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On Artin approximation for formal CR mappings

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Let M be a real-analytic CR submanifold of $\mathbb{C}^N$ and S' be a real-analytic subset of $\mathbb{C}^{N+N'}$. We say that the pair (M, S') has the *Artin approximation property* if for every point $p \in M$ and every positive integer ℓ, if $H\colon (\mathbb{C}^N, p) \to \mathbb{C}^{N'}$ is a formal holomorphic map such that $\operatorname{Graph} H \cap (M \times \mathbb{C}^{N'}) \subset S'$, there exists a germ at p of a holomorphic map $h^\ell\colon (\mathbb{C}^N, p) \to \mathbb{C}^{N'}$ which agrees with H at p up to order ℓ satisfying $\operatorname{Graph} h^\ell \cap (M \times \mathbb{C}^{N'}) \subset S'$. In this paper, we give some sufficient conditions on a pair (M, S') to have the Artin approximation property. We show that if the CR orbits of M are all of the same dimension and at most of codimension one in M and if S' is any partially algebraic subset of $\mathbb{C}^N \times \mathbb{C}^{N'}$, then (M, S') has the Artin approximation property.

1. Introduction

In 1968, Artin [A68] provided a general and powerful tool in order to find solutions of systems of analytic equations : given any system of real-analytic equations, if there exists a formal solution to such a system at a given point, then there exists a real-analytic one that is as close as we want in the Krull topology to the formal one. A question that naturally thereafter arises is whether the conclusion of Artin's approximation theorem is still preserved if the system of equations is coupled with a specific PDE. In 1978, Milman [M78] investigated such a question when the PDE consists of the standard Cauchy-Riemann operator in $\mathbb{R}^{2n} \simeq \mathbb{C}^n$: he showed that any formal solution of a system of real-analytic equations and of the standard CR equations in $\mathbb{C}^n$ can be approximated (in the Krull topology) by a sequence of convergent

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solutions of the system of analytic and CR equations. Hence, Milman's result provided the first CR version of Artin's approximation theorem.

The present paper aims at investigating a similar question when the standard Cauchy-operator in $\mathbb{C}^n$ is replaced by the tangential Cauchy-Riemann operator associated to a general real-analytic CR manifold. As explained e.g. in [M14], one of the motivations for such a study comes from CR geometry itself : given two real-analytic CR manifolds and a formal nondegenerate CR map between them, does there exist a convergent nondegenerate one between them? Establishing a CR version of Milman's approximation theorem implies a corresponding affirmative answer to the above mentioned existence question in CR geometry. Furthermore, it has only been discovered recently by Kossovskiy and Shafikov [KS13] that there exist pairs of real-analytic CR manifolds that are formally but not biholomorphically equivalent. As an immediate consequence, this also shows that the conclusion of Milman's result fails to hold in general if the CR equations in $\mathbb{C}^n$ are replaced by tangential CR equations.

In this paper, we shall provide some sufficient conditions ensuring that approximation of formal solutions of analytic and CR equations by convergent ones is however possible. These conditions are related to the form of the analytic equations to be fulfilled by a formal solution and to the CR geometry of the associated CR manifold.

Our main result will be formulated as an approximation result for formal holomorphic mappings from $\mathbb{C}^N$ to $\mathbb{C}^{N'}$ rather than an approximation result for formal mappings from $\mathbb{R}^k$ to $\mathbb{C}^{N'}$ satisfying some CR equations; the correspondence between these two points of view is explained e.g. in [M14]. To this end, we need to introduce the following terminology. Suppose that M is a real-analytic CR submanifold of $\mathbb{C}^N$ and S' is a real-analytic subset of $\mathbb{C}^{N+N'}$. We say that the pair (M, S') has the *Artin approximation property* if for every point $p \in M$ and every positive integer ℓ, if $H\colon (\mathbb{C}^N, p) \to \mathbb{C}^{N'}$ is a formal holomorphic map such that $\operatorname{Graph} H \cap (M \times \mathbb{C}^{N'}) \subset S'$ (see Section 2.1 for the precise definition), there exists a germ at p of a holomorphic map $h^\ell\colon (\mathbb{C}^N, p) \to \mathbb{C}^{N'}$ which agrees with H at p up to order ℓ with $\operatorname{Graph} h^\ell \cap (M \times \mathbb{C}^{N'}) \subset S'$. We also say that a real-analytic subset $S' \subset \mathbb{C}^N_Z \times \mathbb{C}^{N'}_{Z'}$ is *partially algebraic* if it is locally given by the vanishing of finitely many functions that are real polynomials in Z' with real-analytic coefficients in Z (see [L88] for a similar notion).

