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LIE GROUP STRUCTURES ON AUTOMORPHISM GROUPS OF REAL-ANALYTIC CR MANIFOLDS

By Bernhard Lamel, Nordine Mir, and Dmitri Zaitsev

Abstract. Given any real-analytic CR manifold M, we provide general conditions on M guaranteeing that the group of all its global real-analytic CR automorphisms $\mathrm{Aut}_{\mathrm{CR}}(M)$ is a Lie group (in an appropriate topology). In particular, we obtain a Lie group structure for $\mathrm{Aut}_{\mathrm{CR}}(M)$ when M is an arbitrary compact real-analytic hypersurface embedded in some Stein manifold.

1. Introduction. There exists a wide variety of results concerned with the structure of the automorphism group of a given geometric structure. In Riemannian Geometry, the classical Myers-Steenrod theorem [MS39] states that the group of all isometries of a Riemannian manifold is a Lie group. H. Cartan [Ca35] proved an analogous result for the group of holomorphic automorphisms of a bounded domain in $\mathbb{C}^N$. Cartan's techniques have in turn been used to establish general results for groups of diffeomorphisms of real or complex manifolds, see e.g. [BM45].

In this paper, we consider an analogous question for CR manifolds (that one can think of as a boundary or CR version of Cartan's Theorem mentioned above):

Under what conditions on a real-analytic CR manifold M is the group $\mathrm{Aut}_{\mathrm{CR}}(M)$ *of all real-analytic CR automorphisms of M a Lie group in an appropriate topology?*

Here for every $r \in \mathbb{N} \cup \{\infty, \omega\}$, we equip $\mathrm{Aut}_{\mathrm{CR}}(M)$ with a natural topology that we call "compact-open $\mathcal{C}^r$ topology", which is defined as follows. For open subsets $\Omega \subset \mathbb{R}^n$ and $\Omega' \subset \mathbb{R}^{n'}$, consider the space $\mathcal{C}^r(\Omega, \Omega')$ of all maps of class $\mathcal{C}^r$ from Ω to Ω'. If $r \in \mathbb{N} \cup \{\infty\}$, $\mathcal{C}^r(\Omega, \Omega')$ is equipped with the topology of uniform convergence on compacta together with all partial derivatives of order up to r. In case $r = \omega$, the space $\mathcal{C}^\omega(\Omega, \Omega')$ is equipped with its topology as an inductive limit of Fréchet spaces of holomorphic maps between open neighborhoods of Ω and Ω' in $\mathbb{C}^n$ and $\mathbb{C}^{n'}$ respectively. The compact-open $\mathcal{C}^r$ topology on $\mathrm{Aut}_{\mathrm{CR}}(M)$ is now induced by the appropriate topology relative to the coordinate charts for the maps and their inverses (see e.g. [BRWZ04] for a more detailed discussion). For brevity, we adopt the order $k < \infty < \omega$ for any integer k.

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In this paper, we exhibit general sufficient conditions on M that provide an affirmative answer to the above question. We begin with the following special case of our more general results, which is particularly easy to state:

COROLLARY 1.1. *Let M be a compact real-analytic hypersurface in a Stein manifold of complex dimension at least two. Then the group* $\mathrm{Aut}_{\mathrm{CR}}(M)$ *of all* (*global*) *real-analytic CR automorphisms of M is a Lie group in the compact-open $\mathcal{C}^\omega$ topology and the action* $\mathrm{Aut}_{\mathrm{CR}}(M) \times M \to M$ *is real-analytic. Furthermore, the compact-open $\mathcal{C}^r$ topologies on* $\mathrm{Aut}_{\mathrm{CR}}(M)$ *coincide for $r = \infty, \omega$ and $r \geq k$, where k is an integer depending only on M.*

Corollary 1.1 is a direct consequence of the following more general result that also applies to CR manifolds of higher codimension. In the following statement, the notions of essential finiteness, finite nondegeneracy and minimality must be understood in the sense of [BER99b] (see also Section 3 for more details).

THEOREM 1.2. *Let M be a real-analytic CR manifold. Assume that M has finitely many connected components, is minimal everywhere and that there exists a compact subset $K \subset M$ such that:*

(i) *M is essentially finite at all points of K;*

(ii) *M is finitely nondegenerate at all points of $M \setminus K$.*

Then $\mathrm{Aut}_{\mathrm{CR}}(M)$ *is a Lie group in the compact-open $\mathcal{C}^\omega$ topology and the action* $\mathrm{Aut}_{\mathrm{CR}}(M) \times M \to M$ *is real-analytic. Furthermore, the compact-open $\mathcal{C}^r$ topologies on* $\mathrm{Aut}_{\mathrm{CR}}(M)$ *coincide for $r = \infty, \omega$ and $r \geq k$ for some integer k, where k is an integer depending only on M.*

Theorem 1.2 provides a generalization of all known corresponding results for real-analytic CR manifolds. It also covers new situations, such as in Corollary 1.1; indeed, any real-analytic compact hypersurface in a Stein manifold is essentially finite and minimal at *each* of its points (see e.g. [DF78], [BER99b]).

For the case of real hypersurfaces whose Levi form is nondegenerate at every point, the conclusion of Theorem 1.2 follows from the work of E. Cartan [Ca32a], [Ca32b], Chern-Moser [CM74], Tanaka [Ta67] and Burns-Schnider [BS77]. For the case of Levi-degenerate CR manifolds, the same conclusion was recently obtained by Baouendi, Rothschild, Winkelmann and the third author [BRWZ04] for the class of finitely nondegenerate minimal CR manifolds, which corresponds here to our Theorem 1.2 with $K = \emptyset$. (We should point out that the results in those papers also apply for merely smooth CR manifolds as well, based on the previous work [KZ05], but in this paper we shall focus on the real-analytic category.)

In addition to the compact hypersurface case considered in Corollary 1.1, an important class of CR manifolds for which the previously known results do not apply and for which the conclusion of Theorem 1.2 holds is that of *compact* minimal real-analytic CR submanifolds embedded in a Stein manifold. Again, the condition of essential finiteness holds here at every point, see [DF78], [BER99b] (whereas the condition of finite nondegeneracy holds only outside a proper real-analytic subvariety which need not be empty in general); Therefore taking $K = M$

in Theorem 1.2, we obtain the following extension of Corollary 1.1 to higher codimension:

COROLLARY 1.3. *Let M be a compact real-analytic CR submanifold in a Stein manifold. Assume that M is minimal at every point. Then the group* $\mathrm{Aut}_{\mathrm{CR}}(M)$ *of all (global) real-analytic CR automorphisms of M is a Lie group in the compact-open $\mathcal{C}^\omega$ topology and the action* $\mathrm{Aut}_{\mathrm{CR}}(M) \times M \to M$ *is real-analytic. Furthermore, the compact-open $\mathcal{C}^r$ topologies on* $\mathrm{Aut}_{\mathrm{CR}}(M)$ *coincide for $r = \infty, \omega$ and $r \geq k$, where k is an integer depending only on M.*

On the other hand, Theorem 1.2 also applies to cases with M noncompact also not covered by previously known results. Let us illustrate this with an example:

Example 1.4. The hypersurface $M \subset \mathbb{C}^2$ given by

$$|z|^2 - |w|^4 = 1$$

is Levi-nondegenerate at all its points except the circle $S^1 \times \{0\} \subset M$, where M is essentially finite. Hence, Theorem 1.2 applied with $K := S^1 \times \{0\}$, yields that $\mathrm{Aut}_{\mathrm{CR}}(M)$ is a Lie group. On the other hand, M is not finitely nondegenerate at any point of K and hence the results from [BRWZ04] do not apply to M.

