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ARTIN'S APPROXIMATION THEOREMS AND CAUCHY-RIEMANN GEOMETRY*

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Abstract. Artin's approximation theorems are powerful tools in analytic and algebraic geometry for finding solutions of systems of analytic or algebraic equations whenever a given formal solution exists. In this survey article we describe the recent developments involving the use of Artin's approximation theorems in some problems arising from Cauchy-Riemann geometry. The solution to such problems simultaneously lead to a number of results that can be stated as PDE versions of Artin's approximation theorems. The article is intended to a non-expert audience. A number of examples and open problems are also mentioned.

Key words. Holomorphic map, formal map, algebraic map, CR manifold, Artin approximation.

AMS subject classifications. 32H02, 32V05, 32V20, 32V40, 32C05, 32C07, 14P05, 14P15, 14P20.

1. Introduction.

Artin's approximation theorems [A68, A69] are powerful tools in analytic and algebraic geometry for finding solutions of systems of analytic or algebraic equations whenever a given formal solution exists. Our goal in this survey article is to describe recent developments in which Artin's approximation theorems are used in problems arising from Cauchy-Riemann geometry. The solution to such problems simultaneously lead to PDE versions of Artin's approximation theorems.

The questions discussed in this article originate from the local equivalence problem for real submanifolds in complex manifolds. Suppose that $M, M' \subset \mathbb{C}^N$ are germs (at distinguished points p and p') of real submanifolds, $N \geq 2$. One fundamental question in several complex variables is to understand when such germs are *(bi)holomorphically equivalent*, i.e. when there is a germ of a biholomorphic map $f\colon (\mathbb{C}^N, p) \to (\mathbb{C}^N, p')$ such that $f(M) = M'$ (in the sense of germs). If this is the case, we write $(M, p) \sim (M', p')$. When studying this local equivalence problem, several notions of equivalence naturally appear and it becomes of interest to compare these various notions each other. The notions of equivalence we will be interested in this article are the notions of *formal equivalence*, *algebraic equivalence* and *k-equivalence* for all $k \in \mathbb{Z}_+$, in addition to that of *holomorphic equivalence* just mentioned above. In Section 2, we describe the known relations between these notions for real-analytic and real-algebraic submanifolds. We emphasize on the class of CR submanifolds for which a number of results are known, but for which a number of open questions still remain. These results all use, in a crucial way, Artin's approximation theorems or more elaborate versions of them such as Wavrik's or Popescu's theorems [W75, P86]. Though these comparison questions make sense only for pairs of germs of real submanifolds that are of the same dimension, we show in Section 3 how these questions may be suitably generalized to mappings between real submanifolds in complex space of (a priori) different dimension. This leads us to introduce the notions of *Artin approximation property* and *Nash-Artin approximation property* for real-analytic and real-algebraic submanifolds in complex space. The very few results on these approximation proper-

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ties are discussed for CR manifolds in Section 3 where some open problems are also mentioned. In the last part of this article (Section 4), we show, mostly to the non-expert reader, how the results of Section 3 can explicitly be restated as PDE versions of Artin's approximation theorems.

The reader should observe that the present article does not aim at offering any overview on the developments related to Artin's approximation theorem in algebraic and analytic geometry. For this, we rather refer to the recent survey by Hauser and Rond [HR13] and the references therein. The reader should also note that this paper does not aim at describing all topics in CR geometry where Artin's approximation theorem plays an important role. One example of such a topic that will be left over in this paper is the convergence problem for formal CR mappings for which the reader is referred e.g. to the survey [MMZ03b]. Finally, we would like to mention that Artin's approximation theorem seems to have been applied for the first time to the mapping problems in CR geometry in 1984 by Derridj [D84] in his work on the reflection principle in several complex variables.

2. Comparison of various notions of equivalence for real-analytic submanifolds. Throughout the paper, we shall use the following standard notation: if $\mathbb{K}$ is a field that is either $\mathbb{R}$ or $\mathbb{C}$, we denote by $\mathbb{K}\{x_1,\dots,x_r\}$ and $\mathbb{K}[[x_1,\dots,x_r]]$ the ring of convergent power series and formal power series in r variables.

2.1. Formal, k-equivalence and first results. Let $M, M' \subset \mathbb{C}^N$ be germs at distinguished points p and p' of real-analytic submanifolds. Recall that the germs (M,p) and (M',p') are said to be holomorphically equivalent, i.e. $(M,p)\sim(M',p')$, if there is a germ of a biholomorphic map $f\colon(\mathbb{C}^N,p)\to(\mathbb{C}^{N'},p')$ sending M into M' (in the sense of germs). When $N=1$, it is well known that any germ of a real-analytic curve is holomorphically equivalent to a piece of the real line. Hence, in the rest of this article we will always assume that $N\geq 2$.

For real-analytic submanifolds, we shall define two further notions of equivalence that will be compared to that of holomorphic equivalence. To put things in order, some preliminary terminology must be explained. If k is a fixed nonnegative integer and (M,p) and (M',p') are as above and $f\colon(\mathbb{C}^N,p)\to(\mathbb{C}^N,p')$ is a germ of a holomorphic map, we say that f sends M into M' up to order k if for some (and hence every) local vector-valued real-analytic defining function ρ' for M' near p', $(\rho'\circ f)\big|_M$ vanishes at least up to order k at p.

Definition 2.1. Let $k\in\mathbb{Z}_+$. We say that the germs (M,p) and (M',p') are *k-equivalent* and write $(M,p)\sim_k(M',p')$ if there exists a germ of a biholomorphic map $f\colon(\mathbb{C}^N,p)\to(\mathbb{C}^N,p')$ sending M into M' up to order k.

We need also to define the notion of formal holomorphic map. A formal holomorphic map $h\colon(\mathbb{C}^N_Z,p)\to(\mathbb{C}^{N'},p')$ is an N'-tuple of formal power series in $(Z-p)$ satisfying $h(p)=p'$. We say that *h maps M into M'* if for every germ of a real-analytic function ρ' at $p'\in\mathbb{C}^{N'}$ vanishing on M', and for some (and hence every) local real-analytic parametrization $\psi\colon(\mathbb{R}^d_x,0)\to(M',p')$, $\rho'\circ h\circ\psi$ vanishes identically as a vector valued formal power series in x. (Note that this definition makes sense even when M' is any real-analytic subset of $\mathbb{C}^{N'}$.) If $N=N'$, the mapping h is called invertible if $\det\left(\frac{\partial h}{\partial Z}\right)(p)\neq 0$.

Definition 2.2. We say that the germs (M,p) and (M',p') are *formally equivalent* and write $(M,p)\sim_{\mathcal{F}}(M',p')$ if there exists a germ of a formal holomorphic invertible map $h\colon(\mathbb{C}^N,p)\to(\mathbb{C}^N,p')$ sending M into M'.

It is obvious that $(M,p) \sim (M',p') \Rightarrow (M,p) \sim_{\mathcal{F}} (M',p') \Rightarrow (M,p) \sim_k (M',p')$ for every $k \in \mathbb{Z}_+$. The converse of this second implication is less obvious and was proved recently by Zaitsev [Z11].

Theorem 2.3 (Zaitsev [Z11]). *If (M,p) and (M',p') are two germs of real-analytic submanifolds that are k-equivalent for every integer k, then they are formally equivalent.*

Zaitsev's result is in fact more general than what is stated here. It holds for germs of real-analytic sets and even families of real-analytic sets (in an appropriate sense). The proof of Theorem 2.3 uses some arguments from semi-algebraic and algebraic geometry and reduces the problem to a stabilization property of decreasing sequences of algebraic subgroups of some jet groups (see [Z11] for more details).

Considering now the implication $(M,p) \sim_{\mathcal{F}} (M',p') \Rightarrow (M,p) \sim (M',p')$, we first note that it obviously holds for pairs M, M' of *complex* submanifolds (since they can be biholomorphically straightened to $\mathbb{C}^q \times \{0\}$ where $q = \dim_{\mathbb{C}} M = \dim_{\mathbb{C}} M'$). The same conclusion also holds for *totally real* submanifolds (since they can be biholomorphically straightened to $\mathbb{R}^d \times \{0\}$ where $d = \dim_{\mathbb{R}} M = \dim_{\mathbb{R}} M'$, see [BER99] or Section 2.2 for the definition). However, this implication does not hold in general for real submanifolds. Indeed, Moser and Webster [MW83] showed that there are 2-dimensional real-analytic surfaces in $\mathbb{C}^2$ that are formally equivalent but not biholomorphically equivalent. More precisely, the following is proved in [MW83]:

Theorem 2.4 (Moser-Webster [MW83]). *There exists a real-analytic surface in $M \subset \mathbb{C}^2_{z,w}$ of the form*

$$w = |z|^2 + \gamma \bar{z}^2 + \gamma z^3 \bar{z},$$

for some $\gamma > 2$, such that $(M,0)$ is formally equivalent to the germ at 0 of some real-analytic surface $M' \subset \mathbb{C} \times \mathbb{R}$ but such that $(M,0)$ and $(M',0)$ are not holomorphically equivalent.

The existence of other examples of 2-dimensional real surfaces in $\mathbb{C}^2$ that are formally but not biholomorphically equivalent was later established by Gong [Go04]. These examples are of the same nature as those given by Theorem 2.4 in the sense that they are 2 dimensional real surfaces presenting a so-called CR singularity at the origin (see the next paragraph for further explanations). Such surfaces are called *Bishop surfaces* and the equivalence problem for such surfaces has been studied by a number of authors. We refer the reader to the surveys [H04, HY10] for more detailed account on this topic. We shall now focus on real submanifolds for which CR singularities do not appear.