Finally, recall that given any real-analytic CR submanifold $M \subset \mathbb{C}^N$ with complex tangent bundle $T^cM \subset TM$, for every point $p \in M$, there exists a unique germ of a real-analytic CR submanifold $\mathcal{O}_p$ through p, called the *CR orbit* of M at p, such that every point $q \in \mathcal{O}_p$ can be reached from

p by following a piecewise differentiable curve in M whose tangent vectors are in T^cM (see e.g. [BER99a, BCH08]).

We may now state our main result:

Theorem 1.1. *Let $M \subset \mathbb{C}^N$ be a connected real-analytic CR submanifold. Assume that the CR orbits of M are all of the same dimension and at most of codimension one in M. Then for every partially algebraic real-analytic subset $S' \subset \mathbb{C}^N \times \mathbb{C}^{N'}$, the pair (M, S') has the Artin approximation property.*

In view of the above mentioned examples of CR manifolds found in [KS13], the assumption about the CR orbits be all of the same dimension is quite natural in Theorem 1.1; indeed, these examples correspond to germs of real-analytic CR manifolds whose dimension of CR orbits does jump. In the case where all CR orbits are all open pieces of M (i.e. M is everywhere *minimal*), then Theorem 1.1 was proved by Meylan, Zaitsev and the author [MMZ03]. Hence, the main new contribution of Theorem 1.1 deals with the case of nowhere minimal real-analytic CR submanifolds whose CR orbits are all one-codimensional. It is an open problem to decide whether the conclusion of Theorem 1.1 remains valid for CR orbits of arbitrary codimension or for arbitrary real-analytic subsets S'. For other related results (in a CR or non-CR setting), we refer the reader to [M12, HY09, HY10].

Let us now describe the proof of Theorem 1.1. Given a real-analytic CR submanifold $M \subset \mathbb{C}^N$, a partially algebraic subset $S' \subset \mathbb{C}^N \times \mathbb{C}^{N'}$, a point $p \in M$ and a formal holomorphic mapping $H\colon (\mathbb{C}^N, p) \to \mathbb{C}^{N'}$ such that $\operatorname{Graph} H \cap (M \times \mathbb{C}^{N'}) \subset S'$, the induced formal map $H\big|_M$ satisfies both the CR equations on M and the system of analytic equations defining S' near $(p, H(p))$. The main step of the proof consists of reducing the desired approximation property for the original system of analytic and CR equations to a so-called "nested" approximation property for a certain system of real-analytic equations. By "nested" approximation property, we mean that the obtained system of equations admits a formal solution such that some of its components depend on fewer variables and that it is possible to approximate such a formal solution by convergent ones whose components satisfy the same property. We would like to point out that this step of the proof is valid without any assumption on the codimension of the CR orbits. The "nested" approximation property for systems of polynomial or analytic equations has been quite well studied, see e.g. [R15] for a complete account on this topic. While such a property is known to be true for polynomial systems (see [P86]), this is not the case for general systems of real-analytic equations as an example due to Gabrielov [Ga71] shows. However, when the so-called "nested" part of a formal solution as above does depend only on one variable, Denef and Lipshitz [DL80] have shown that any system of analytic equations does have the "nested" approximation property. In our context, it happens that when CR orbits of M are of codimension one, one may use the result by Denef-Lipshitz, yielding the desired final result.