Our proof of Theorem 1.2 makes use of the recent developments providing a relationship between various notions and results concerning jet parametrization of local CR diffeomorphisms [BER97], [Z97], [BER99a], [E01], [KZ05], [LM07] and Lie group structures on (local and) global groups of automorphisms of CR manifolds [BRWZ04]. In the next section, we give in Theorem 2.2 new sufficient conditions on a connected real-analytic CR manifold M, in terms of local jet parametrization properties of CR automorphisms, that ensure that $\mathrm{Aut}_{\mathrm{CR}}(M)$ is a Lie group. Then the remainder of the paper is devoted to prove that under the assumptions of Theorem 1.2 the conditions of Theorem 2.2 are fulfilled. To this end, we establish, following the analysis of the first two authors' paper [LM07], a new parametrization theorem (Theorem 3.1) for local CR automorphisms that may be of independent interest. The proof of Theorem 1.2 is given in Section 5. We conclude the paper by giving in Section 6 an alternative proof of Corollary 1.3 following [Z97] that does not make use of Theorem 2.2 but requires compactness of the manifold M.

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2. New sufficient conditions for the automorphism group being a Lie group. Let M be a real-analytic manifold and k a positive integer. We use the notation $G^k(M)$ for the fiber bundle of all k-jets of local real-analytic diffeomorphisms of M. For every point $p \in M$, we denote by $G^k_p(M)$ the fiber of $G^k(M)$

at p. Given a germ of a local real-analytic diffeomorphism $h\colon (M,p) \to M$, we write $j_p^k h \in G_p^k(M)$ for the corresponding k-jet. For instance, $j_p^k\,\mathsf{id}$ is the k-jet of the identity map of M, regarded as a germ at p. In local coordinates, $j_p^k h$ is given by the source p, the target $h(p)$ and the collection of all partial derivatives of h at p up to order k. (See e.g. [GG73] for more details on this terminology.)

We now fix an arbitrary set $\mathcal{S}$ of germs of local real-analytic diffeomorphisms $h\colon (M,p) \to M$ with possibly varying reference point $p \in M$ and, as in [BRWZ04], consider the following condition.

Definition 2.1. Let k be a positive integer and $p_0 \in M$. We say that $\mathcal{S}$ *has the real-analytic jet parametrization property of order k at p_0* if there exist open neighborhoods Ω' of p_0 in M, Ω'' of $j_{p_0}^k\mathsf{id}$ in $G^k(M)$ and a real-analytic map $\Psi\colon \Omega' \times \Omega'' \to M$ such that, for every germ $h\colon (M,p) \to M$ in $\mathcal{S}$ with $p \in \Omega'$ and $j_p^k h \in \Omega''$, the identity $h(\cdot) \equiv \Psi(\cdot, j_p^k h)$ holds in the sense of germs at p.

The following theorem is one of the key ingredients in the proof of Theorem 1.2.

THEOREM 2.2. *Let M be a connected real-analytic CR manifold. Assume that there exist an integer k and a compact subset $K \subset M$ such that the following holds:*

(i) *For every $p_0 \in K$, the set of all germs at p_0 of local CR diffeomorphisms of M has the real-analytic jet parametrization property of order k at p_0;*

(ii) *The set of all germs at all points of local CR diffeomorphisms has the real-analytic jet parametrization property at every point $p_0 \in M \setminus K$ of some finite order possibly depending on p_0.*

Then $\mathrm{Aut}_{\mathrm{CR}}(M)$ *is a Lie group in the compact-open $\mathcal{C}^\omega$ topology and the action* $\mathrm{Aut}_{\mathrm{CR}}(M) \times M \to M$ *is real-analytic. Furthermore, the compact-open $\mathcal{C}^r$ topologies on* $\mathrm{Aut}_{\mathrm{CR}}(M)$ *coincide for $r = \infty, \omega$ and $r \geq k$, where k is an integer depending only on M.*

In the case $K = \emptyset$, Theorem 2.2 is contained in [BRWZ04]. Heuristically speaking the points of $M \setminus K$ in Theorem 2.2 (ii) fulfill a "strong" jet parametrization property (namely, a so-called complete system in the sense of [KZ05], [BRWZ04]). In Theorem 2.2, we allow some points to satisfy a weaker property (namely condition (i)), but we have to pay the price by requiring that all these points lie in a compact subset of M.

Proof of Theorem 2.2. Let $K \subset M$ be the compact subset as in Theorem 2.2. We first apply the parametrization property for the set of all germs at a fixed point $p_0 \in K$, which holds in view of (i); without loss of generality we may assume that K is nonempty. By Definition 2.1, for every fixed $p_0 \in K$, we can find open neighborhoods Ω' of p_0 in M, Ω'' of $j_{p_0}^k\mathsf{id}$ in $G^k(M)$ and a real-analytic map $\Psi\colon \Omega' \times \Omega'' \to M$ such that, for every germ $h\colon (M,p_0) \to M$ of a local CR diffeomorphism of M with $j_{p_0}^k h \in \Omega''$, we have the identity $h(\cdot) \equiv \Psi(\cdot, j_{p_0}^k h)$ in the sense of germs at p_0. Let $\widetilde{\Omega}'$ (resp. $\widetilde{\Omega}''$) be a smaller neighborhood of p_0 in

Ω' (resp. of $j^k_{p_0}\mathsf{id}$ in Ω'') which is relatively compact in Ω' (resp. Ω''), chosen for every $p_0 \in K$. Without loss of generality, all neighborhoods here are connected. Using the compactness of K and passing to a finite subcovering, we obtain a finite collection of points $p_1, \ldots, p_s \in K$, the corresponding neighborhoods

$$\Omega'_m \supset\supset \widetilde{\Omega}'_m \ni p_m, \quad \Omega''_m \supset\supset \widetilde{\Omega}''_m \ni j^k_{p_m}\mathsf{id},$$

and real-analytic maps $\Psi_m \colon \Omega'_m \times \Omega''_m \to M$ for $m = 1, \ldots, s$, such that $(\widetilde{\Omega}'_m)$ is a covering of K.

We next define neighborhoods $\mathcal{U}$ and $\widetilde{\mathcal{U}}$ of the identity mapping in $\mathrm{Aut}_{\mathrm{CR}}(M)$ with respect to the compact-open $\mathcal{C}^k$ topology as follows:

$$\begin{aligned} \widetilde{\mathcal{U}} &:= \{g \in \mathrm{Aut}_{\mathrm{CR}}(M) : j^k_{p_m} g \in \widetilde{\Omega}''_m,\ 1 \le m \le s\}, \\ \mathcal{U} &:= \{g \in \widetilde{\mathcal{U}} : g^{-1} \in \widetilde{\mathcal{U}}\}. \end{aligned} \tag{2.0.1}$$

It is clear from the definition of the topology chosen that $\mathcal{U}$ is indeed an open set. Obviously the same conclusion holds for the compact-open $\mathcal{C}^\infty$ topology as well as for the compact-open $\mathcal{C}^r$ topology for any $r \ge k$.

Our main step of the proof will be to show that $\mathcal{U}$ is relatively compact in $\mathrm{Aut}_{\mathrm{CR}}(M)$. We shall prove it with respect to the compact-open $\mathcal{C}^k$ topology, which is Fréchet and hence, in particular, metrizable. Thus it suffices to prove that the closure of $\mathcal{U}$ is sequentially compact. Let (f_n) be any sequence in $\mathcal{U}$, for which we shall prove that there exists a convergent subsequence. In view of (2.0.1), we have

$$j^k_{p_m} f_n \in \widetilde{\Omega}''_m \subset\subset \Omega''_m \subset G^k(M), \quad m = 1, \ldots, s,$$

for every n. Hence, passing to a subsequence, we may assume that $j^k_{p_m} f_n$ converges to some $\Lambda_m \in \Omega''_m$ for each $m = 1, \ldots, s$.