2.2. CR submanifolds. Let $M \subset \mathbb{C}^N$ be a (real-analytic) submanifold and $J\colon \mathbb{C}^N \to \mathbb{C}^N$ the complex structure map. For every point $p \in M$, define the *complex tangent space* $T_p^c M := T_p M \cap J(T_p M)$. We say that M is a *Cauchy-Riemann (or CR) submanifold* of $\mathbb{C}^N$ if $\dim_{\mathbb{R}} T_p^c M$ is independent of the point p. Complex submanifolds are obviously CR. Real submanifolds for which the complex tangent space is zero-dimensional at each of their points are special CR submanifolds called *totally real* submanifolds. The prototype of such submanifolds is given by $\mathbb{R}^k \times \{0\} \subset \mathbb{C}^N$. The first important class of examples of CR submanifolds (that are neither complex nor totally real) is given by all real hypersurfaces in $\mathbb{C}^N$. Examples of real submanifolds that are not CR must therefore be taken from submanifolds of higher codimension. Bishop surfaces fall into this category: these are two dimensional real surfaces in $\mathbb{C}^2$

for which $T_p^c M = \{0\}$ for all points $p \in M$ except at one point $p_0 \in M$ for which one has $\dim_{\mathbb{R}} T_{p_0}^c M = 2$ (a so-called complex tangent at p_0). For more details about CR manifolds, we refer the reader to the books [B91, BER99, BCH08].

Coming back to our question of deciding when formal equivalence of two germs of real-analytic submanifolds implies their biholomorphic equivalence, we have the following positive result due to Baouendi, Rothschild and Zaitsev.

Theorem 2.5 (Baouendi, Rothschild, Zaitsev [BRZ01a]). *Let $M \subset \mathbb{C}^N$ be a real-analytic CR submanifold. Then there exists a closed proper real-analytic subvariety $V \subset M$ such that for every $p \in M \setminus V$ and for every real-analytic CR submanifold $M' \subset \mathbb{C}^N$ and every $p' \in M'$, if $(M,p) \sim_{\mathcal{F}} (M',p')$ then $(M,p) \sim (M',p')$.*

Let us discuss briefly the explicit construction of the subvariety V in Theorem 2.5. For this, we need to introduce some basic geometric objects associated to CR manifolds. If $M \subset \mathbb{C}^N$ is a CR submanifold, we denote by T^cM the complex tangent bundle whose fiber at some point $p \in M$ is the complex tangent space T_p^cM as defined above. For every such p, denote by $T_p^{0,1}M$ the complex subspace of $\mathbb{C}T_pM$ consisting of those vectors of the form $v + iJ(v)$ for $v \in T_p^cM$. Then $T^{0,1}M := \cup_{p\in M} T_p^{0,1}M$ forms a subbundle of $\mathbb{C}TM$ that is called the *CR bundle* of M. Sections of $T^{0,1}M$ are called *CR vector fields* of M.

For every point $p \in M$, we denote by $\mathcal{G}_M(p)$ the Lie algebra evaluated at p generated by the (local) sections of $T^{0,1}M$ and $T^{1,0}M := \overline{T^{0,1}M}$. By a theorem of Nagano (see e.g. [BER99, BCH08]), for every point $p \in M$, there is a well-defined unique germ at p of a real-analytic submanifold $\mathcal{O}_p$ satisfying $\mathbb{C}T_q\mathcal{O}_p = \mathcal{G}_M(q)$ for all $q \in \mathcal{O}_p$. This unique submanifold is necessarily CR and is called the *CR orbit* of M at p. It is not difficult to see that on any connected component of M, the dimension of the CR orbits is constant except possibly on a proper real-analytic subvariety of this component (see e.g. [BRZ01a]). We shall say that M is of *constant orbit dimension* if on every connected component of M, all CR orbits of M have the same dimension. If $\dim \mathcal{O}_p = \dim M$ for some point $p \in M$, we say that M is *minimal at* p.

The real-analytic subvariety V in Theorem 2.5 is the union of two closed proper real-analytic subvarieties V_1 and V_2 of M. The first subvariety V_1 is defined as the set of points $p \in M$ such that $M \ni q \mapsto \dim \mathcal{O}_q$ is not locally constant at p. (This coincides with the set of points of M where the foliation by CR orbits is not singular). The complement in M of the second subvariety V_2 is defined as the set of points $p \in M$ for which there exist an integer $\ell \in \{0, \ldots, N-1\}$ and a germ at 0 of a "finitely nondegenerate" real-analytic CR submanifold $\widehat{M} \subset \mathbb{C}^{N-\ell}$ such that $(M,p) \sim (\widehat{M} \times \mathbb{C}^\ell, 0)$. Finite nondegeneracy is a nondegeneracy condition (generalizing the well known Levi-nondegeneracy condition) that we will not recall in this article and we refer e.g. to [BER99, BRZ01a] for the definition.

Let us now illustrate how Theorem 2.5 applies in some elementary examples.

Example 2.6. If M, M' are Levi-nondegenerate real-analytic hypersurfaces in $\mathbb{C}^N$, we have $V = \emptyset$. Hence for all points $(p,p') \in M \times M'$, it holds that $(M,p) \sim_{\mathcal{F}} (M',p') \Rightarrow (M,p) \sim (M',p')$. In fact, in the present situation one has the following stronger conclusion: any formal holomorphic equivalence sending (M,p) into (M',p') is necessarily convergent and hence a local biholomorphic map (see [CM74, BMR02]).

Example 2.7. Suppose that $M = M_1 \times \mathbb{R}^k \times \mathbb{C}^\ell$ and $M = M_1' \times \mathbb{R}^k \times \mathbb{C}^\ell$ where M_1, M_1' are strongly pseudoconvex real-analytic hypersurfaces in $\mathbb{C}^q$. Here $k, \ell \geq 0$ and $q \geq 2$. So, in this case one has $V = \emptyset$. But, in contrast to Example 2.6, if

$(p,p') \in M \times M'$, a formal equivalence between (M,p) and (M',p') need not be convergent. Indeed, if $(M,p) \sim_{\mathcal{F}} (M',p')$, it is not difficult to construct divergent formal equivalences between (M,p) and (M',p'). Theorem 2.5 then tells that even if (M,p) and (M',p') are formally equivalent (via a divergent formal map), then (M,p) and (M',p') are necessarily biholomorphically equivalent.

Example 2.8. Suppose that $M = \{(z,w) \in \mathbb{C}^2 : |z|^2 + |w|^{2r} = 1\}$ with $r \geq 2$. Then M is a real-analytic strongly pseudoconvex hypersurface outside the submanifold $S^1 \times \{0\}$, where S^1 is the unit circle in the complex plane and $V = S^1 \times \{0\}$. Hence if $p \in M \setminus (S^1 \times \{0\})$ and if (M',p') is any germ of any real-analytic hypersurface in $\mathbb{C}^2$ such that $(M,p) \sim_{\mathcal{F}} (M',p')$, Theorem 2.5 yields that $(M,p) \sim (M',p')$. We will observe later that the conclusion also holds for all points $p \in V$ (in this example) by applying another result (Theorem 2.9 below).

One could ask whether it is really necessary to exclude from the conclusion of Theorem 2.5 all points p belonging to the subvariety V. In fact, up to now, and in contrast with the situation of surfaces with CR singularities previously discussed in Section 2.1, there is no known pairs of germs of real-analytic CR submanifolds for which formal equivalence does not imply biholomorphic equivalence. One may therefore formulate the conjecture stating that the conclusion given by Theorem 2.5 should hold with $V = \emptyset$, i.e.

Conjecture A. *If $M, M' \subset \mathbb{C}^N$ are real-analytic CR submanifolds and if $(p,p') \in M \times M'$, then $(M,p) \sim_{\mathcal{F}} (M',p')$ implies that $(M,p) \sim (M',p')$.*

This conjecture is also supported by the following result by Baouendi, Rothschild and the author [BMR02].

Theorem 2.9 (Baouendi, Mir, Rothschild [BMR02]). *Let $M \subset \mathbb{C}^N$ be a real-analytic CR submanifold that is everywhere minimal. Then for every real-analytic CR submanifold $M' \subset \mathbb{C}^N$ and for every $p \in M$ and $p' \in M'$, $(M,p) \sim_{\mathcal{F}} (M',p')$ implies that $(M,p) \sim (M',p')$.*

A few words putting Theorems 2.5 and 2.9 into proper perspective should be added here. While Theorem 2.9 does not provide, in contrast to Theorem 2.5, any conclusion for nowhere minimal CR submanifolds (as e.g. in Example 2.7), it does provide a solution to the above conjecture for everywhere minimal CR submanifolds that can not be derived from Theorem 2.5. Indeed, for such submanifolds M, note that the associated subvariety V_1 is empty and hence the subvariety V (given by Theorem 2.5) consists only of the subvariety V_2 that is in general not empty (see e.g. below).

Example 2.10. Suppose that M, M' are arbitrary compact real-analytic hypersurfaces in $\mathbb{C}^N$. Then it can be shown (see e.g. [BER99]) that they are everywhere minimal. Hence, Theorem 2.9 applies to this situation. Note that in this case the subvariety V associated to M is in general not empty (as shown by considering for instance Example 2.8).

Example 2.11. Suppose that M is a germ of a real-analytic hypersurface through the origin in $\mathbb{C}^N$ given by

$$M = \{(z,w) \in (\mathbb{C}^N, 0) : \operatorname{Im} w = \varphi(z, \bar{z}, \operatorname{Re} w)\},$$

where φ is a real-analytic function vanishing at least to order one at the origin. Then M is minimal at 0 if and only if $\varphi(z,\bar{z},0) \not\equiv 0$ (see e.g. [BER99]). Hence if $\varphi(z,\bar{z},0) \not\equiv 0$, then M is minimal in a sufficiently small neighborhood Ω of 0 in M and for every point $p \in \Omega$ and for any germ of a real-analytic hypersurface (M',p') in $\mathbb{C}^N$, one has $(M,p) \sim_{\mathcal{F}} (M',p') \Rightarrow (M,p) \sim (M',p')$. On the other hand if $\varphi(z,\bar{z},0) \equiv 0$ but if $\varphi \not\equiv 0$, the locus of non minimal points in M is given by the complex hypersurface $\{w=0\}$ (see e.g. [BER99]). Hence, in this situation, Theorem 2.9 gives that for every $p=(z_p,w_p) \in M$ such that $w_p \neq 0$, $(M,p) \sim_{\mathcal{F}} (M',p') \Rightarrow (M,p) \sim (M',p')$.