The paper is organized as follows. Section 2 is devoted to the proof of a key result, Proposition 2.4, which can be seen as an algebraic dependence propagation property for formal maps. Such a property has been proved in some different and simpler situations in [MMZ03, Proposition 4.3] and [M12, Proposition 3.7]. Our proof in this more general setting differs and is simpler from the ones given in [MMZ03, M12] by avoiding the use of "CR ratios" and of any of Artin's approximation theorems [A68, A69]. In Section 3, we set up a certain system of real-analytic equations associated to any formal holomorphic mapping and to any real-analytic generic submanifold with CR orbits of constant dimension. We then prove that the desired approximation property for the formal mapping follows from the "nested" approximation property of the constructed system (Proposition 3.3). We should mention that the arguments used in Section 3 are adapted from those developed in an algebraic context in [M12]. The proof of Theorem 1.1 is finalized in Section 4.

2. An algebraic dependence propagation property

In order to state and prove the main result of this section, Proposition 2.4 below, we need to set-up the notation used throughout this paper as well as recall some facts about generic submanifolds in complex space. For basic background about CR manifolds, we refer the reader to e.g. [BER99a, BCH08].

2.1. Notation

Throughout the paper, for $\mathbb{K} = \mathbb{R}, \mathbb{C}$, we shall denote the rings of convergent and formal power series in r variables with coefficients in $\mathbb{K}$ by $\mathbb{K}\{x\}$ and $\mathbb{K}[[x]]$ respectively, $x = (x_1, \ldots, x_r)$. For $\theta \in \mathbb{C}[[x]]$, we denote by $\bar{\theta}$ the formal power series obtained from θ by taking complex conjugates of its coefficients.

Let $\theta \in \mathbb{K}[[x]]$ and m be a nonnegative integer. We write $j^m\theta$ for the collection of all partial derivatives of θ up to order m. If we can split the indeterminates $x = (\hat{x}, \tilde{x})$, $(j^m\theta)(0, \tilde{x})$ stands for the power series mapping $j^m\theta$ evaluated at $\hat{x} = 0$. Furthermore, if X is a germ at the origin of a $\mathbb{K}$-analytic submanifold in $\mathbb{K}^r$, we say that θ vanishes on X and write $\theta\big|_X = 0$ or

$\theta(x)\big|_X = 0$ to mean that for some (and hence any) $\mathbb{K}$-analytic parametrization $\varphi\colon (\mathbb{K}^{\dim X}, 0) \to (X, 0) \subset \mathbb{K}^r$, one has $\theta \circ \varphi = 0$.

Let $\psi\colon (\mathbb{R}^r_x, 0) \to \mathbb{R}^q$ be a formal mapping. Let $X \subset \mathbb{R}^r$ be a germ at the origin of a real-analytic submanifold and let Y be a real-analytic subset of $\mathbb{R}^{r+q}$. We write $\operatorname{Graph} \psi \cap (X \times \mathbb{R}^q) \subset Y$ to mean that for every germ at $(0, \psi(0)) \in \mathbb{R}^{r+q}$ of a real-analytic function ρ vanishing on Y, $\rho(x, \psi(x))|_X = 0$. When ψ is convergent, this latter definition coincides with the usual set theoretic notion.

Finally, we set up appropriate notation for coordinates in jets spaces. Recall that for every integer k, $J^k_0(\mathbb{C}^N, \mathbb{C}^{N'})$ denotes the jet space of order k at the origin of holomorphic maps from $\mathbb{C}^N$ to $\mathbb{C}^{N'}$. Throughout the paper, we will use Λ^k and Γ^k as coordinates in $J^k_0(\mathbb{C}^N, \mathbb{C}^{N'})$ (where k is allowed to vary) and write $\Lambda^k = (\Lambda^k_\alpha)_{|\alpha| \le k}$ with $\Lambda^k_\alpha \in \mathbb{C}^{N'}$ and similarly for Γ^k.

2.2. Generic real-analytic submanifolds of constant orbit dimension

Suppose that $M \subset \mathbb{C}^N$ is a real-analytic generic submanifold and denote by $T^cM \subset TM$ its complex tangent bundle. Recall that for every point $p \in M$, there exists a unique germ of a real-analytic CR submanifold $\mathcal{O}_p$ through p such that every point $q \in \mathcal{O}_p$ can be reached from p by following a piecewise differentiable curve in M whose tangent vectors are in T^cM (see e.g. [BER99a, BCH08]). This germ is called the *CR orbit* of M at p and M is called *minimal* at p if this CR orbit is a neighborhood of p in M. It is a standard fact (see e.g. [BRZ01]) that if M is connected, the dimension of the CR orbits is maximal except possibly on some proper real-analytic subvariety of Σ_M of M.