Following the strategy of [BRWZ04], we denote by $\mathcal{O}$ the open set of points $q \in M$ with the property that (f_n) converges in the compact-open $\mathcal{C}^\omega$ (and hence any $\mathcal{C}^r$ with $r \ge k$) topology in a neighorhood V of q in M to a map $f \colon V \to M$ such that the Jacobian of f at q is nonzero. We want to show that $\mathcal{O}$ is nonempty and closed in M. By our construction, we have $f_n(\cdot) \equiv \Psi_1(\cdot, j^k_{p_1} f_n)$ in the sense of germs at p_1 and hence, by the identity principle for real-analytic functions, all over Ω'_1. Since $j^k_{p_1} f_n$ converges to $\Lambda_1 \in \Omega''_1$, it clearly follows that $f_n|_{\Omega'_1} \to f := \Psi_1(\cdot, \Lambda_1)$ as $n \to +\infty$ in the compact-open $\mathcal{C}^\omega$ topology on Ω'_1. In particular, we also have $j^k_{p_1} f_n \to j^k_{p_1} f$ and since $j^k_{p_1} f \in \Omega''_1 \subset G^k(M)$, we immediately see that $p_1 \in \mathcal{O}$, proving that $\mathcal{O}$ is nonempty. To show that $\mathcal{O}$ is closed, let q_0 be any point in the closure of $\mathcal{O}$ in M. We now distinguish two cases.

Case 1. $q_0 \notin K$. Here we can repeat the arguments of the proof of [BRWZ04, Lemma 3.3] to show that $q_0 \in \mathcal{O}$.

Case 2. $q_0 \in K$. Here we only have the restricted parametrization given by (i) and hence cannot use the same arguments as in Case 1; instead, we use our construction. Since the neighborhoods $\widetilde{\Omega}'_m$, $m = 1, \ldots, s$, cover K, we have $q_0 \in \widetilde{\Omega}'_{m_0} \subset\subset \Omega'_{m_0}$ for some m_0 and let $p_{m_0} \in K$ be the corresponding point. The sequence of the k-jets $\Lambda^n_{m_0} := j^k_{p_{m_0}} f_n$ converges to Λ_{m_0} by our assumptions above and therefore

$$f_n(\cdot) \equiv \Psi_{m_0}(\cdot, \Lambda^n_{m_0}) \to \Psi_{m_0}(\cdot, \Lambda_{m_0}), \tag{2.0.2}$$

which immediately implies that $q_0 \in \mathcal{O}$.

Summarizing, we have shown that $\mathcal{O}$ is nonempty, open and closed in M and therefore $\mathcal{O} = M$, i.e. (f_n) converges on M to a real-analytic map $f : M \to M$ which is automatically CR. Furthermore, by our construction of $\mathcal{U}$, also the sequence of the inverses f_n^{-1} is in $\mathcal{U}$. Hence similar arguments show that this sequence converges to another real-analytic CR self-map g of M. Then it follows that $g \circ f = f \circ g = \mathsf{id}$ and therefore $f \in \mathrm{Aut}_{\mathrm{CR}}(M)$. This completes the proof that the chosen neighborhood $\mathcal{U}$ of id in $\mathrm{Aut}_{\mathrm{CR}}(M)$ is relatively compact. Since any $g \in \mathrm{Aut}_{\mathrm{CR}}(M)$ has $g\mathcal{U}$ as its neighborhood, it follows that the whole group $\mathrm{Aut}_{\mathrm{CR}}(M)$ is locally compact.

As in [BRWZ04], we make use of the following theorem of Bochner-Montgomery [BM46], [MZ55, Theorem 2, p. 208]:

THEOREM 2.3. (Bochner-Montgomery) *Let G be a locally compact topological group acting effectively and continuously on a smooth manifold M by smooth diffeomorphisms. Then G is a Lie group and the action $G \times M \to M$ is smooth.*

Indeed, we have just shown that $G := \mathrm{Aut}_{\mathrm{CR}}(M)$ is locally compact. Since the action $\mathrm{Aut}_{\mathrm{CR}}(M) \times M \to M$ is obviously effective, Theorem 2.3 shows that $\mathrm{Aut}_{\mathrm{CR}}(M)$ is a Lie group and its action is smooth. The coincidence of the compact-open $\mathcal{C}^r$ topologies on $\mathrm{Aut}_{\mathrm{CR}}(M)$ for $r \geq k$ also follows from the proof. Finally the analyticity of the action follows from another result of Bochner-Montgomery [BM45]:

THEOREM 2.4. (Bochner-Montgomery) *Let G be a Lie group acting continuously on a real-analytic manifold M by real-analytic diffeomorphisms. Then the action $G \times M \to M$ is real-analytic.*

The proof of Theorem 2.2 is complete. □

3. Parametrization of local CR diffeomorphisms. In order to deduce Theorem 1.2 from Theorem 2.2, we will establish a jet parametrization property of local CR diffeomorphisms for a certain class of real-analytic CR submanifolds in complex space. Such a property has already been established by the first two authors in [LM07] for an appropriate class of CR manifolds for local CR diffeomorphisms *which furthermore fix a given point of the manifold*. However, in view of Definition 2.1, we need to extend such a parametrization property to local CR diffeomorphisms *which do not necessarily fix a base point*. In what follows, we make the above statements precise and show how they may be derived from the analysis given in the paper [LM07].

The class of germs of real-analytic generic submanifolds we shall consider in this paper is the one introduced by the first two authors in [LM07], denoted by $\mathcal{C}$, whose definition we now recall. Denote by (M, p_0) a germ of a real-analytic generic submanifold of $\mathbb{C}^N$ (or, more generally, of any complex manifold) of CR dimension n and real codimension d, i.e. $N = n + d$, $T_{p_0}M + iT_{p_0}M = T_{p_0}\mathbb{C}^N$ and $n = \dim_{\mathbb{C}} (T_{p_0}M \cap iT_{p_0}M)$. Let $\rho = (\rho_1, \ldots, \rho_d)$ be a real-analytic vector valued defining function for M in some neighborhood U of p_0 in $\mathbb{C}^N$ satisfying $\partial\rho_1 \wedge \ldots \wedge \partial\rho_d \neq 0$. Using standard notation, we write ρ as a convergent power series (after shrinking U if necessary)

$$\rho(Z, \overline{Z}) = \sum_{\alpha,\beta\in\mathbb{N}^N} \rho_{\alpha\beta}(Z - p_0)^{\alpha}(\overline{Z - p_0})^{\beta}, \quad Z \in U,$$

where $\rho_{\alpha,\beta} \in \mathbb{C}^d$ satisfy $\rho_{\alpha,\beta} = \overline{\rho_{\beta,\alpha}}$, and complexify it to the power series

$$\rho(Z, \zeta) = \sum \rho_{\alpha\beta}(Z - p_0)^{\alpha}(\zeta - \overline{p}_0)^{\beta}, \quad \partial\rho_1 \wedge \cdots \wedge \partial\rho_d \neq 0,$$

with $(Z, \zeta) \in \mathbb{C}^N \times \mathbb{C}^N$, which we still denote by ρ. It is easy to see that the complexification $\rho(Z, \zeta)$ is still convergent in a suitable neighborhood of $(p_0, \overline{p}_0)$ that (after shrinking U again if necessary) can be chosen of the form $U \times \overline{U} \subset \mathbb{C}^N \times \mathbb{C}^N$. Recall that the *Segre variety* S_q of a point $q \in U$ is the n-dimensional complex submanifold of U given by $S_q := \{Z \in U : \rho(Z, \overline{q}) = 0\}$. Furthermore, the *complexification of* M is defined to be the $2n+d$-dimensional complex submanifold of $U \times \overline{U}$ given by

$$\mathcal{M} := \{(Z, \zeta) \in U \times \overline{U} : \rho(Z, \zeta) = 0\} = \left\{(Z, \zeta) \in U \times \overline{U} : Z \in S_{\bar{\zeta}}\right\}. \tag{3.0.3}$$

For every integer k and for $q \in \mathbb{C}^N$, we denote by $J_q^{k,n}(\mathbb{C}^N)$ the space of all jets at q of order k of n-dimensional complex submanifolds of $\mathbb{C}^N$ passing through q. For every $q \in M$ sufficiently close to p_0, we consider the anti-holomorphic map π_q^k defined as follows:

$$\pi_q^k : S_q \to J_q^{k,n}(\mathbb{C}^N), \quad \pi_q^k(\xi) = j_q^k S_\xi, \tag{3.0.4}$$

where $j_q^k S_\xi$ denotes the k-jet at q of the submanifold S_ξ (see e.g. [Z99] for more details on jets of complex submanifolds used here, and also [LM07]).