Example 2.11 shows that Conjecture A is still open for real hypersurfaces. Indeed, if M is an arbitrary (connected) real-analytic hypersurface in $\mathbb{C}^N$ then one of the following three situations holds:

(1) M is Levi-flat (i.e. its Levi form vanishes identically);
(2) M is everywhere minimal;
(3) there exists a complex hypersurface $S \subset M$ such that M is minimal at every point of $M \setminus S$.

Fix a germ (M',p') of a real-analytic hypersurface. If M satisfies (1) and $p \in M$, then M is locally biholomorphically equivalent to a real hyperplane near p. It is not difficult to see that if $(M,p) \sim_{\mathcal{F}} (M',p')$ then the same property also holds for (M',p') (see e.g. [BER99]). It therefore holds that $(M,p) \sim (M',p')$ in this case. If M satisfies (2), then Theorem 2.9 provides that $(M,p) \sim_{\mathcal{F}} (M',p') \Rightarrow (M,p) \sim (M',p')$. If M satisfies (3), then Theorem 2.9 provides that $(M,p) \sim_{\mathcal{F}} (M',p') \Rightarrow (M,p) \sim (M',p')$ if $p \notin S$. Hence, to establish Conjecture A for real-analytic hypersurfaces, it remains to deal with all non-minimal points $p \in S$ in situation (3).

Let us say a few words about some of the arguments involved in the proofs of Theorems 2.5 and 2.9. In both cases, one starts with a given formal invertible holomorphic map $f\colon (\mathbb{C}^N,p) \to (\mathbb{C}^N,p')$ sending M into M', with M,M',p,p' satisfying the appropriate assumptions of the theorems. Under such assumptions, the reader should note that the map f might very well be divergent to start with. Indeed, such divergent formal equivalences always exist e.g. whenever M is *holomorphically degenerate* in the sense of Stanton [St96], meaning that there is a germ of a nontrivial holomorphic vector field tangent to M near p (see e.g. [BER97]). From that map f, one has to find a local biholomorphic map $\widetilde{f}\colon (\mathbb{C}^N,p) \to (\mathbb{C}^N,p')$ sending M into M'. In fact, the proof of Theorems 2.5 and 2.9 does not only provide the existence of such a map $\widetilde{f}$ but provides the much stronger conclusion that f is the limit, in the Krull topology, of germs at p of biholomorphic maps: there exists a sequence of germs of biholomorphic maps $f_j\colon (\mathbb{C}^N,p) \to (\mathbb{C}^N,p')$ sending M into M', $j \in \mathbb{Z}_+$, such that the Taylor series of f_j at p coincides with that of f up to order j. Such a conclusion is obtained by applying at some stages of the proof Artin's approximation theorem [A68] or, possibly, a very useful variation of such a result due to Wavrik [W75]. Let us recall the statement of this fundamental result since it is the main motivation of this survey.

Theorem 2.12 (Artin [A68]). *Let $S(x,y) = (S_i(x,y))_{i\in I}$, $S_i(x,y) \in \mathbb{K}\{x,y\}$, be a family of convergent power series, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, $x \in \mathbb{K}^n$, $y \in \mathbb{K}^p$. Suppose that there exist a formal solution $y(x) \in (\mathbb{K}[[x]])^p$, vanishing at the origin, of the system of equations*

$$S(x,y(x)) = 0. \tag{2.1}$$

Then for any nonnegative integer k, there exists a convergent solution of the system (2.1), for which the Taylor series agrees with that of $y(x)$ up to order k.

The above mentioned approximation property obtained in Theorems 2.5 and 2.9 enables one to formulate a more general problem related to Conjecture A. This more general approximation problem is valid for arbitrary formal mappings between complex spaces of different dimension and will be discussed in detail in Section 3.1.

2.3. Algebraic equivalence of real-algebraic submanifolds. Recall that a real-analytic submanifold M of $\mathbb{C}^N$ is called *real-algebraic* if it is contained in a real-algebraic set of the same dimension as M. Recall also that if $\mathbb{K} = \mathbb{R}, \mathbb{C}$, an analytic function $\Omega \subset \mathbb{K}^n \to \mathbb{K}$ defined on an open set of $\mathbb{K}^n$ is called $\mathbb{K}$-*algebraic* (or *algebraic over* $\mathbb{K}$) if it satisfies a nontrivial polynomial identity with polynomial coefficients (over $\mathbb{K}$). When $\mathbb{K} = \mathbb{R}$, we call such a function *real-algebraic* and $\mathbb{K} = \mathbb{C}$, we call it *complex-algebraic (or algebraic holomorphic).* The same terminology applies in a straightforward way to mappings valued in $\mathbb{R}^k$ or $\mathbb{C}^k$.

Suppose that $M, M' \subset \mathbb{C}^N$ are real-algebraic submanifolds and $(p, p') \in M \times M'$. Another useful notion of equivalence between the germs (M, p) and (M', p') is that of *algebraic (or Nash) equivalence* that is defined as follows.

Definition 2.13. We say that the germs (M, p) and (M', p') are *algebraically equivalent* and write $(M, p) \sim_{\mathcal{A}} (M', p')$ if there exists a germ of an algebraic biholomorphic map $h\colon (\mathbb{C}^N, p) \to (\mathbb{C}^{N'}, p')$ sending M into M'.

Obviously, one has the implication $(M, p) \sim_{\mathcal{A}} (M', p') \Rightarrow (M, p) \sim (M', p')$. The main question is to decide when the converse does hold. The converse does clearly hold when M, M' are complex-algebraic or totally real-algebraic submanifolds of $\mathbb{C}^N$ (since M and M' can be complex-algebraically strenghtened in these cases). In fact, it is still an open problem to decide whether this is also true for arbitrary real-algebraic submanifolds, i.e.

Question B. *If $M, M' \subset \mathbb{C}^N$ are real-algebraic submanifolds and if $(p, p') \in M \times M'$, does it hold that $(M, p) \sim (M', p') \Rightarrow (M, p) \sim_{\mathcal{A}} (M', p')$?*

One should note that, in contrast to the situation discussed in Section 2.1, there is no known pair of real-algebraic submanifolds for which the answer to Question B is negative. A number of positive results for CR submanifolds though exist regarding such a question. The first result is somewhat analogous to Theorem 2.5 and provides a positive answer at all points in Zariski open subset of an arbitrary real-algebraic CR submanifold. The precise statement is as follows.

Theorem 2.14 (Baouendi, Rothschild, Zaitsev [BRZ01b]). *Let $M \subset \mathbb{C}^N$ be a real-algebraic CR submanifold. Then there exists a closed proper real-algebraic subvariety $\widetilde{V} \subset M$ such that for every $p \in M \setminus \widetilde{V}$ and for every real-algebraic CR submanifold $M' \subset \mathbb{C}^N$ and every $p' \in M'$, if $(M, p) \sim (M', p')$ then $(M, p) \sim_{\mathcal{A}} (M', p')$.*

For a real-algebraic CR submanifold $M \subset \mathbb{C}^N$, the associated subvariety $\widetilde{V}$ in Theorem 2.14 is the same as the subvariety V obtained by applying Theorem 2.5 to M. Lamel and the author [LM10] improved later Theorem 2.14 by exhibiting another real-algebraic subvariety $V' \subset \widetilde{V} \subset M$, that is in general strictly contained in the subvariety $\widetilde{V}$, and for which the conclusion of Theorem 2.14 still holds with $\widetilde{V}$ replaced by V'. This subvariety V' is however in general strictly larger than the subvariety V_1 consisting of the set of points of $p \in M$ where the dimension of the CR orbits is not constant in any neighborhood of p (as defined in Section 2.2). The conclusion of Theorem 2.14 with $\widetilde{V}$ replaced by V_1 has been established more recently by the author in [Mir12]. The result can be stated as follows:

THEOREM 2.15 ([Mir12]). *Let $M, M' \subset \mathbb{C}^N$ be two real-algebraic CR submanifolds of constant orbit dimension. Then for all points $p \in M$ and $p' \in M'$, $(M,p) \sim (M',p')$ implies that $(M,p) \sim_{\mathcal{A}} (M',p')$.*

Theorem 2.15 is a consequence of a more general result proved in [Mir12] that is developed in detail later in this article in Section 3.2. The following elementary example shows that V_1 is in general strictly contained in $\widetilde{V}$.

EXAMPLE 2.16. Consider the real-algebraic CR submanifold in $\mathbb{C}^3_{z,w_1,w_2}$ given by

$$M := \begin{cases} \operatorname{Im} w_1 = |z|^4 \\ \operatorname{Im} w_2 = 0 \end{cases}$$

Then one may easily check that M is of constant orbit dimension and hence $V_1 = \emptyset$ here. On the other hand, one can show that $\widetilde{V}$ is the two dimensional real plane given by $=\{(0,s,t) : (s,t) \in \mathbb{R}^2\}$.

We would like to point out that if M, M', p, p' satisfy the assumptions given by Theorems 2.14, 2.15 or 2.17, a germ of a local holomorphic map $h\colon (\mathbb{C}^N,p) \to (\mathbb{C}^N,p')$ sending M into M' need not be algebraic. In fact, additional geometric assumptions on M and M' are needed in order to force any such mapping h to be algebraic. Necessary and sufficient conditions on M and M' so that this rigidity property holds have been found by Baouendi, Ebenfelt and Rothschild [BER96].

Hence, up to now, the largest Zariski open subset on arbitrary real-algebraic CR submanifolds for which one knows that local holomorphic equivalence implies algebraic equivalence is the complement of the set of points where the dimension of CR orbits jump. On the other hand, in the specific situation where these submanifolds are generically minimal, one has the following stronger conclusion.