From now on, we assume that M is a connected real-analytic generic submanifold of $\mathbb{C}^N$ of CR dimension n and codimension d. We have the following result that provides suitable holomorphic (normal) coordinates near every point $p \in M \setminus \Sigma_M$.

Lemma 2.1. *([BRZ01, Proposition 3.4]) Let $M \subset \mathbb{C}^N$ be a connected generic real-analytic submanifold through a point $p \in M$ whose CR orbit at p is of maximal dimension and let $c \in \{0, \dots, d\}$ be the codimension of this CR orbit in M. Then there exists normal holomorphic coordinates $Z = (z, \eta) \in \mathbb{C}^n \times \mathbb{C}^d$, $\eta = (w, u) \in \mathbb{C}^{d-c} \times \mathbb{C}^c$, such that M is given near the origin by an equation of the form*

$$\eta = (w, u) = \Theta(z, \bar z, \bar\eta) := (Q(z, \bar z, \bar w, \bar u), \bar u), \tag{2.1}$$

where Q is a $\mathbb{C}^{d-c}$-valued holomorphic map near $0 \in \mathbb{C}^{n+N}$. Furthermore, there exist neighborhoods U, V of the origin in $\mathbb{R}^c$ and $\mathbb{C}^{N-c}$ respectively such that for every $u \in U$, the real-analytic submanifold given by

$$M_u := \{(z,w) \in V : w = Q(z,\bar{z},\bar{w},u)\} \tag{2.2}$$

is generic in $\mathbb{C}^{N-c}$ and minimal at 0.

As a real-analytic submanifold, we may complexify M which gives rise to its so-called *complexification* $\mathcal{M}$. For a fixed point $p \in M \setminus \Sigma_M$, using the coordinates provided by Lemma 2.1, the germ of $\mathcal{M}$ at 0 is the complex submanifold of $\mathbb{C}^{2N}$ given by

$$\{(Z,\zeta) \in (\mathbb{C}^N \times \mathbb{C}^N, 0) : \sigma = \bar{\Theta}(\chi, z, \eta)\}, \tag{2.3}$$

where $Z = (z,\eta) \in \mathbb{C}^n \times \mathbb{C}^d$ and $\zeta = (\chi,\sigma) \in \mathbb{C}^n \times \mathbb{C}^d$. It is an easy and standard fact (see e.g. [BER99a]) that our choice of normal coordinates implies the following two identities:

$$\Theta(z,0,\sigma) = \Theta(0,\chi,\sigma) = \sigma, \qquad \Theta(z,\chi,\bar{\Theta}(\chi,z,\eta)) = \eta. \tag{2.4}$$

We now define the *iterated Segre mappings* attached to M near p (see e.g. [BER99b, BRZ01]). For any nonnegative integer j, we denote by t^j a variable lying in $\mathbb{C}^n$ and also introduce the variable $t^{[j]} := (t^1, \ldots, t^j) \in \mathbb{C}^{nj}$. Then we set

$$V_0(u) := (0,u) \in \mathbb{C}^N$$

for $u \in \mathbb{C}^c$ and define the map $V_j \colon (\mathbb{C}^{nj} \times \mathbb{C}^c, 0) \to \mathbb{C}^N$ for $j \geq 1$ inductively as follows:

$$\begin{aligned} &V_j(t^{[j]},u) := (t^j, U_j(t^{[j]},u)), \\ &\text{where} \quad U_j(t^{[j]},u) := \Theta(t^j, \overline{V}_{j-1}(t^{[j-1]},u)). \end{aligned} \tag{2.5}$$

