1}, \bar{F}_\nu \circ \bar{v}^j, F_\nu \circ v^{j-1}, \ldots\big), \quad \nu = 1, 2. \tag{40}$$

Since $M' \in \mathcal{C}$, we may choose multiindices $\alpha^{(1)}, \ldots, \alpha^{(n)} \in \mathbb{N}^n$ and $s_1, \ldots, s_n \in \{1, \ldots, d\}$ with $|\alpha^j| \leqslant \kappa_{M'}$ such that the formal map $\bar{\Theta}' : z' \mapsto (\bar{\Theta}'^{s_1}_{\alpha^{(1)}}(z'), \ldots, \bar{\Theta}'^{s_n}_{\alpha^{(n)}}(z'))$ is of generic rank n. Denote by Ψ the formal map $(T^{j+1}, \ldots, T^1) \mapsto (\bar{Q}'^{s_i}_{\chi'^{\alpha^{(i)}}}(T^j, V^{j+1}(T^{j+1}, T^j, \ldots, T^1)))_{1 \leqslant i \leqslant n}$. As in the proof of Corollary 19, we write

$$\Psi\big(u, T^{[j]}\big) - \Psi\big(v, T^{[j]}\big) = (u - v) \cdot \int_0^1 \Psi_{T^{j+1}}\big(tu + (1-t)v, T^{[j]}\big)\, dt,$$

and note that it follows from (39), (40) and (38) that

$$\begin{aligned} 0 &= \Psi\big(F_1 \circ v^{j+1}, \bar{F}_1 \circ \bar{v}^j, \ldots\big) - \Psi\big(F_2 \circ v^{j+1}, \bar{F}_2 \circ \bar{v}^j, \ldots\big) \\ &= \big(F_1 \circ v^{j+1} - F_2 \circ v^{j+1}\big) \\ &\quad \cdot \int_0^1 \Psi_{T^{j+1}}\big(tF_1 \circ v^{j+1} + (1-t)F_2 \circ v^{j+1}, \bar{F}_1 \circ \bar{v}^j, F_1 \circ v^{j-1}, \ldots\big)\, dt. \end{aligned}$$

We now claim that $\det \int_0^1 \Psi_{T^{j+1}}(tF_1 \circ v^{j+1} + (1-t)F_2 \circ v^{j+1}, \bar{F}_1 \circ \bar{v}^j, \ldots)\, dt \not\equiv 0$. Indeed if it were not the case we would in particular have, after setting $t^{[j]} = 0$ in the above determinant, that

$$\det\left(\frac{\partial \bar{\Theta}'}{\partial z'}\big(F_1 \circ v^1\big)\right) \equiv 0. \tag{41}$$

But since H_1 is not totally degenerate, (41) implies that $\operatorname{Rk} \bar{\Theta}' < n$, a contradiction. This proves the claim and hence that $F_1 \circ v^{j+1} = F_2 \circ v^{j+1}$ and therefore that $H_1 \circ v^{j+1} = H_2 \circ v^{j+1}$ in view of (40). This completes the proof of the proposition for the case $\ell = 0$.

Now assume that $\ell > 0$ and suppose that $(\partial^\alpha H_1) \circ v^j = (\partial^\alpha H_2) \circ v^j$ for all $\alpha \in \mathbb{N}^N$ with $|\alpha| \leqslant \kappa_{M'} + \ell$. From the induction assumption, we know that

$$\big(\partial^\beta H_1\big) \circ v^{j+1} = \big(\partial^\beta H_2\big) \circ v^{j+1}, \quad \forall \beta \in \mathbb{N}^N,\ |\beta| \leqslant \ell - 1. \tag{42}$$

It remains therefore to show the equality of the ℓth order derivatives restricted to the $(j+1)$th Segre set. We first prove that this is so for the pure transversal derivatives i.e. that

$$\forall \mu \in \mathbb{N}^d, \quad |\mu| = \ell, \quad \frac{\partial^{|\mu|} H_1}{\partial w^\mu} \circ v^{j+1} = \frac{\partial^{|\mu|} H_2}{\partial w^\mu} \circ v^{j+1}. \tag{43}$$