Following [LM07], we say that the germ (M, p_0) belongs to the class $\mathcal{C}$ if *the anti-holomorphic map* $\pi^k_{p_0}$ *is generically of full rank* $n = \dim S_{p_0}$ *in any neighborhood of* p_0*, for* k *sufficiently large*. For $(M, p_0) \in \mathcal{C}$, we denote by $\kappa_M(p_0)$ the smallest integer k for which the map $\pi^k_{p_0}$ is of generic rank n. Since the Segre varieties are associated to (M, p_0) in a biholomorphically invariant way, the integer $\kappa_M(p_0)$ is a biholomorphic invariant of the germ (M, p_0). Note furthermore that the condition for a germ of real-analytic generic submanifold to belong to the class $\mathcal{C}$ is an *open* condition in the sense that, if $M \in \mathcal{C}$ is given by the equation $\rho(Z, \bar{Z}) = 0$ as above, then $\widetilde{M} \in \mathcal{C}$ for any $\widetilde{M}$ given by the equation $\widetilde{\rho}(Z, \bar{Z}) = 0$ with $\widetilde{\rho}$ sufficiently close to ρ in the $\mathcal{C}^\infty$ topology (see e.g. [GG73] for details on this topology; here it is enough to assume that $\widetilde{\rho}$ is close to ρ in the $\mathcal{C}^k$-topology for a suitable k). In particular, there exists a neighborhood V of p_0 in M such that $(M, q) \in \mathcal{C}$ for all $q \in V$ and moreover, it is clear from the definition that $\kappa_M(q)$ is *upper semi-continuous* on V.

We also recall that M is *essentially finite* (resp. *finitely nondegenerate*) at p_0 if the map $\pi^k_{p_0}$ is *finite* near p_0 (resp. an *immersion* at p_0) for k sufficiently large (see [BHR96], [BER99b] for more details). It follows that finite nondegeneracy of M at p_0 implies essential finiteness of M at p_0 which in turn implies that $(M, p_0) \in \mathcal{C}$. Recall also that M is *minimal* at p_0 if there does not exist any CR submanifold of lower dimension contained in M and passing through p_0 with the same CR dimension as that of M (see [Tu88], [BER99b]).

For a real-analytic CR submanifold $M \subset \mathbb{C}^N$ which is not necessarily generic and for a point $p_0 \in M$, we say that (M, p_0) is in the class $\mathcal{C}$ if it is in the class $\mathcal{C}$ when considered as a generic submanifold of its intrinsic complexification, i.e. the minimal germ of a complex submanifold of $\mathbb{C}^N$ containing (M, p_0) (see e.g. [BER99b] for this notion). Finally we should also note that the local nondegeneracy conditions defined above are defined in the same way for abstract real-analytic CR manifolds since such manifolds can always be locally embedded in some complex euclidean space $\mathbb{C}^q$ for some integer q, see e.g. [BER99b].

Finally, we refer the reader to [LM07] for examples of manifolds that belong to the class $\mathcal{C}$, as well as for a more thorough discussion of the relation between this nondegeneracy condition and other well-known nondegeneracy conditions such as essential finiteness and finite nondegeneracy. We only stress in this paper the following fact that will be used implicitly in the proofs of Corollaries 1.1 and 1.3 and that follows from a result of [DF78]: *for every compact real-analytic CR submanifold* Σ *embedded in some Stein manifold and for every* $q \in \Sigma$, Σ *is essentially finite at* q*, and in particular* $(\Sigma, q) \in \mathcal{C}$ (see [LM07] for more details).

The following parametrization theorem is the second main ingredient of the proof of Theorem 1.2.

THEOREM 3.1. *Let* $M \subset \mathbb{C}^N$ *be a real-analytic CR submanifold of codimension* d *and* $p_0 \in M$*. Assume that* (M, p_0) *is minimal and belongs to the class* $\mathcal{C}$ *and set* $\ell_0 := 2(d+1)\kappa_M(p_0)$*. Then the set of all germs* $h\colon (M, p_0) \to M$ *of local CR* *diffeomorphisms of M has the real-analytic jet parametrization property of order ℓ_0 at p_0.*

As mentioned above, the difference between Theorem 3.1 and [LM07, Theorem 7.3] is due to the fact that the parametrization theorem given in [LM07] is obtained for the set of germs $h\colon (M,p_0) \to (M,p_0)$ of local CR diffeomorphisms with fixed source p_0 but also with fixed target p_0. The version given here by Theorem 3.1 allows to parametrize local CR diffeomorphisms which send the point p_0 to a varying target point $p \in M$ (close to p_0) that has to be regarded as an additional parameter. We will provide a deformation version of [LM07, Theorem 7.3] which allows us to treat this additional parameter in Theorem 3.2 below. (Note that it is not always possible to parametrize in a proper sense the germs of all local CR diffeomorphisms with varying both source and targets, see Remark 3.3 below).

Before we proceed, we need to introduce some additional terminology. Given a real-analytic manifold E and a point $p_0 \in \mathbb{C}^N$, a *real-analytic family of germs at p_0 of real-analytic generic submanifolds* $(M_\epsilon)_{\epsilon\in E}$ of $\mathbb{C}^N$, is given by a family of convergent power series mapping in Z and $\bar{Z}$ centered at p_0, $\rho(Z,\bar{Z};\epsilon) = (\rho_1(Z,\bar{Z};\epsilon),\ldots,\rho_d(Z,\bar{Z};\epsilon))$ with $\rho(p_0,\bar{p}_0;\epsilon) = 0$ and $\partial\rho_1(\cdot;\epsilon)\wedge\cdots\wedge\partial\rho_d(\cdot;\epsilon) \neq 0$ for every $\epsilon \in E$ such that there exists a neighborhood of $\{p_0\}\times E \subset \mathbb{C}^N \times E$ on which $\rho(Z,\bar{Z};\epsilon)$ is real-analytic in all its arguments. In particular, for each $\epsilon \in E$, the set $\{Z \in \mathbb{C}^N : \rho(Z,\bar{Z};\epsilon) = 0\}$ defines a germ at p_0 of a real-analytic generic submanifold $M_\epsilon \subset \mathbb{C}^N$ of codimension d. Given a fixed germ of a real-analytic generic submanifold $M \subset \mathbb{C}^N$ through p_0 and $(M_\epsilon)_{\epsilon\in E}$ a real-analytic family through p_0 as defined above, we say that $(M_\epsilon)_{\epsilon\in E}$ is a real-analytic *deformation* of (M,p_0) if there exists $\epsilon_0 \in E$ such that $(M,p_0) = (M_{\epsilon_0},p_0)$.

We are now ready to state the following result.