THEOREM 2.17 (Baouendi, Mir, Rothschild [BMR02]). *Let $M \subset \mathbb{C}^N$ be a real-algebraic CR submanifold that is generically minimal. Then for every real-analytic CR submanifold $M' \subset \mathbb{C}^N$ and for every $p \in M$ and $p' \in M'$, $(M,p) \sim (M',p')$ implies that $(M,p) \sim_{\mathcal{A}} (M',p')$.*

By generically minimal, we mean that M contains a dense open subset of minimal points. This result can be seen as an algebraic version of Theorem 2.9 but with a weaker geometric condition than everywhere minimality. In fact, if M is connected, it is not difficult to see that if M is minimal at some point, then it is generically minimal (and in fact, minimal except a proper real-algebraic subvariety of M, see e.g. [BER99]). The weakening of the minimality assumption in Theorem 2.17 (compared to Theorem 2.9) allows one to deal with one important situation where CR orbits need not all have the same dimension. This also allows us to deduce the following solution to Question B for all one codimensional real-algebraic submanifolds.

COROLLARY 2.18 (Baouendi, Mir, Rothschild [BMR02]). *If $M, M' \subset \mathbb{C}^N$ are real-algebraic hypersurfaces and if $(p,p') \in M \times M'$, then $(M,p) \sim (M',p')$ implies that $(M,p) \sim_{\mathcal{A}} (M',p')$.*

Corollary 2.18 is a straightforward consequence of Theorem 2.17. Indeed it follows from the trichotomy mentioned after Example 2.11 describing all possible situations that a connected real-analytic hypersurface of $\mathbb{C}^N$ might have regarding its set of minimal points.

The proofs of Theorems 2.14 and 2.17 provide a more general conclusion than what is mentioned in the theorems. One indeed gets that any given germ of a biholomorphic map $f\colon (\mathbb{C}^N, p) \to (\mathbb{C}^N, p')$ sending M into M', with the appropriate assumptions on M, M', p, p', is the limit in the Krull topology of a sequence $(f_j)_{j\in\mathbb{Z}_+}$ of germs at p of algebraic biholomorphisms sending M into M'. This kind of approximation property is established by making use of the approximation theorem for algebraic systems established by Artin in 1969, which can be considered as the algebraic analog of Theorem 2.12. Its statement is as follows.

Theorem 2.19. (Artin [A69]) *Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$, and let $P(x,y) = (P_1(x,y), \ldots, P_m(x,y))$ be m polynomials in the ring $\mathbb{K}[x,y]$ with $y = (y_1, \ldots, y_q)$. Suppose that $Y(x) = (Y_1(x), \ldots, Y_q(x)) \in (\mathbb{K}[[x]])^q$ is a formal solution of the system*

$$P(x, Y(x)) = 0. \tag{2.2}$$

Then, for every integer ℓ, there exists $Y^\ell(x) \in (\mathbb{K}\{x\})^q$, that is algebraic over $\mathbb{K}$, satisfying the system (2.2) *such that $Y^\ell(x)$ agrees with $Y(x)$ up to order ℓ.*

The reader should note that the conclusion of Theorem 2.19 still holds if one assumes that the polynomial P is merely a polynomial mapping in y with coefficients that are convergent power series in x and algebraic over $\mathbb{K}$.

The approximation property obtained in the proofs of Theorems 2.14 and 2.17 leads us to generalize Question B to asking when germs of holomorphic maps sending real-algebraic submanifolds embedded in complex spaces of different dimension into each other can be approximated in the Krull topology by complex-algebraic maps. This will be discussed in Section 3.2.

3. The Artin and Nash-Artin approximation property.

3.1. The Artin approximation property for real-analytic submanifolds.

Let $M \subset \mathbb{C}^N$ be a real-analytic submanifold and $\Sigma' \subset \mathbb{C}^{N+N'}$ be a real-analytic set, $N, N' \geq 1$. If $p \in M$ and $h\colon (\mathbb{C}^N, p) \to \mathbb{C}^{N'}$ is a formal holomorphic map, we say that the graph of h along M is contained in Σ' and write $\Gamma_h \cap (M \times \mathbb{C}^{N'}) \subset \Sigma'$ to mean that the formal holomorphic map $\widetilde{h}(z) := (z, h(z))$ sends M into Σ'. Note that if h is local holomorphic map, this definition coincides with the standard meaning involving the graph of h.

Definition 3.1. Let M, Σ' be as above and $p \in M$. We say that the pair (M, Σ') has the *Artin approximation property* at p if for every formal holomorphic map $h\colon (\mathbb{C}^N, p) \to \mathbb{C}^{N'}$ satisfying $\Gamma_h \cap (M \times \mathbb{C}^{N'}) \subset \Sigma'$, there exists a sequence of germs of holomorphic maps $h_j\colon (\mathbb{C}^N, p) \to \mathbb{C}^{N'}$ satisfying $\Gamma_{h_j} \cap (M \times \mathbb{C}^{N'}) \subset \Sigma'$, $j \in \mathbb{Z}_+$, with the property that h_j agrees with h up to order j at p. We also say that

(i) (M, Σ') has the Artin approximation property if it has the Artin approximation property at every point $p \in M$.

(ii) M has the Artin approximation property at p if (M, Σ') has the Artin approximation property at p for every real-analytic set $\Sigma' \subset \mathbb{C}^{N'}$ and for every positive integer N'.

(iii) M has the Artin approximation property if (M, Σ') has the Artin approximation property for every real-analytic set $\Sigma' \subset \mathbb{C}^{N'}$ and for every positive integer N'.

Note that if M has the Artin approximation property at some point $p \in M$, then given a germ (M', p') of a real-analytic submanifold in $\mathbb{C}^N$ one has $(M, p) \sim_{\mathcal{F}}$

$(M',p') \Rightarrow (M,p) \sim (M',p')$. Hence deciding which real-analytic submanifolds have the Artin approximation property immediately provides a solution to the equivalence problem discussed in Section 2.1. In what follows, we write AA property for Artin approximation property.

Let us first observe that any real-analytic submanifold $M \subset \mathbb{C}$ has the AA property. Indeed, if M is germ of a real-analytic curve in the complex plane, then it can biholomorphically mapped to a piece of the real line. Hence one is reduced to consider the case where $M = \mathbb{R}$. The fact that $\mathbb{R}$ has the AA property at each of its points follows immediately from applying Artin's approximation theorem given by Theorem 2.12 (in the real setting) and from a complexification argument. From now on, we shall therefore assume that $N \geq 2$.

The first class of real-analytic submanifolds having the AA property consists of the totally real real-analytic submanifolds. This is a direct consequence of Theorem 2.12. Indeed, if M is a totally real real-analytic submanifold in $\mathbb{C}^N$, then for any given $p \in M$, M can be locally biholomorphically mapped near p to $\mathbb{R}^k \times \{0\}$. Hence we may assume without loss of generality that $M = \mathbb{R}^k \times \{0\} \subset \mathbb{C}^k_z \times \mathbb{C}^{N-k}_w$ and that $p = 0$. Fix a real-analytic set $\Sigma' \subset \mathbb{C}^N_z \times \mathbb{C}^{N'}_{z'}$ and a formal holomorphic map $h\colon (\mathbb{C}^N, p) \to \mathbb{C}^{N'}$ satisfying $\Gamma_h \cap (M \times \mathbb{C}^{N'}) \subset \Sigma'$. We may assume that Σ' is given near $h(0)$ by the vanishing of a vector-valued real-analytic map $\rho'(z,\bar z, z', \bar z')$. For all $x \in \mathbb{R}^k$ (sufficiently close to the origin), we have $\rho'(x, h(x,0), x, \overline{h(x,0)}) = 0$. Hence, by Theorem 2.12, there exists a sequence $(g_j)\colon (\mathbb{R}^k, 0) \to \mathbb{R}^{2N'} \simeq \mathbb{C}^{N'}$ of germs of real-analytic mappings satisfying the system of equations fulfilled by $h(x,0)$ and such that $g_j(x)$ and $h(x,0)$ agree at 0 up to order j. For all $j \in \mathbb{Z}_+$, we complexify the map g_j to get a germ of a holomorphic map from $(\mathbb{C}^k_z, 0)$ to $\mathbb{C}^{N'}$ and keep the same notation for the complexified map. Denote also by $\theta_j = \theta_j(z,w)$ the Taylor polynomial of order j of the mapping $h(z,w) - h(z,0)$. Then one can check that if we set for every $j \in \mathbb{Z}_+$, $h_j(z,w) = \theta_j(z,w) + g_j(z)$, the sequence (h_j) satisfies the required conclusion of the AA property at 0.

Another class of real-analytic submanifolds for which the AA property is known to hold is that of complex submanifolds. In this case, this property is a rather direct application of the following (Cauchy-Riemann) version of Artin's approximation theorem proved by Milman:

Theorem 3.2 (Milman [Mil78]). *Let $S(x,y) = (S_1(x,y), \ldots, S_q(x,y))$ be q convergent power series in the ring $\mathbb{R}\{x,y\}$ with $x = (x_1, \ldots, x_{2r})$, $y = (y_1, \ldots, y_{2m})$. Suppose that $Y(x) \in (\mathbb{R}[[x]])^{2m}$ vanishes at the origin and is a solution of the system*

$$S(x, Y(x)) = 0, \quad \bar\partial(Y_{2k-1} + iY_{2k}) = 0, \ 1 \leq k \leq m, \tag{3.1}$$

where $\bar\partial$ is the Cauchy-Riemann operator in $\mathbb{R}^{2r} \simeq \mathbb{C}^r$. Then, for every integer ℓ, there exists a convergent map $Y^\ell(x)$ satisfying the system (3.1) *such that $Y^\ell(x)$ agrees with $Y(x)$ up to order ℓ.*

Indeed, if $M \subset \mathbb{C}^N$ is a complex submanifold, it can be locally biholomorphically straightened to $\mathbb{C}^k \times \{0\}$. Then a direct application of Theorem 3.2 together with an adaptation of the last argument used for the case of totally real submanifolds shows that M has the AA property.