Let μ be such a multiindex. Setting $Z = v^{j+1}(t^{[j+1]})$ and $\zeta = \bar{v}^j(t^{[j]})$ in (35) applied to both H_1 and H_2 and using (42), we get for all $\alpha \in \mathbb{N}^n$ with $|\alpha| \leqslant \kappa_{M'}$

$$\left(\frac{\partial^{|\mu|} F_1}{\partial w^\mu} \circ v^{j+1}\right) \cdot \Gamma_\alpha = \left(\frac{\partial^{|\mu|} F_2}{\partial w^\mu} \circ v^{j+1}\right) \cdot \Gamma_\alpha, \tag{44}$$

where

$$\begin{aligned}\Gamma_\alpha = \Gamma_\alpha\big(t^{[j+1]}\big) := {} & \bar{Q}'_{\chi'^\alpha, z'}\big(\bar{F}_1 \circ \bar{v}^j, H_1 \circ v^{j+1}\big) \\ & + Q'_{z'}\big(F_1 \circ v^{j+1}, \bar{H}_1 \circ \bar{v}^j\big) \cdot \bar{Q}_{\chi'^\alpha, w'}\big(\bar{F} \circ \bar{v}^j, H \circ v^{j+1}\big).\end{aligned}$$

To conclude from (44) that $\frac{\partial^{|\mu|} F_1}{\partial w^\mu} \circ v^{j+1} = \frac{\partial^{|\mu|} F_2}{\partial w^\mu} \circ v^{j+1}$, it is enough to show that the generic rank of the family of matrices $(\Gamma_\alpha)_{|\alpha| \leqslant \kappa_{M'}}$ is n. This holds trivially since $\Gamma_\alpha(0, \ldots, 0, t^{j+1}) = \frac{\partial \bar{\Theta}'_\alpha}{\partial z'}(t^{j+1})$ and since $M' \in \mathcal{C}$. Next using the identity (25) given by Lemma 21 applied to $\zeta = \bar{v}^{j+1}(t^{[j+1]})$ and $Z = v^j(t^{[j]})$, we immediately get that $\frac{\partial^{|\mu|} \bar{G}_1}{\partial \tau^\mu} \circ \bar{v}^{j+1} = \frac{\partial^{|\mu|} \bar{G}_2}{\partial \tau^\mu} \circ \bar{v}^{j+1}$ which yields (43).

To complete the proof of the induction, we need to show that $(\partial^\beta H_1) \circ v^{j+1} = (\partial^\beta H_2) \circ v^{j+1}$ for arbitrary $\beta = (\beta_1, \ldots, \beta_N) \in \mathbb{N}^N$ with $|\beta| = \ell$. We prove it by induction on the number $c_\beta := \beta_1 + \cdots + \beta_n$. For $c_\beta = 0$, this follows from (43) proved above. Now if $c_\beta > 0$, we may assume without loss of generality that $\beta_1 > 0$ and write $\beta = (1, 0, \ldots, 0) + \tilde{\beta}$ with $|\tilde{\beta}| = \ell - 1$. By (42) we know that $\partial^{\tilde{\beta}} H_1 \circ v^{j+1} = \partial^{\tilde{\beta}} H_2 \circ v^{j+1}$ and hence by differentiating this latter identity with respect to first variable of $t^{j+1} \in \mathbb{C}^n$, we get

$$\begin{aligned}&\partial^\beta H_1 \circ v^{j+1} + Q_{z_1}\big(t^{j+1}, \bar{v}^j\big) \cdot \left(\left(\frac{\partial^\ell H_1}{\partial w \partial z^{\tilde{\beta}}}\right) \circ v^{j+1}\right) \\ &\quad = \partial^\beta H_2 \circ v^{j+1} + Q_{z_1}\big(t^{j+1}, \bar{v}^j\big) \cdot \left(\left(\frac{\partial^\ell H_2}{\partial w \partial z^{\tilde{\beta}}}\right) \circ v^{j+1}\right),\end{aligned} \tag{45}$$

from which the desired equality $\partial^\beta H_1 \circ v^{j+1} = \partial^\beta H_2 \circ v^{j+1}$ follows by using the induction assumption. The proof of the proposition is therefore complete. □

Combining now Propositions 27, 24, Corollaries 12 and 19, one gets the following.