THEOREM 3.2. *Let (M,p_0) be a germ of a real-analytic generic submanifold of codimension d that is minimal and in the class $\mathcal{C}$ and set $\ell_0 = 2(d+1)\kappa_M(p_0)$. Let $(M_\epsilon)_{\epsilon\in E}$ be a real-analytic deformation of the germ (M,p_0) (parametrized by some real-analytic manifold E) with $(M_{\epsilon_0},p_0) = (M,p_0)$ for some $\epsilon_0 \in E$. Then there exist open neighborhoods U_0 of ϵ_0 in E, U_1 of $p_0 \in \mathbb{C}^N$ and Ω of $j_{p_0}^{\ell_0}\mathsf{Id}$ in $G_{p_0}^{\ell_0}(\mathbb{C}^N)$ and a real-analytic map $\Psi(Z,\Lambda;\epsilon)\colon U_1\times\Omega\times U_0 \to \mathbb{C}^N$, holomorphic in its first factor such that for every germ of a biholomorphic map $H\colon(\mathbb{C}^N,p_0)\to(\mathbb{C}^N,p_0)$ sending (M_ϵ,p_0) for some $\epsilon \in U_0$ into (M,p_0) with $j_{p_0}^{\ell_0}H \in \Omega$, we have*

$$H(Z) = \Psi(Z, j_{p_0}^{\ell_0}H;\epsilon), \text{ for } Z \in \mathbb{C}^N \text{ close to } p_0.$$

Remark 3.3. It is natural to ask whether Theorem 3.2 remains true with the target manifold (M,p_0) also varying. Such a result holds for finitely nondegenerate manifolds [BER99a], [KZ05]. However, it *cannot* hold for the more general class $\mathcal{C}$ (even in the real-analytic case) as the example with $M \subset \mathbb{C}^2_{(z,w)}$ given by $\mathsf{Im}\, w = |z|^4$ shows, see [KZ05, Example 1.5].

Let us now show how Theorem 3.1 follows from Theorem 3.2.

Proof of Theorem 3.1 *assuming Theorem* 3.2. Without loss of generality, we may assume that M is generic. Let $\rho = \rho(Z, \bar{Z})$ be a real-analytic vector valued defining equation for M in a neighborhood U of 0. Consider the real-analytic deformation of the germ (M, p_0) obtained by varying the base point in some small neighborhood $\widetilde{U} \subset U$ i.e. defined by the family $(M_p)_{p\in\widetilde{U}}$ where M_p is the germ at p_0 of the real-analytic generic submanifold given by the equation $\{Z \in \mathbb{C}^N : \rho(Z-p_0+p, \bar{Z}-\bar{p}_0+\bar{p}) = 0\}$. Applying Theorem 3.2 to this deformation, it is not difficult to derive the following:

PROPOSITION 3.4. *Under the assumptions of Theorem* 3.1*, the set of all germs* $h\colon (M, p) \to (M, p_0)$ *of local CR diffeomorphisms of M with variable source point p has the real-analytic jet parametrization property of order* ℓ_0 *at* p_0.

The conclusion of Theorem 3.1 then follows easily from Proposition 3.4 and an application of the inverse function theorem. We leave the details to the reader. □

4. Proof of Theorem 3.2. We assume that we are in the setting of Theorem 3.2. Without loss of generality, suppose that p_0 coincides with the origin in $\mathbb{C}^N$ and set $n = N - d$. Consider the given real-analytic family $(M_\epsilon)_{\epsilon\in E}$ and $\epsilon_0 \in E$ satisfying $(M_{\epsilon_0}, 0) = (M, 0)$.

4.1. Normal coordinates and Segre mappings for the deformation. The first basic fact needed for the construction of a mapping Ψ satisfying the conclusion of Theorem 3.2 is the choice of a certain set of coordinates (the so-called "normal coordinates") for each manifold M_ϵ near the origin and depending real-analytically on ϵ for ϵ close to ϵ_0.

The coordinates we need are obtained from the standard construction of the normal coordinates (cf. e.g. [BER99b]):

LEMMA 4.1. *Let* $(M_\epsilon)_{\epsilon\in E}$ *be a real-analytic family of real-analytic generic submanifolds through the origin in* $\mathbb{C}^N$ *of codimension d and* $\epsilon_0 \in E$ *as above. Then there exist germs of real-analytic maps*

$$Z\colon (\mathbb{C}^N \times E, (0, \epsilon_0)) \to (\mathbb{C}^N, 0) \text{ and } Q\colon (\mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}^d \times E, (0, 0, 0, \epsilon_0)) \to (\mathbb{C}^d, 0),$$

holomorphic in all their components except E, such that for every fixed $\epsilon \in E$ *sufficiently close to* ϵ_0*, the following hold:*

(i) $Z(0; \epsilon) = 0$ *and the map* $Z(\cdot; \epsilon)\colon (\mathbb{C}^N, 0) \to (\mathbb{C}^N, 0)$ *is locally biholomorphic near* 0*;*

(ii) *in the local coordinates* $Z(\cdot; \epsilon) = (z, w) \in \mathbb{C}^n \times \mathbb{C}^d$ *near* 0*, the manifold* M_ϵ *is given by*

$$w - Q(z, \bar{z}, \overline{w}; \epsilon) = 0; \tag{4.1.1}$$

(iii) *one has* $Q(z, 0, \tau; \epsilon) \equiv Q(0, \chi, \tau; \epsilon) \equiv \tau$.

We fix an open neighborhood U_0 of ϵ_0 in E so that Lemma 4.1 holds. After possibly shrinking U_0 we may assume that for every $\epsilon \in U_0$, M_ϵ is minimal at 0 and that $\kappa_{M_\epsilon}(0) \leq \kappa_M(0)$; we set $Q(z, \chi, \tau) := Q(z, \chi, \tau, \epsilon_0)$.

The next tools we need are the Segre mappings associated with the manifolds M_ϵ, $\epsilon \in U_0$. Recall that for every integer $k \geq 1$, the k-th Segre (germ of a) mapping

$$v_\epsilon^k : (\mathbb{C}^{kn}, 0) \to (\mathbb{C}^n \times \mathbb{C}^d, 0)$$

associated to $(M_\epsilon, 0)$ and the chosen normal coordinates is defined inductively as follows (see [BER99a]):

$$v_\epsilon^1(t^1) := (t^1, 0), \quad v_\epsilon^{k+1}(t^{[k+1]}) := \left(t^{k+1}, Q\big(t^{k+1}, \overline{v_\epsilon^k}(t^{[k]}); \epsilon\big)\right), \tag{4.1.2}$$

where $t^k \in \mathbb{C}^n$, $t^{[k]} := (t^1, \ldots, t^k) \in \mathbb{C}^{kn}$. Here and throughout the paper, for any power series mapping θ, we denote by $\bar{\theta}$ the power series obtained from θ by taking complex conjugates of its coefficients.

For every $\epsilon \in U_0$ and a germ of a biholomorphic map $H : (\mathbb{C}^N, 0) \to (\mathbb{C}^N, 0)$ sending $(M_\epsilon, 0)$ into $(M, 0)$, we define

$$H_\epsilon : (\mathbb{C}^N, 0) \to (\mathbb{C}^N, 0), \quad H_\epsilon := H(Z(\cdot; \epsilon)^{-1}), \tag{4.1.3}$$

where $Z(\cdot; \epsilon)^{-1} : (\mathbb{C}^N, 0) \to (\mathbb{C}^N, 0)$ is the local inverse of $Z(\cdot; \epsilon)$; H_ϵ sends $(M_\epsilon, 0)$ written in the Z-coordinates into $(M, 0)$, and M_ϵ is given by (4.1.1). It is clear from the construction of the above coordinates and from the Inverse Function Theorem that it is enough to prove the parametrization property for all our mappings H_ϵ for ϵ sufficiently close to ϵ_0 to obtain the conclusion of Theorem 3.2.

After choosing normal coordinates $Z' = (z', w') \in \mathbb{C}^n \times \mathbb{C}^d$ for the target manifold M at p_0, which are fixed here, we write $H_\epsilon = (f_\epsilon, g_\epsilon) \in \mathbb{C}^n \times \mathbb{C}^d$ and also denote by $\mathcal{M}_\epsilon$ the germ at the origin of the complexification of M_ϵ. In our coordinates, it is defined as the germ of the complex submanifold of $\mathbb{C}^N \times \mathbb{C}^N$ at the origin given by

$$\mathcal{M}_\epsilon = \{(Z, \zeta) = ((z, w), (\chi, \tau)) \in \mathbb{C}^n \times \mathbb{C}^d \times \mathbb{C}^n \times \mathbb{C}^d : w = Q(z, \chi, \tau; \epsilon)\}.$$

4.2. Reflection identities with parameters. We now want to state a version with parameters of the reflection identities given in [LM07, Propositions 9.1 and 9.2, Lemma 9.3 and Proposition 9.4]. For this, as in [LM07], it is convenient to introduce the following notation.