Besides the cases of totally real and complex submanifolds, there is, to the author's knowledge, no other known classes of real-analytic submanifolds having the AA property. The question seems for instance open for real-analytic strongly pseudoconvex hypersurfaces or even for the real unit sphere. On the other hand, if we are interested in pairs (M, Σ') satisfying the AA property, one has the following result:

THEOREM 3.3 (Meylan, Mir, Zaitsev [MMZ03a]). *Let $M \subset \mathbb{C}^N$ be a real-analytic CR submanifold that is everywhere minimal and Σ' be a real-algebraic subset of $\mathbb{C}^N \times \mathbb{C}^{N'}$. Then (M, Σ') has the Artin approximation property.*

Though Theorem 3.3 is stated in [MMZ03a] only for real-algebraic sets Σ' of the form $M \times M'$ where M' is a real-algebraic subset in $\mathbb{C}^{N'}$, the proof given in [MMZ03a] applies also to the case where Σ' is an arbitrary real-algebraic subset of $\mathbb{C}^{N+N'}$. It is still an open problem to determine whether the conclusion of Theorem 3.3 holds for arbitrary real-analytic sets Σ' (instead of real-algebraic ones).

Let us describe the main ideas of the proof of Theorem 3.3 to see how Theorem 2.12 is used. Let us therefore fix M, Σ' with the corresponding assumptions. Fix also $p \in M$, which we assume to be the origin and $h\colon (\mathbb{C}^N, 0) \to \mathbb{C}^{N'}$ a formal holomorphic map whose graph along M is contained in $\Sigma' \subset \mathbb{C}^N_z \times \mathbb{C}^{N'}_{z'}$. The proof is divided in three steps:

(1) Consider the *Zariski closure* of Γ_h with respect to the ring $\mathbb{C}\{z\}[z']$. It is defined as the germ $\mathcal{Z}_h \subset \mathbb{C}^N \times \mathbb{C}^{N'}$ at $(0, h(0))$ of a complex-analytic set defined as the zero-set of all elements in $\mathbb{C}\{z\}[z']$ vanishing on Γ_h.

(2) Apply Theorem 2.12 to the system of holomorphic equations defining $\mathcal{Z}_h$ and to the formal solution h satisfying this system. One then gets a sequence of germs of holomorphic maps $(h_j)\colon (\mathbb{C}^N, 0) \to \mathbb{C}^{N'}$ such that $h_j - h$ vanishes up to order (at least) j at 0 for all $j \in \mathbb{Z}_+$.

(3) Show that for j large enough, one necessarily has $\Gamma_{h_j} \cap (M \times \mathbb{C}^{N'}) \subset \Sigma'$.

The Zariski closure $\mathcal{Z}_h$ is a useful tool that measures the lack of convergence of the mapping h. Indeed, one may show (see [MMZ03a]) that h is convergent if and only if $\mathcal{Z}_h$ is N-dimensional. Under the assumptions of Theorem 3.3, this Zariski closure need not be N-dimensional, but it is if one adds some additional geometric conditions on Σ' (see [MMZ03a] for more on this issue).

As one can see steps, (1) and (2) are easy to implement. The heart of the proof boils down to proving step (3). It relies on the following proposition [MMZ03a, Proposition 4.3]:

PROPOSITION 3.4. *Let $M \subset \mathbb{C}^N$ be a generic real-analytic submanifold through the origin and assume that M is minimal at 0. Let $g\colon (\mathbb{C}^N, 0) \to \mathbb{C}^k_w$ be a formal holomorphic map with $N, k \geq 1$. Suppose that there exists $\mathcal{Q}(z, \bar{z}; w, \bar{w}) \in \mathbb{C}\{z, \bar{z}\}[w, \bar{w}]$ such that $\mathcal{Q}(z, \bar{z}; w, \bar{w})|_{M \times \mathbb{C}^k} \not\equiv 0$ and such that $\mathcal{Q}(z, \bar{z}; g(z), \overline{g(z)}) = 0$ for $z \in M$ close to 0. Then there exists a nontrivial holomorphic function $\mathcal{P}(z, w) \in \mathbb{C}\{z\}[w]$ such that $\mathcal{P}(z, g(z)) = 0$ for $z \in \mathbb{C}^N$ close to 0.*

We should mention here that a CR submanifold $M \subset \mathbb{C}^N$ is called *generic* if for every point $p \in M$, one has $T_pM + J(T_pM) = T_p\mathbb{C}^N$.

Proposition 3.4 is the key result for establishing step (3) above and may also be used to show that the Zariski closure defined in step (1) is not the whole space $\mathbb{C}^N \times \mathbb{C}^{N'}$ whenever Σ' is any proper real-algebraic subset of $\mathbb{C}^N \times \mathbb{C}^{N'}$. The main ingredients involved in the proof of Proposition 3.4 are the theory of Segre sets developed by Baouendi, Ebenfelt and Rothschild [BER96] for generic minimal submanifolds and a repeated use of, again, Artin's approximation theorem given by Theorem 2.12.

Theorem 3.3 deals with the AA property for CR submanifolds. For such submanifolds, one may formulate a variant of the AA property for $\mathcal{C}^\infty$-smooth CR mappings as follows. Recall that given a real-analytic CR submanifold $M \subset \mathbb{C}^N$, a $\mathcal{C}^\infty$-smooth $H\colon M \to \mathbb{C}^{N'}$ is called CR if each component of H is annihilated by all CR vector fields of M. We denote in what follows by Γ_H the graph of H. Fix M, H as above and a real-analytic set $\Sigma' \subset \mathbb{C}^{N'}$ such that $\Gamma_H \subset \Sigma'$. Following Definition 3.1, we say that the triple (M, Σ', H) has the *CR Artin approximation property (CR-AA property for short)* at some point $p \in M$ if for all $j \in \mathbb{Z}_+$, there exists a germ of a holomorphic map $H_j \colon (\mathbb{C}^N, p) \to \mathbb{C}^{N'}$ such that $\Gamma_{H_j|_M} \subset \Sigma'$ and such that $H - H_j|_M$ vanishes up to order j at p.

One can easily show that if (M, Σ') has the AA property at some point $p \in M$, then for any $\mathcal{C}^\infty$-smooth CR mapping H, the triple (M, Σ', H) has the CR-AA property at p. Not much is known about triples having the CR-AA property. The only existing result valid for arbitrary real-analytic sets Σ' was proved by Sunyé [Su10] and is given as follows.

Theorem 3.5 (Sunyé [Su10]). *Let $M \subset \mathbb{C}^N$ be a connected real-analytic CR submanifold that is somewhere minimal, Σ' a real-analytic subset of $\mathbb{C}^N \times \mathbb{C}^{N'}$ and $H \colon M \to \mathbb{C}^{N'}$ a $\mathcal{C}^\infty$-smooth CR mapping such that $\Gamma_H \subset \Sigma'$. Then there exists a dense open subset Ω of M such that (M, Σ', H) has the CR Artin approximation property at each point of Ω.*

The reader might note that Theorem 3.5 is not stated for arbitrary real-analytic sets in [Su10] but for real-analytic sets of the form $M \times M'$ where M' is some real-analytic set in $\mathbb{C}^{N'}$. The proof given in [Su10] applies nevertheless also in the more general situation described in Theorem 3.5.

3.2. The Nash-Artin approximation property for real-algebraic submanifolds. We shall now discuss an algebraic version of the Artin approximation property for real-algebraic submanifolds in complex space.

Definition 3.6. Let $M \subset \mathbb{C}^N$ be a real-algebraic submanifold, $\Sigma' \subset \mathbb{C}^N \times \mathbb{C}^{N'}$ a real-algebraic set and $p \in M$. We say that the pair (M, Σ') has the *Nash-Artin approximation property at p* if for every germ of a holomorphic map $h \colon (\mathbb{C}^N, p) \to \mathbb{C}^{N'}$ satisfying $\Gamma_h \cap (M \times \mathbb{C}^{N'}) \subset \Sigma'$, there exists a sequence of germs of algebraic holomorphic maps $h_j \colon (\mathbb{C}^N, p) \to \mathbb{C}^{N'}$ satisfying $\Gamma_{h_j} \cap (M \times \mathbb{C}^{N'}) \subset \Sigma'$, $j \in \mathbb{Z}_+$, with the property that h_j agrees with h up to order j at p. We also say that:

- (i) (M, Σ') has the Nash-Artin approximation property if it has the Nash-Artin approximation property at every point $p \in M$.
- (ii) M has the Nash-Artin approximation property at p if (M, Σ') has the Nash-Artin approximation property at p for every real-algebraic set $\Sigma' \subset \mathbb{C}^{N'}$ and for every positive integer N'.
- (iii) M has the Nash-Artin approximation property if (M, Σ') has the Nash-Artin approximation property for every real-algebraic set $\Sigma' \subset \mathbb{C}^{N'}$ and for every positive integer N'.

Note again here that if M has the Nash-Artin approximation property at some point $p \in M$, then given a germ (M', p') of a real-algebraic submanifold in $\mathbb{C}^N$, the holomorphic equivalence of the germs (M, p) and (M', p') implies their algebraic equivalence.

As in Section 3.1, we can show that every real-algebraic curve in the complex plane has the Nash-Artin approximation property (NAA property for short). Similarly, one may also check that complex-algebraic submanifolds and totally real real-algebraic submanifolds in $\mathbb{C}^N$ have the NAA property.

On the other hand, and contrarily to the situation discussed in the previous section, there are other classes of real-algebraic submanifolds for which the NAA property is known to hold. These classes concern only CR manifolds. The first general class of CR submanifolds for which this property is known to hold is that of generically minimal submanifolds i.e.