Proposition 28. *Let $M, M' \subset \mathbb{C}^N$ be formal generic submanifolds of the same dimension. Assume that M belongs to the class $\mathcal{C}$. Then for every positive integer j, the integer*

$$k_j = k_1 + \kappa_M (j-1),$$

where k_1 is the integer defined in Proposition 24, has the following property: If $H_1, H_2 : (\mathbb{C}^N, 0) \to (\mathbb{C}^N, 0)$ are two formal CR-transversal holomorphic mappings sending M into M' such that $j_0^{k_j} H_1 = j_0^{k_j} H_2$, then necessarily $H_1 \circ v^j = H_2 \circ v^j$. Furthermore, k_j depends upper-semicontinuously on continuous deformations of M.

Completion of the proof of Theorem 9. Firstly, we may assume that M and M' are given in normal coordinates as above and we denote by d the codimension of M. By Proposition 28, if $H_1, H_2 : (\mathbb{C}^N, 0) \to (\mathbb{C}^N, 0)$ are two formal CR-transversal holomorphic mappings sending M into M' with the same k_{d+1}-jet, then necessarily $H_1 \circ v^{d+1} = H_2 \circ v^{d+1}$. By the finite type assumption on M, we have from Theorem 8 that $\operatorname{Rk} v^{d+1} = N$ and hence that $H_1 = H_2$. Furthermore, it also follows from Proposition 28 that the integer k_{d+1} depends upper-semicontinuously on perturbations of M, which completes the proof of Theorem 9. $\square$

5. Application to smooth generic submanifolds and proofs of Theorem 1 and Corollary 2

We have the following result obtained from Theorem 9 by considering the smooth deformation of M given by varying its base point as explained in Section 2.2.

Theorem 29. *Let $M \subset \mathbb{C}^N$ be a smooth generic submanifold that is in the class $\mathcal{C}$ and of finite type at each of its points. Then for every point $p \in M$ there exists an integer ℓ_p, depending upper-semicontinuously on p, such that for every smooth generic submanifold $M' \subset \mathbb{C}^N$ of the same dimension as that of M, if $h_1, h_2 : (M, p) \to M'$ are two germs of smooth CR-transversal mappings with the same ℓ_p jet at p, then necessarily $j_p^k h_1 = j_p^k h_2$ for all positive integers k.*

We also have the following slightly stronger version of Corollary 2 which is an immediate consequence of Corollary 10 and the fact that any smooth real hypersurface of $\mathbb{C}^N$ that is of D'Angelo finite type at some point $p \in M$ necessarily does not contain any formal curve at that point.

Corollary 30. *Let $M, M' \subset \mathbb{C}^N$ be smooth real hypersurfaces. Assume that $M \in \mathcal{C}$ and that M' is of D'Angelo finite type at each of their points. Then for every $p \in M$, there exists an integer $\ell = \ell(p)$, depending upper-semicontinuously on p, such that if $h_1, h_2 : (M, p) \to M'$ are two germs of smooth CR mappings with the same ℓ-jet at p, then necessarily $j_p^k h_1 = j_p^k h_2$ for all positive integers k.*

Proof of Theorem 1. Theorem 1 follows immediately from Theorem 29 since in the setting of Theorem 1 smooth CR finite mappings are automatically CR-transversal (see [22]) and since every germ of an essentially finite smooth generic submanifold of $\mathbb{C}^N$ is necessarily in the class $\mathcal{C}$. $\square$

Proof of Corollary 2. Corollary 2 follows immediately from Corollary 30 since any smooth real hypersurface of $\mathbb{C}^N$ that is of D'Angelo finite at some point $p \in M$ is necessarily in the class $\mathcal{C}$ at that point. (We note here that Corollary 2 could also be derived directly from Theorem 1 using some results from [7,22].) The last part of the corollary follows from the first part after applying the regularity result given in [19] (see also [25] for the case $N = 2$). $\square$

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