For every positive integer k, we denote by $J_{0,0}^k(\mathbb{C}^N)$ the space of all jets at the origin of order k of holomorphic mappings from $\mathbb{C}^N$ into itself and fixing the origin. In our normal coordinates $Z = (Z_1, \ldots, Z_N)$ in $\mathbb{C}^N$, we identify a jet

$\mathcal{J} \in J^k_{0,0}(\mathbb{C}^N)$ with a polynomial map of the form

$$\mathcal{J} = \mathcal{J}(Z) = \sum_{\alpha\in\mathbb{N}^r,\ 1\leq|\alpha|\leq k} \frac{\Lambda^k_\alpha}{\alpha!} Z^\alpha, \tag{4.2.1}$$

where $\Lambda^k_\alpha \in \mathbb{C}^N$. We thus have for a jet $\mathcal{J} \in J^k_{0,0}(\mathbb{C}^N)$, the coordinates

$$\Lambda^k := (\Lambda^k_\alpha)_{1\leq|\alpha|\leq k} \tag{4.2.2}$$

given by (4.2.1). Given a germ of a holomorphic map $h\colon(\mathbb{C}^N,0) \to (\mathbb{C}^N,0)$, $h = h(t)$, for t sufficiently small we use for the k-jet of h at t the notation $j^k_t h =: (t, h(t), \widehat{j}^k_t h)$ (which is defined as a germ at 0). Moreover, since $h(0) = 0$, we may also identify $j^k_0 h$ with $\widehat{j}^k_0 h$, which we will freely do in the sequel.

Given the normal coordinates $(z,w) \in \mathbb{C}^n \times \mathbb{C}^d = \mathbb{C}^N$, we consider a special component of a jet $\Lambda^k \in J^k_{0,0}(\mathbb{C}^N)$ defined as follows. Denote the set of all multiindices of length one having 0 from the $n+1$-th to the N-th component by S, and the projection onto the first n coordinates by $\mathrm{proj}_1\colon \mathbb{C}^N \to \mathbb{C}^n$ (that is, $\mathrm{proj}_1(z,w) = z$). Then set

$$\widetilde{\Lambda}^1 := (\mathrm{proj}_1(\Lambda_\alpha))_{\alpha\in S}. \tag{4.2.3}$$

Note that for any local holomorphic map

$$(\mathbb{C}^n \times \mathbb{C}^d, 0) \ni (z,w) \mapsto h(z,w) = (f(z,w), g(z,w)) \in (\mathbb{C}^n \times \mathbb{C}^d, 0),$$

if $j^k_0 h = \Lambda^k$, then $\widetilde{\Lambda}^1 = (\frac{\partial f}{\partial z}(0))$. We can therefore identify $\widetilde{\Lambda}^1$ with an $n\times n$ matrix or equivalently with an element of $J^1_{0,0}(\mathbb{C}^n)$. Throughout the paper, given any jet $\lambda^k \in J^k_{0,0}(\mathbb{C}^N)$, $\widetilde{\lambda}^1$ will always denote the component of λ^k defined by (4.2.3).

In addition, for every positive integer r and an open neighborhood U_0 of ϵ_0 in E, we denote by $\mathcal{S}_r = \mathcal{S}_r(U_0)$ the ring of germs at $\{0\} \times U_0$ of real-analytic functions on $\mathbb{C}^r \times E$ that are holomorphic in their first argument. Recall that this is the space of all real-analytic functions that are defined in a connected open neighborhood (depending on the function) of $\{0\}\times U_0$ in $\mathbb{C}^r \times E$ (and holomorphic in their first argument).

We now collect the following versions of the reflection identities of [LM07, Section 9] with parameters that are necessary in order to complete the proof of Theorem 3.2. The first basic identity given by (4.2.4) is standard and may obtained by complexifying the identity $H_\epsilon(M_\epsilon) \subset M$ and applying the vector fields tangent to $\mathcal{M}_\epsilon$.

PROPOSITION 4.2. *In the above setting, there exists a polynomial* $\mathcal{D} = \mathcal{D}(Z,\zeta,\epsilon,\Lambda^1) \in \mathcal{S}_{2N}[\Lambda^1]$ *and, for every* $\alpha \in \mathbb{N}^n \setminus \{0\}$, *a* $\mathbb{C}^d$*-valued polynomial* *map* $\mathcal{P}_\alpha = \mathcal{P}_\alpha(Z,\zeta,\epsilon,\Lambda^{|\alpha|})$ *whose components are in the ring* $\mathcal{S}_{2N}[\Lambda^{|\alpha|}]$ *such that for* $\epsilon \in U_0$ *and for every map* $H_\epsilon\colon (M_\epsilon,0)\to(M,0)$ *the following hold:*

(i) $\mathcal{D}(0,0,\epsilon,\Lambda^1) = \det \widetilde{\Lambda}^1$*;*

(ii) *for all* $(Z,\zeta)\in\mathcal{M}_\epsilon$ *near* 0*,*

$$(\mathcal{D}(Z,\zeta,\epsilon,\widehat{j}^1_\zeta\overline{H}_\epsilon))^{2|\alpha|-1}\,\bar{Q}_{\chi^\alpha}(\bar{f}_\epsilon(\zeta),H_\epsilon(Z)) = \mathcal{P}_\alpha(Z,\zeta,\epsilon,\widehat{j}^{|\alpha|}_\zeta\overline{H}_\epsilon). \tag{4.2.4}$$

The next identity given by (4.2.5) involves the (transversal) derivatives of the mappings H_ϵ and follows easily from differentiating (4.2.4) and applying the chain rule.

PROPOSITION 4.3. *For any* $\mu\in\mathbb{N}^d\setminus\{0\}$ *and* $\alpha\in\mathbb{N}^n\setminus\{0\}$*, there exist a* $\mathbb{C}^d$*-valued polynomial map* $\mathcal{T}_{\mu,\alpha}(Z,\zeta,Z',\zeta',\epsilon,\lambda^{|\mu|-1},\Lambda^{|\mu|})$ *whose components belong to the ring* $\mathcal{S}_{4N}[\lambda^{|\mu|-1},\Lambda^{|\mu|}]$ *and a* $\mathbb{C}^d$*-valued polynomial map* $\mathcal{Q}_{\mu,\alpha}(Z,\zeta,\epsilon,\Lambda^{|\alpha|+|\mu|})$ *whose components are in the ring* $\mathcal{S}_{2N}[\Lambda^{|\alpha|+|\mu|}]$*, such that for* $\epsilon\in U_0$*, for every map* $H_\epsilon\colon(M_\epsilon,0)\to(M,0)$ *and for any* $(Z,\zeta)\in\mathcal{M}_\epsilon$ *close to the origin, the following relation holds:*

$$\frac{\partial^{|\mu|}H_\epsilon}{\partial w^\mu}(Z)\cdot\bar{Q}_{\chi^\alpha,Z}(\bar{f}_\epsilon(\zeta),H_\epsilon(Z)) = (*)_1+(*)_2, \tag{4.2.5}$$

where

$$(*)_1 := \mathcal{T}_{\mu,\alpha}\left(Z,\zeta,H_\epsilon(Z),\overline{H}_\epsilon(\zeta),\epsilon,\widehat{j}^{|\mu|-1}_Z H_\epsilon,\widehat{j}^{|\mu|}_\zeta\overline{H}_\epsilon\right) \tag{4.2.6}$$

and

$$(*)_2 := \frac{\mathcal{Q}_{\mu,\alpha}(Z,\zeta,\epsilon,\widehat{j}^{|\alpha|+|\mu|}_\zeta\overline{H}_\epsilon)}{(\mathcal{D}(Z,\zeta,\epsilon,\widehat{j}^1_\zeta\overline{H}_\epsilon))^{2|\alpha|+|\mu|-1}}. \tag{4.2.7}$$

In the next lemma, we observe that for any given map $H_\epsilon=(f_\epsilon,g_\epsilon)$, the (transversal) derivatives of the normal component g_ϵ can be expressed (in an universal way) through the (transversal) derivatives of the components of f_ϵ and some other terms that have to be seen as remainders. In particular, this lemma will allow us (as in [LM07]) to derive the desired parametrizations of the maps H_ϵ and their derivatives on each Segre set from the corresponding parametrizations of the maps f_ϵ and their derivatives.