Theorem 3.7 (Meylan, Mir, Zaitsev [MMZ03b]). *Let $M \subset \mathbb{C}^N$ be a real-algebraic CR submanifold that is generically minimal. Then M has the Nash-Artin approximation property.*

The strategy of the proof of Theorem 3.7 is quite analogous to that of the proof of Theorem 3.3. Indeed suppose that M, Σ' are as in Theorem 3.7 and $p = 0 \in M$. Assume also that $h\colon (\mathbb{C}^N, 0) \to \mathbb{C}^{N'}$ a is a germ of holomorphic map whose graph along M is contained in $\Sigma' \subset \mathbb{C}^N_z \times \mathbb{C}^{N'}_{z'}$. One first considers the *Zariski closure* of Γ_h with respect to the ring $\mathbb{C}[z, z']$, which is defined as the smallest complex-algebraic variety $\mathcal{V}_h \subset \mathbb{C}^N \times \mathbb{C}^{N'}$ containing Γ_h. This Zariski closure measures the lack of algebraicity of the mapping h. Indeed, such a map need not be algebraic to start with, unless some additional assumptions on Σ' are added (for this see [CMS99, Z99]). One then applies Theorem 2.19 (in the complex setting) to the system of complex-algebraic equations defining $\mathcal{V}_h$ and to the holomorphic solution h satisfying this system. This gives a sequence of germs of algebraic holomorphic maps $(h_j)\colon (\mathbb{C}^N, 0) \to \mathbb{C}^{N'}$ such that $h_j - h$ vanishes up to order (at least) j at 0 for all $j \in \mathbb{Z}_+$. The last step consists of proving that for j large enough, one necessarily has $\Gamma_{h_j} \cap (M \times \mathbb{C}^{N'}) \subset \Sigma'$. The argument given in [MMZ03b] to prove this last step is again based on the Segre sets theory of minimal generic submanifolds and the use of Theorem 2.19. We should mention here that the weakening of the minimality assumption on M in Theorem 3.7 compared to its analogue Theorem 3.3 is due to the fact that being algebraic for a holomorphic function is a non-local property; this allows one to move to points nearby 0, which cannot be done while dealing with formal power series mappings.

Theorem 3.7 provides a complete solution to our problem for an important class of real-algebraic submanifolds but does not provide any result for nowhere minimal submanifolds. We shall now describe a recent result of the author providing a sufficient condition ensuring that a real-algebraic CR submanifold has the NAA property. Such a condition has the advantage to be satisfied generically on an arbitrary real-algebraic CR submanifold.

Let $M \subset \mathbb{C}^N$ be a real-algebraic CR submanifold. Recall that M is of constant orbit dimension if on every connected component of M, all CR orbits of M have the same dimension (see Section 2.2). We have the following:

Theorem 3.8 ([Mir12]). *Let $M \subset \mathbb{C}^N$ be a real-algebraic CR submanifold of constant orbit dimension. Then M has the Nash-Artin approximation property.*

As mentioned above, the constant orbit dimension assumption is a geometric condition that is generically satisfied. More precisely, as observed in Section 2.2, on every connected component $\widehat{M}$ of an arbitrary real-algebraic CR submanifold $M \subset \mathbb{C}^N$, there exists a closed proper real-algebraic subvariety $\widehat{V}$ such that the orbits of M have the same dimension on $\widehat{M} \setminus \widehat{V}$. Hence, Theorem 3.8 immediately implies the following:

Theorem 3.9 ([Mir12]). *For every real-algebraic CR submanifold $M \subset \mathbb{C}^N$, there exists a closed proper real-algebraic subvariety Σ_M of M such that $M \setminus \Sigma_M$ has the Nash-Artin approximation property.*

One noteworthy consequence of Theorems 3.8 and 3.7 provides a complete solution of the Nash-Artin approximation problem for real-algebraic hypersurfaces. Indeed, any connected component of an real-algebraic hypersurface in $\mathbb{C}^N$ must be either generically minimal, in which case Theorem 3.7 applies, or must be Levi-flat and therefore of constant orbit dimension in which case Theorem 3.8 applies. Hence, one has:

Theorem 3.10 ([Mir12]). *Any real-algebraic hypersurface of* $\mathbb{C}^N$ *has the Nash-Artin approximation property.*

The proof of Theorem 3.8 is a version of the proof of Theorem 3.7 with parameters. Indeed, suppose that M, Σ' are as in Theorem 3.8 and $p = 0 \in M$. Assume also that $h\colon (\mathbb{C}^N, 0) \to \mathbb{C}^{N'}$ is a germ of holomorphic map whose graph along M is contained in Σ'. Then by e.g. [BER96], one may view the germ of M at 0 as a (small) algebraic deformation of its CR orbits, in the sense that, there exists an integer $c \in \{0, \dots, N\}$ and a real-algebraic submersion $\pi\colon (M, 0) \to (\mathbb{R}^c, 0)$ such that $\pi^{-1}(\pi(q)) = \mathcal{O}_q$ for all $q \in M$ near 0. The level sets of π are the CR orbits and foliate M near the origin by *minimal* real-algebraic CR submanifolds. We may then identify the germ of M at 0 with an algebraic deformation (M_t) for $t \in \mathbb{R}^c$ close to 0, where $M_t \subset \mathbb{C}^{N-c}$ is a germ at 0 of a *minimal* real-algebraic CR submanifold. Writing $\mathbb{C}^N = \mathbb{C}^{N-c}_z \times \mathbb{C}^c_u$, then for every $t \in \mathbb{R}^c$ sufficiently small, the holomorphic map $h_t\colon (\mathbb{C}^{N-c}_z, 0) \ni z \mapsto h(z,t) \in \mathbb{C}^{N'}$ satisfies $\operatorname{Graph} h_t \cap (M_t \times \mathbb{C}^{N'}) \subset \Sigma'_t$ where $\Sigma'_t := \{(z, z') \in \mathbb{C}^{N-c} \times \mathbb{C}^{N'} : (z, t, z') \in \Sigma'\}$. Since each submanifold M_t is minimal, the conclusion of Theorem 3.8 boils down to providing a deformation version of Theorem 3.7 associated to the deformation $(M_t)_{t \in \mathbb{R}^c}$, the analytic family of holomorphic maps $(f_t)_{t \in \mathbb{R}^c}$ and the family of real-algebraic subsets $(S'_t)_{t \in \mathbb{R}^c}$.

There are several new ingredients in order to achieve the proof of Theorem 3.8. We will mention only two of them. The first one is the use of Artin's approximation theorem [A69] on a ground field that is different from the usual fields of real and complex numbers. Indeed, one needs to apply such a result to some field extension over $\mathbb{C}$ defined by some ratios of formal power series (see [Mir12] for more details). The other important ingredient is the use of a version of Artin's approximation theorem (for so-called nested subrings) proved by Popescu [P86]. We shall state a version of Popescu's theorem that is sufficient for the purpose of the proof of Theorem 3.8.

Theorem 3.11 (Popescu [P86]). *Let* $\mathbb{K} = \mathbb{R}, \mathbb{C}$, $P(x,y) = (P_1(x,y), \dots, P_m(x,y))$ *be a polynomial mapping with components in the ring* $\mathbb{K}[x,y]$, *with* $x = (x_1, \dots, x_r)$, $y = (y_1, \dots, y_q)$. *Suppose that* $Y(x) = (Y_1(x), \dots, Y_q(x)) \in (\mathbb{K}[[x]])^q$ *is a formal solution of the system*

$$P(x, Y(x)) = 0. \tag{3.2}$$

Suppose furthermore that for every $j = 1, \dots, q$, *there exists* $s_j \in \{1, \dots, r\}$ *such that* $Y_j(x) \in \mathbb{K}[[x_1, \dots, x_{s_j}]]$. *Then, for every integer* ℓ, *there exists* $Y^\ell(x) \in (\mathbb{K}\{x\})^q$, *that is algebraic over* $\mathbb{K}$, *satisfying the system* (3.2), *such that* $Y^\ell(x)$ *agrees with* $Y(x)$ *up to order* ℓ *and each* $Y^\ell_j(x) \in \mathbb{K}\{x_1, \dots, x_{s_j}\}$, $j = 1, \dots, q$.

At this point, we should mention that the use of Theorem 3.11 in the equivalence and approximation problem in CR geometry appeared for the first time in [LM10]. One should also note that Theorem 3.11 is specific to the algebraic category. Indeed, the corresponding result for *analytic* systems of equations does not hold in general (see [Ga71]).

In view of Theorems 3.7 and 3.8, we are lead to formulate the following conjecture regarding the NAA property:

Conjecture C. *Any real-algebraic CR submanifold of* $\mathbb{C}^N$ *has the Nash-Artin approximation property.*

Summarizing the above described results, we see that the last step that remains to be proved towards the solution to Conjecture C is to prove that a connected real-algebraic CR submanifold in $\mathbb{C}^N$ has the NAA property at each point where the CR orbit is not of maximal dimension.

3.3. Strong approximation property. In this last paragraph, we indicate one other possible approximation property that may be of interest for real-analytic submanifolds in complex space. This is motivated by the following "strong" approximation theorem proved by Wavrik [W75].

THEOREM 3.12 (Wavrik [W75]). *Let* $S(x,y) = (S_i(x,y))_{i\in I}$*,* $S_i(x,y) \in \mathbb{K}[[x,y]]$*, be a family of formal power series, where* $\mathbb{K} = \mathbb{R}$ *or* $\mathbb{C}$*,* $x \in \mathbb{K}^n$*,* $y \in \mathbb{K}^p$*. Then there exists a map* $\beta\colon \mathbb{Z}_+ \to \mathbb{Z}_+$ *with the following property: given a positive integer* ℓ *and* $y(x) \in (\mathbb{K}[[x]])^p$*,* $y(0) = 0$*, such that* $S(x, y(x))$ *vanishes at* 0 *up to order* $\beta(\ell)$*, there exists* $\widehat{y}(x) \in (\mathbb{K}[[x]])^p$*,* $\widehat{y}(0) = 0$*, for which* $S(x,\widehat{y}(x)) = 0$ *and such that the Taylor series at* 0 *of* $\widehat{y}(x)$ *agrees with that of* $y(x)$ *up to order* ℓ*.*

Let M, Σ' be, for simplicity, two real-analytic submanifolds in $\mathbb{C}^N$ and $\mathbb{C}^N \times \mathbb{C}^{N'}$ respectively. Let $p \in M$, ℓ a positive integer and $h\colon (\mathbb{C}^N_z, p) \to \mathbb{C}^{N'}_{z'}$ be a formal holomorphic map with $(p, h(p)) \in \Sigma'$. We shall say that the graph of h along M *has order of contact at least* ℓ *with* Σ' *at* p if for one (and hence every) real-analytic vector-valued defining function $\rho' = \rho'(z,\bar{z},z',\bar{z}')$ for Σ' near $(p, h(p))$, the restriction to M of the formal power series mapping $\rho'(z,\bar{z},h(z),\overline{h(z)})$ vanishes at least up to order ℓ at p.

We say that (M, Σ') has the *strong Artin approximation property* at p, and write SAA property at p for short, if there exists a map $\beta\colon \mathbb{Z}_+ \to \mathbb{Z}_+$ satisfying the following: given a positive integer ℓ and $h\colon (\mathbb{C}^N, p) \to \mathbb{C}^{N'}$ a formal holomorphic map whose graph along M has order of contact at least $\beta(\ell)$ with Σ' at p, there exists a formal holomorphic map $h_\ell\colon (\mathbb{C}^N, p) \to \mathbb{C}^{N'}$ whose graph along M is contained in Σ' and such that h_ℓ and h agree at p up to order ℓ.

Note that in the previous definition, we may assume, without loss of generality, that the mapping h is already holomorphic (and in fact polynomial) to start with. Observe also that one may formulate the above definition assuming that M and Σ' are merely "formal real" manifolds instead of real-analytic ones (see e.g. [BMR02]).

As in previous sections, it is easy to see that Theorem 3.12 imply that (M, Σ') has the SAA property at every point of a totally real real-analytic submanifold M of $\mathbb{C}^N$ and every real-analytic submanifold Σ' of $\mathbb{C}^{N+N'}$. The same conclusion also holds when M is any complex submanifold of $\mathbb{C}^N$ due to a recent result of Hickel and Rond [HR12] providing a strong approximation version of Theorem 3.2. Besides these cases, the author does not know whether the other classes of real-analytic submanifolds discussed in Section 3.1 and having the AA property also have the SAA property.

4. Reformulation as PDE versions of Artin's approximation theorems. In this last part of this article, we want to describe, especially to the non-expert reader, the results of the previous sections as PDE versions of Artin's approximation theorems.

4.1. Systems of complex vector fields and analytic systems. The first PDE version of Theorem 2.12 is Milman's result given by Theorem 3.2. In addition to the system of analytic equations that a given formal power series mapping must satisfy, one adds in Theorem 3.2 the PDE given by the homogeneous (standard) Cauchy-Riemann operator in some complex euclidean space. Milman's setting is therefore a special case of the following situation: let $S(x,y)=(S_1(x,y),\dots,S_q(x,y))$ be q convergent power series in the ring $\mathbb{R}\{x,y\}$ with $x=(x_1,\dots,x_s)$, $y=(y_1,\dots,y_{2m})$. Let $L_1,\dots,L_n$ be a family of complex vector fields, defined in a neighbhorhood of $0\in\Omega\subset\mathbb{R}^s$, with real-analytic coefficients i.e. of the form

$$L_j=\sum_{\nu=1}^{s}a_{j\nu}(x)\cdot\frac{\partial}{\partial x_\nu},\ j=1,\dots,n$$

with $a_{j\nu}$ being complex-valued real-analytic functions in Ω. Suppose that $Y(x)\in(\mathbb{R}[[x]])^{2m}$ vanishes at the origin and is a solution of the system

$$S(x,Y(x))=0,\quad L_j(Y_{2k-1}+iY_{2k})=0,\ 1\leq k\leq m,\ 1\leq j\leq n. \tag{4.1}$$

The question is then to decide whether, for every integer ℓ, there exists a convergent map $Y^\ell(x)$ satisfying the system (4.1) such that $Y^\ell(x)$ agrees with $Y(x)$ up to order ℓ (at 0).

It is clear that Milman's theorem corresponds to the situation where s is even, i.e. $s=2r$ and

$$L_j=\frac{\partial}{\partial x_{2j-1}}+i\frac{\partial}{\partial x_{2j}},\quad j=1,\dots,r.$$

We need the following definition.

Definition 4.1. Let $L_1,\dots,L_n$ be a family of n complex vector fields defined in a neighborhood of 0 in $\mathbb{R}^s$. We say that $(L_1,\dots,L_n)$ has the *Artin approximation property* (AA property for short) if for every positive integer q and every convergent power series mapping S as above, any formal solution Y of the system (4.1) is the limit, in the Krull topology, of a sequence of convergent solutions of (4.1).

The AA property for systems of complex vector fields and the AA property for real-analytic CR submanifolds are directly linked as follows.

Let $M\subset\mathbb{C}^N$ be a real-analytic CR submanifold of dimension s of CR dimension n and $p\in M$. It is a standard fact (see e.g. [BER99]) that we may choose a real-analytic parametrization $\varphi\colon(\mathbb{R}^s_x,0)\to\mathbb{R}^{2N}$ of M near p and a family of complex vector fields $L_1,\dots,L_n$ defined in a neighborhood of 0 in $\mathbb{R}^s$, with real-analytic coefficients, such that $\varphi_*(L_1),\dots,\varphi_*(L_n)$ forms a basis of the local sections of $T^{0,1}M$ near p. Standard arguments about CR manifolds (that may be found in [BER99]) provide the following correspondence:

Proposition 4.2. *With the above notation, M has the AA property at p if and only if $(L_1,\dots,L_n)$ has the AA property.*

In view of Proposotion 4.2, one may formulate Theorem 3.3 in a form that looks closer to the statement of Artin's or Milman's approximation theorem. Let us make the following additional definitions:

Definition 4.3. Let $L_1,\dots,L_n$ be a system of n complex vector fields, with real-analytic coefficients, defined in a connected neighborhood Ω of 0 in $\mathbb{R}^s$.

(i) We say that $(L_1,\dots,L_n)$ is *analytically admissible* if there exists a local real-analytic immersion $\varphi\colon\mathbb{R}^s\supset\Omega\supset\omega\to\mathbb{C}^N$ such that $M:=\varphi(\omega)$ is a real-analytic CR submanifold of $\mathbb{C}^N$ with $\varphi_*(L_1),\dots,\varphi_*(L_n)$ being a local basis of the sections of $T^{0,1}M$ near $\varphi(0)$.

(ii) We say that $(L_1, \dots, L_n)$ is of *finite type at (a point)* $x_0 \in \Omega$ if the Lie algebra generated by $L_1, \dots, L_n$ and by $\overline{L}_1, \dots, \overline{L}_n$ is of maximal dimension at x_0 i.e. if its dimension is equal to s.

(iii) We say that $(L_1, \dots, L_n)$ is of *constant orbit dimension at (a point)* $x_0 \in \Omega$ if the dimension of the Lie algebra generated by $L_1, \dots, L_n$ and by $\overline{L}_1, \dots, \overline{L}_n$ is the same at all points in some sufficiently small neighborhood of x_0.

REMARK 4.4. Necessary and sufficient conditions are known for a system of complex vector fields to be analytically admissible. If $(L_1, \dots, L_n)$ is a system of n complex vector fields, with real-analytic coefficients, defined in a connected neighborhood Ω of 0 in $\mathbb{R}^s$, then $(L_1, \dots, L_n)$ is analytically admissible if and only if $L_1, \dots, L_n$ are $\mathbb{C}$-linearly independent in a sufficiently small neighborhood $V \subset \Omega$ of 0 and the bundle $\mathcal{V}$ over V formed by the span of $(L_1, \dots, L_n)$ satisfies $[\mathcal{V}, \mathcal{V}] \subset \mathcal{V}$ (integrability condition) and $\mathcal{V} \cap \overline{\mathcal{V}} = \{0\}$ (see e.g. [BER99, BCH08]).

We can now reformulate Theorem 3.3 as follows:

THEOREM 4.5. *Let $P(x, y) = (P_1(x, y), \dots, P_q(x, y))$ be q polynomials in the ring $\mathbb{R}[x, y]$ with $x = (x_1, \dots, x_s)$, $y = (y_1, \dots, y_{2m})$. Let $L_1, \dots, L_n$ be an analytically admissible system of complex vector fields defined in a neighborhood of 0 in $\mathbb{R}^s$ and assume that $(L_1, \dots, L_n)$ is of finite type at 0. Suppose that $Y(x) \in (\mathbb{R}[[x]])^q$ is a solution of the system*

$$P(x, Y(x)) = 0,\ L_j(Y_{2k-1} + iY_{2k}) = 0,\ 1 \le k \le m,\ 1 \le j \le n. \tag{4.2}$$

Then, for every integer ℓ, there exists $Y^\ell(x) \in (\mathbb{R}\{x\})^q$ satisfying the system (4.2) such that $Y^\ell(x)$ agrees with $Y(x)$ (at 0) up to order ℓ.

If we consider the case where case $s = 2r$ and $L_j = \dfrac{\partial}{\partial \bar{z}_j}$, $j = 1, \dots, r$, in $\mathbb{C}^r \simeq \mathbb{R}^s$, then such a system automatically fulfills the required conditions of the theorem. In this case, Theorem 4.5 is a special case of Milman's theorem (Theorem 3.2) since it holds for polynomial systems of equations instead of analytic ones. Let us mention, on the other hand, the following class of examples (different from the standard Cauchy-Riemann setting) where Theorem 4.5 applies.

EXAMPLE 4.6. Let $\varphi \colon \mathbb{R}^{2n+1} \supset \Omega \to \mathbb{R}$ be a real-analytic function defined in some neighborhood Ω of 0 such that $\varphi(0) = d\varphi(0) = 0$. Identifying $\mathbb{R}^{2n+1}$ with $\mathbb{C}^n \times \mathbb{R}$, we use $(z, \bar{z}, u)$ as coordinates in $\mathbb{R}^{2n+1}$ where $z = (z_1, \dots, z_n) \in \mathbb{C}^n$, $u \in \mathbb{R}$ and $z = x + iy$ with $x, y \in \mathbb{R}^n$. We set

$$\begin{aligned} L_j &= \frac{\partial}{\partial \bar{z}_j} - i \frac{\varphi_{\bar{z}_j}(z', \bar{z}', u)}{1 + i\varphi_u(z', \bar{z}', u)} \frac{\partial}{\partial u}, \\ &= \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right) - i \frac{\varphi_{\bar{z}_j}(z', \bar{z}', u)}{1 + i\varphi_u(z', \bar{z}', u)} \frac{\partial}{\partial u},\ 1 \le j \le n. \end{aligned} \tag{4.3}$$

Then this system of complex vectors fields is analytically admissible and the finite type assumption at 0 is equivalent to say that the real-analytic function $\varphi(z, \bar{z}, 0)$ does not vanish identically near 0 (see e.g. [BER99]). Under this condition, Theorem 4.5 therefore applies.