LEMMA 4.4. *For any* $\mu\in\mathbb{N}^d\setminus\{0\}$*, there exists a* $\mathbb{C}^d$*-valued polynomial map*

$$W_\mu = W_\mu\left(Z,\zeta,Z',\zeta',\epsilon,\lambda^{|\mu|-1},\Lambda^{|\mu|}\right),$$

whose components belong to the ring $\mathcal{S}_{4N}[\lambda^{|\mu|-1},\Lambda^{|\mu|}]$ *and such that for* $\epsilon\in U_0$*, for every map* $H_\epsilon\colon(M_\epsilon,0)\to(M,0)$ *and for any* $(Z,\zeta)\in\mathcal{M}_\epsilon$ *close to the origin,* *the identity*

$$\frac{\partial^{|\mu|} g_\epsilon}{\partial w^\mu}(Z) = \frac{\partial^{|\mu|} f_\epsilon}{\partial w^\mu}(Z) \cdot Q_z(f_\epsilon(Z), \overline{H}_\epsilon(\zeta)) + (*)_3 \tag{4.2.8}$$

holds with

$$(*)_3 := W_\mu \left(Z, \zeta, H_\epsilon(Z), \overline{H}_\epsilon(\zeta), \epsilon, \widehat{j}_Z^{|\mu|-1} H_\epsilon, \widehat{j}_\zeta^{|\mu|} \overline{H}_\epsilon \right). \tag{4.2.9}$$

The next statement is obtained as a direct combination of Lemma 4.4 and Proposition 4.3 and provides the form of the system of equations fulfilled by any (transversal) derivative of f_ϵ.

PROPOSITION 4.5. *For any $\mu \in \mathbb{N}^d \setminus \{0\}$ and $\alpha \in \mathbb{N}^n \setminus \{0\}$, there exist a $\mathbb{C}^d$-valued polynomial map $\mathcal{T}'_{\mu,\alpha}(Z, \zeta, Z', \zeta', \epsilon, \lambda^{|\mu|-1}, \Lambda^{|\mu|})$ whose components belong to the ring $\mathcal{S}_{4N}[\lambda^{|\mu|-1}, \Lambda^{|\mu|}]$ such that for $\epsilon \in U_0$ and for every map $H_\epsilon\colon (M_\epsilon, 0) \to (M, 0)$ the following relation holds for $(Z, \zeta) \in \mathcal{M}_\epsilon$ close to 0:*

$$\begin{aligned} \frac{\partial^{|\mu|} f_\epsilon}{\partial w^\mu}(Z) \cdot (\bar{Q}_{\chi^\alpha, z}(\bar{f}_\epsilon(\zeta), H_\epsilon(Z)) \\ + Q_z(f_\epsilon(Z), \overline{H}_\epsilon(\zeta)) \cdot \bar{Q}_{\chi^\alpha, w}(\bar{f}_\epsilon(\zeta), H_\epsilon(Z))) = (*)'_1 + (*)_2, \end{aligned} \tag{4.2.10}$$

where $(*)_2$ is given by (4.2.7) and $(*)'_1$ is given by

$$(*)'_1 := \mathcal{T}'_{\mu,\alpha} \left(Z, \zeta, H_\epsilon(Z), \overline{H}_\epsilon(\zeta), \epsilon, \widehat{j}_Z^{|\mu|-1} H_\epsilon, \widehat{j}_\zeta^{|\mu|} \overline{H}_\epsilon \right). \tag{4.2.11}$$

Since the proof of the above relations is analogous to those derived in [LM07], we leave the details to the reader. We should point out that, in the reflection identities with parameters mentioned above, the most relevant fact is the location of the parameter ϵ in the identities. Indeed, the parameter ϵ appears always in an appropriate place so that the results concerning the parametrization of solutions of singular analytic systems given in the next paragraph will be applicable. This crucial fact explains why we can follow the analysis of [LM07] in order to derive Theorem 3.2.

4.3. Parametrization of solutions of singular analytic systems. We state here the two versions of the parametrization results for singular systems needed for the proof of Theorem 3.2. The first one is needed to have a parametrization of the compositions $H_\epsilon \circ v_\epsilon^j$ for all integers j, where v_ϵ^j is defined by (4.1.2).

THEOREM 4.6. *Let $A\colon (\mathbb{C}^m, 0) \to \mathbb{C}^m$ be a germ of a holomorphic map of generic rank m, X a real-analytic manifold, Y a complex manifold and $b = b(z, x, y)$ a $\mathbb{C}^m$-valued real-analytic map defined on an open neighborhood V of $\{0\} \times X \times Y$ in $\mathbb{C}^m \times X \times Y$, holomorphic in (z, y). Then there exists a real-analytic map $\Gamma = \Gamma(z, \lambda, x, y)\colon \mathbb{C}^m \times \mathsf{GL_m}(\mathbb{C}) \times X \times Y \to \mathbb{C}^m$, defined on an open neighborhood Ω*

of $\{0\} \times \mathsf{GL_m}(\mathbb{C}) \times X \times Y$*, holomorphic in all its components except* X*, satisfying the following properties:*

(i) *If* $u : (\mathbb{C}^m, 0) \to (\mathbb{C}^m, 0)$ *is a germ of a biholomorphism satisfying* $A(u(z)) = b(z, x_0, y_0)$ *for some* $(x_0, y_0) \in X \times Y$*, then necessarily* $u(z) = \Gamma(z, j_0^1 u, x_0, y_0)$*;*

(ii) *For every* $\lambda \in \mathsf{GL_m}(\mathbb{C})$ *and* $(x_0, y_0) \in X \times Y$*, the map* Γ *satisfies* $\Gamma(0, \lambda, x_0, y_0) = 0$ *and* $\dfrac{\partial \Gamma}{\partial z}(0, \lambda, x_0, y_0) = \lambda$.

The statement given by Theorem 4.6 follows directly from [LM07, Corollary 3.2] after an obvious complexification argument.

The second version given below is needed to get a parametrization of the mappings $(\partial^\beta H_\epsilon) \circ v_\epsilon^j$ for all integers j and all multiindices $\beta \in \mathbb{N}^N \setminus \{0\}$.

Proposition 4.7. *Let* Θ *be an* $r \times r$ *matrix with holomorphic coefficients near the origin in* $\mathbb{C}^m$*,* $m, r \geq 1$*, such that* Θ *is of generic rank* r*. Let* X *be a real-analytic manifold and* Y *a complex manifold. Assume that* $c \colon \mathbb{C}^m \times X \times Y \to \mathbb{C}^m$ *and* $b \colon \mathbb{C}^m \times X \times Y \to \mathbb{C}^r$ *are real-analytic maps defined on some neighborhood* V *of* $\{0\} \times X \times Y$ *such that* $(z, y) \mapsto b(z, x, y)$ *and* $(z, y) \mapsto c(z, x, y)$ *are holomorphic on* $V_x = \{(z, y) \in \mathbb{C}^m \times Y \colon (z, x, y) \in V\}$ *for each* $x \in X$*. Assume furthermore that* c *satisfies*

$$c(0, x, y) = 0, \quad \det c_z(0, x, y) \neq 0, \quad \text{for every } (x, y) \in X \times Y.$$

Then there exists a real-analytic map $\Gamma \colon \mathbb{C}^m \times X \times Y \to \mathbb{C}^r$ *defined on a neighborhood of* $\{0\} \times X \times Y$*, holomorphic in all its components except* X*, such that if* $u \colon (\mathbb{C}^m, 0) \to \mathbb{C}^r$ *is a germ of a holomorphic map satisfying* $\Theta(c(z, x_0, y_0)) \cdot u(z) = b(z, x_0, y_0)$ *for some* $(x_0, y_0) \in X \times Y$*, then* $u(z) = \Gamma(z, x_0, y_0)$*.*

The statement given by Proposition 4.7 follows from [LM07, Proposition 6.3] and again a simple complexification argument.

4.4. Completion of the proof of Theorem 3.2. With the statements given in Sections 4.2 and 4.3 at our disposal, we can follow the plan of the proof of [LM07, Theorem 7.3] to get the needed parametrization of the maps H_ϵ restricted to any Segre set. More precisely, the reader may verify that after applying the above statements as in [LM07], one obtains the following:

Proposition 4.8. *In the above setting and shrinking the neighborhood* U_0 *if necessary, for every positive integer* j*, there exists a real-analytic map*

$$\Psi_j \colon \mathbb{C}^{nj} \times U_0 \times J_{0,0}^{j\kappa_M(0)}(\mathbb{C}^N) \to \mathbb{C}^N,$$

defined in a neighborhood of $\{0\} \times U_0 \times W_j$ *where* W_j *is an open set in the jet space containing all the jets (at* 0*) of the maps* H_ϵ *for* $\epsilon \in U_0$*, that is holomorphic in its* *first factor and satisfying in addition*

$$\left(H_\epsilon \circ v_\epsilon^j\right)\left(t^{[j]}\right) = \Psi_j\left(t^{[j]}, \epsilon, j_0^{j\kappa_M(0)} H_\epsilon\right), \tag{4.4.1}$$

for all $t^{[j]}$ sufficiently close to the origin.

We are now in a position to finish the proof of Theorem 3.2. For this, recall first that $\ell_0 = 2(d+1)\kappa_M(0)$ and consider the equation (4.4.1) for $j = 2(d+1)$ that we localize near the point $(\epsilon_0, j_0^{\ell_0}\mathsf{Id}) \in E \times J_{0,0}^{\ell_0}(\mathbb{C}^N)$. Shrinking U_0 if necessary, there exist open neighborhoods $O \subset \mathbb{C}^{2n(d+1)}$ of the origin and $O' \subset J_{0,0}^{\ell_0}(\mathbb{C}^N)$ of $j_0^{\ell_0}\mathsf{Id}$ such that $\Psi_{2(d+1)}$ is defined over $O \times U_0 \times O'$ and such that for every $\epsilon \in U_0$ satisfying $j_0^{\ell_0} H_\epsilon \in O'$, the identity (4.4.1) holds (with $j = 2(d+1)$) for $t^{[2(d+1)]}$ sufficiently close to the origin.

The rest of the proof closely follows the lines of [KZ05, Section 4]; it consists of using a version of the implicit function with singularities [KZ05, Lemma 3.4] and resolving the obtained singularities by using [KZ05, Lemma 4.3]. The differences between the situation treated in the present paper and that of [KZ05] are the parameter dependence which is real-analytic in our case (instead of smooth in [KZ05]) and the absence of the error terms in the formula (4.4.1) (in contrast to [KZ05]). The details are left to the reader.

5. Proof of Theorem 1.2.

Proof of Theorem 1.2. Suppose first that M is a connected real-analytic CR-submanifold in $\mathbb{C}^N$. Then we claim that the conclusion of Theorem 1.2 follows from the conjunction of Theorem 2.2 and Theorem 3.1. Indeed, assumption (i) of Theorem 1.2 and Theorem 3.1 imply that assumption (i) of Theorem 2.2 is satisfied. (Note that the upper semi-continuity of the integer $\kappa_M(p)$ on $p \in K \subset M$ in Theorem 3.1 is also used here in order to deduce the existence of the integer k satisfying the conclusions of Theorem 2.2 (i)). Furthermore, assumption (ii) of Theorem 1.2 together with the results of [BER99a], [KZ05] imply that assumption (ii) of Theorem 2.2 is also satisfied. This proves the claim.

If M is not connected, we may repeat the arguments of the proof of [BRWZ04, Theorem 6.2] since M is assumed to have finitely many connected components.

Finally, when M is an abstract real-analytic CR manifold, the proof is the same as before since it is based on purely local arguments and since any such manifold can locally be embedded as a CR submanifold of some complex euclidean space $\mathbb{C}^q$ for some integer q (see e.g. [BER99b]). The proof of the theorem is complete. □

6. An elementary proof of Corollary 1.3. We conclude this paper by providing an elementary proof of Corollary 1.3 which avoids the use of Theorem 2.2

and rather follows the proof of [Z97, Corollary 1.3]. Note that in any case one has to make use of Theorem 3.1.

Proof of Corollary 1.3. Since M is compact (and everywhere minimal) and embedded in some Stein manifold, we may apply Theorem 3.1 to conclude that there exists a finite number of points $p_1, \ldots, p_k \in M$ and open neighborhoods Ω_j' of p_j in M covering M such that for every $h \in \mathrm{Aut}_{\mathrm{CR}}(M)$ sufficiently close to the identity mapping, say in an open neighborhood $\mathcal{N}$ of it, Theorem 3.1 holds at all points p_j with a parametrization Ψ_j defined in a neighborhood of $\Omega_j' \times \{p_j\}$ with the jet order ℓ_j. Write $\ell = \max \ell_j$. As in [Z97], our goal is to show that the image of the neighborhood $\mathcal{N} \subset \mathrm{Aut}_{\mathrm{CR}}(M)$ under the homeomorphism (onto its image)

$$\begin{aligned} h \mapsto \eta(h) &= \left(j_{p_1}^\ell h, \ldots, j_{p_k}^\ell h, j_{p_1}^\ell h^{-1}, \ldots, j_{p_k}^\ell h^{-1}\right) \in \left(G_{p_1}^\ell(M) \times \ldots \times G_{p_k}^\ell(M)\right)^2 \\ &=: \mathcal{Y}^2 \end{aligned}$$

is a real-analytic subset of the target space, and that the group law is real-analytic. But it is easy to single out the points in the image which give rise to a global automorphism of M. Any $(\alpha, \beta) = (\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_k) \in \mathcal{Y}^2$ belongs to $\eta(\mathcal{N})$ if and only if for every $j, m = 1, \ldots, k$, the following identities are satisfied:

$$\begin{aligned} \Psi_j(\cdot, \alpha_j) &= \Psi_m(\cdot, \alpha_m),\ \Psi_j(\cdot, \beta_j) = \Psi_m(\cdot, \beta_m) \text{ on } \Omega_j' \cap \Omega_m', \\ \Psi_m\left(\Psi_m\left(\cdot, \alpha_m\right), \beta_m\right) &= \Psi_m\left(\Psi_m\left(\cdot, \beta_m\right), \alpha_m\right) = \mathsf{Id} \quad \text{ near } p_m, \\ \alpha_j &= j_{p_j}^\ell\left(\Psi_j(\cdot, \alpha_j)\right),\ \beta_m = j_{p_m}^\ell\left(\Psi_m(\cdot, \beta_m)\right). \end{aligned}$$

From this, it is clear that $\eta(\mathcal{N})$ is a real-analytic subset of $\mathcal{Y}^2$, and again following [Z97], we see that the group law is indeed real-analytic. This concludes the proof of Corollary 1.3. □

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