4.2. Algebraic systems. For systems of polynomial equations, notice first that Popescu's theorem (Theorem 3.11) can be seen as a PDE version of Artin's approximation theorem given by Theorem 2.19. Let us now describe the reformulations of some of the results of Section 3.2 in terms of Artin type approximation results for systems of complex vector fields.

So, let $P(x,y)=(P_1(x,y),\dots,P_q(x,y))$ be q polynomials in the ring $\mathbb{R}[x,y]$ with $x=(x_1,\dots,x_s)$, $y=(y_1,\dots,y_{2m})$. Let $L_1,\dots,L_n$ be a family of complex vector fields, defined in a neighbhorhood of $0\in\Omega\subset\in\mathbb{R}^s$, with real-analytic coefficients in Ω i.e. of the form

$$L_j=\sum_{\nu=1}^{s} a_{j\nu}(x)\cdot\frac{\partial}{\partial x_\nu},\ j=1,\dots,n$$

with $a_{j\nu}$ being a complex-valued real-analytic functions. Suppose that $Y(x)\in(\mathbb{R}\{x\})^{2m}$ is a solution of the system

$$P(x,Y(x))=0,\quad L_j(Y_{2k-1}+iY_{2k})=0,\ 1\le k\le m,\ 1\le j\le n. \tag{4.4}$$

The question here is to decide whether, for every integer ℓ, there exists a germ at 0 of a real-algebraic map $Y^\ell(x)$ satisfying the system (4.4) such that $Y^\ell(x)$ agrees with $Y(x)$ up to order ℓ at 0.

Analogously to what has been done in the previous section, we need to define the following.

Definition 4.7. Let $L_1,\dots,L_n$ be a family of n complex vector fields defined in a neighborhood of 0 in $\mathbb{R}^s$. We say that $(L_1,\dots,L_n)$ has the *Nash-Artin approximation property* (NAA property for short) if for every positive integer q and every polynomial map P as above, any convergent solution Y of the system (4.4) is the limit, in the Krull topology, of a sequence of germs at 0 of real-algebraic solutions of (4.4) .

Let $M\subset\mathbb{C}^N$ be a real-algebraic CR submanifold of dimension s of CR dimension n and $p\in M$. Then we may choose a real-algebraic parametrization $\varphi\colon(\mathbb{R}^s_x,0)\to\mathbb{R}^{2N}$ of M near p and $L_1,\dots,L_n$ a family of complex vector fields defined a neighborhood of 0 in $\mathbb{R}^s$ with real-algebraic coefficients such that $\varphi_*(L_1),\dots,\varphi_*(L_n)$ forms a local basis of the sections of $T^{0,1}M$ near p. We have the following link between the Nash-Artin approximation property for real-algebraic CR submanifolds and systems of complex vector fields. It can be proved by using basic arguments about CR manifolds and their mappings (see e.g. [BER99]).

Proposition 4.8. *With the above notation, M has the NAA property at p if and only if $(L_1,\dots,L_n)$ has the NAA property.*

Proposition 4.8 naturally leads us to make the following definition.

Definition 4.9. Let $L_1,\dots,L_n$ be a system of n complex vector fields, with real-analytic coefficients, defined in a connected neighborhood Ω of 0 in $\mathbb{R}^s$. We say that $(L_1,\dots,L_n)$ is *algebraically admissible* if there exists a local real-algebraic immersion $\varphi\colon\Omega\supset\omega\subset\mathbb{R}^s\to\mathbb{C}^N$ such that $M:=\varphi(\omega)$ is a real-algebraic CR submanifold of $\mathbb{C}^N$ with $\varphi_*(L_1),\dots,\varphi_*(L_n)$ being local basis of the sections of $T^{0,1}M$ near $\varphi(0)$.

We may then reformulate Theorems 3.7 and 3.8 as follows.

Theorem 4.10. *Let $P(x,y)=(P_1(x,y),\dots,P_q(x,y))$ be q polynomials in the ring $\mathbb{R}[x,y]$ with $x=(x_1,\dots,x_s)$, $y=(y_1,\dots,y_{2m})$. Let $L_1,\dots,L_n$ be an algebraically admissible system of complex vector fields defined in a connected neighborhood Ω of 0 in $\mathbb{R}^s$. Assume that one of the following two conditions hold:*

(i) $(L_1, \ldots, L_n)$ *is of finite type at some point of* Ω.
(ii) $(L_1, \ldots, L_n)$ *is of constant orbit dimension at* 0.
Then if $Y(x) \in (\mathbb{R}\{x\})^q$ *is a solution of the system*

$$P(x, Y(x)) = 0, \ L_j(Y_{2k-1} + iY_{2k}) = 0, \ 1 \leq k \leq m, \ 1 \leq j \leq n, \tag{4.5}$$

for every integer ℓ, *there exists* $Y^\ell(x)$ *a germ at* 0 *of a real-algebraic mapping solution of the system* (4.5) *such that* $Y^\ell(x)$ *agrees with* $Y(x)$ *(at* 0*) up to order* ℓ.

The first situation where Theorem 4.10 applies is when s is even and the complex vector fields are the standard Cauchy-Riemann vector fields. Then, in this special case, Theorem 4.10 can be seen as the algebraic analog of Milman's theorem (Theorem 3.2). It may be stated as follows:

Corollary 4.11. *Let* $P(x, y) = (P_1(x, y), \ldots, P_q(x, y))$ *be* q *polynomials in the ring* $\mathbb{R}[x, y]$ *with* $x = (x_1, \ldots, x_{2r})$, $y = (y_1, \ldots, y_{2m})$. *Suppose that* $Y(x) \in (\mathbb{R}\{x\})^{2m}$ *is a solution of the system*

$$P(x, Y(x)) = 0, \quad \overline{\partial}(Y_{2k-1} + iY_{2k}) = 0, \ 1 \leq k \leq m. \tag{4.6}$$

Then, for every integer ℓ, *there exists a germ at* 0 *of a real-algebraic map* $Y^\ell(x)$ *satisfying the system* (4.6) *and such that* $Y^\ell(x)$ *agrees with* $Y(x)$ *(at* 0*) up to order* ℓ.

Let us illustrate Theorem 4.10 with other examples.

Example 4.12. Suppose that $s = 2r + k$ so that we may write $\mathbb{R}^s \simeq \mathbb{C}^r_z \times \mathbb{R}^k_t$. Consider the complex vector fields

$$L_j = \frac{\partial}{\partial \bar{z}_j}, \ 1 \leq j \leq r.$$

Then the family $(L_1, \ldots, L_r)$ is an algebraically admissible system of complex vector fields that is (everywhere) of constant orbit dimension and Theorem 4.10 applies. In this specific case, it can be stated as the following parameter version of Corollary 4.11:

Corollary 4.13. *Let* $P(x, y) = (P_1(x, y), \ldots, P_q(x, y))$ *be* q *polynomials in the ring* $\mathbb{R}[x, y]$ *with* $x = (x_1, \ldots, x_{2r})$, $y = (y_1, \ldots, y_{2m})$. *Suppose that* $\mathbb{R}^k \times \mathbb{R}^{2r} \ni (t, x) \mapsto Y_t(x)$ *is a convergent power series mapping near* 0 *such that for every fixed* $t \in \mathbb{R}^k$ *sufficiently close to* 0, $Y_t(x)$ *is a solution of* (4.6). *Then, for every integer* ℓ, *there exists a real-algebraic map* $(t, x) \mapsto Y_t^\ell(x)$ *such that the family* $(Y_t^\ell(x))_t$ *satisfies the system* (4.6) *and such that* $Y_t^\ell(x)$ *agrees with* $Y_t(x)$ *up to order* ℓ *(as powers series of* (t, x)*) at* $0 \in \mathbb{R}^{k+2r}$.

Example 4.14. Consider Example 4.6 where we now suppose that φ is a real-algebraic function defined in some connected neighborhood Ω of 0 in $\mathbb{R}^{2n+1}$. Consider the system of complex vector fields $L_1, \ldots, L_n$ given by (4.3). Such a system is algebraically admissible. Furthermore, it is of finite type at some point of Ω if and only if $\varphi \not\equiv 0$. And when $\varphi \equiv 0$, it is of constant orbit dimension (in fact it is a special case of Example 4.12). Hence Theorem 4.10 applies to any system of complex vector fields of the form (4.3) (for any real-algebraic function φ).

Let us conclude with a reformulation of Theorem 3.9.

Theorem 4.15. *Let* $L_1, \ldots, L_n$ *be an algebraically admissible system of complex vector fields defined in an open connected neighborhood* Ω *of* 0 *in* $\mathbb{R}^s$. *Then there* *exists a Zariski open subset $U \subset \Omega$ such that the following holds: for every point $x_0 \in U$ and every polynomial mapping $P(x,y) = (P_1(x,y), \ldots, P_q(x,y)) \in (\mathbb{R}[x,y])^q$ with $x = (x_1, \ldots, x_s)$, $y = (y_1, \ldots, y_{2m})$, if $Y(x) \in (\mathbb{R}\{x - x_0\})^q$ is a solution of the system*

$$P(x, Y(x)) = 0, \; L_j(Y_{2k-1} + iY_{2k}) = 0, \; 1 \le k \le m, \tag{4.7}$$

then, for every integer ℓ, there exists a germ at x_0 of a real-algebraic mapping $Y^\ell(x)$ satisfying the system (4.7) *and such that $Y^\ell(x)$ agrees with $Y(x)$ up to order ℓ at x_0.*

By a Zariski open subset of Ω, we mean here the complement of a proper real-algebraic subvariety of Ω. Note that Conjecture C states that Theorem 4.15 should hold with $U = \Omega$.

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