From this definition, it follows that each iterated Segre mapping V_j defines a holomorphic map in a neighbhorhood of 0 in $\mathbb{C}^{nj+c}$. Note furthermore that for every point $(0,u) \in \mathbb{C}^d$ sufficiently close to the origin, the map $V_j(\cdot,u)$ parametrizes the Segre set of order j attached to the point $(0,u)$ (see e.g. [BER96, BER99a]). Observe that, thanks to (2.4), one has the following useful identities

$$V_j(0,u)=(0,u),\quad V_{j+2}(t^{[j+2]},u)|_{t^{j+2}=t^j}=V_j(t^{[j]},u),\quad j\geq 0, \tag{2.6}$$

and that for every $j\geq 0$, the germ at 0 of the holomorphic map $(V_{j+1},\overline{V}_j)$ takes its values in $\mathcal{M}$, where $\mathcal{M}$ is the complexification of M as defined by (2.3). Note also that in view of (2.1), one may write $V_j(t^{[j]},u)=:(v_j(t^{[j]},u),u)$ where v_j is a $\mathbb{C}^{N-c}$-valued holomorphic map in a neighbhorhood of 0 in $\mathbb{C}^{nj+c}$. In fact, for every sufficiently small $u\in\mathbb{R}^c$, the map $t^{[j]}\mapsto v_j(t^{[j]},u)$ is the iterated Segre mapping (of order j) attached to the generic submanifold M_u at the origin.

We will need the following property about these iterated Segre mappings which provides a precise version of the "minimality criterion" due to Baouendi-Ebenfelt-Rothschild [BER99b] (see also [BRZ01] :

Proposition 2.2. *With the above notation, there exists a positive integer ℓ_0 such that the holomorphic map v^{ℓ_0} is of generic rank $N-c$ (near the origin) and a connected neighborhood Ω of 0 in $\mathbb{C}^{2n\ell_0}$ such that*

$$\mathrm{Rk}\,\frac{\partial v_{2\ell_0}}{\partial t^1\partial t^{\ell_0+2}\partial t^{\ell_0+3}\dots\partial t^{2\ell_0}}(t^{[2\ell_0]},0)=N-c$$

at a generic point $t^{[2\ell_0]}\in\Omega\cap E$ where E is the complex subspace of $\mathbb{C}^{2n\ell_0}$ given by

$$E:=\{t^1=0,t^{\ell_0+2}=t^{\ell_0},t^{\ell_0+3}=t^{\ell_0-1},\dots,t^{2\ell_0}=t^2\}.$$

2.3. Propagation of algebraic dependence through iterated Segre mappings

We now state and prove one key result, Proposition 2.4 below. Quite analogous statements appear in [MMZ03, Proposition 4.3] and in [M12, Proposition 3.7] but in different and somewhat simpler settings. The approach used to prove such results in [MMZ03, M12] relies on arguments involving "CR ratios" and Artin's approximation theorems [A68, A69]. It seems to us that such arguments can not be generalized to provide a proof of Proposition 2.4 below. Instead, we will provide a direct proof of Proposition 2.4 avoiding the above mentioned arguments. As a byproduct, such a proof can be easily adapted to give simpler proofs of [MMZ03, Proposition 4.3] and [M12, Proposition 3.7].

In what follows, we assume that $M\subset\mathbb{C}^N$ is a germ of a real-analytic generic submanifold through the origin with $N\geq 2$. We also assume that the CR orbit of M at the origin is of maximal dimension and choose normal coordinates $Z = (z, w, u)$ as in Lemma 2.1. Such a choice of coordinates is fixed for the rest of this section. Let $H\colon (\mathbb{C}^N, 0) \to \mathbb{C}^{N'}$ be a fixed formal holomorphic map.

Recall that for any integer k, Λ^k and Γ^k denote coordinates in $J_0^k(\mathbb{C}^N, \mathbb{C}^{N'})$.

Definition 2.3. With the above notation, let $\mathbb{A}^H$ be the subring of $\mathbb{C}[[z, w, u]]$ consisting of those power series T for which there exists an integer k and $B \in \mathbb{C}\{z, w, u\}[\Lambda^k, \Gamma^k]$ such that

$$T(z, w, u) = B(z, w, u, (j^k H)(0, u), (j^k \overline{H})(0, u)).$$

We further denote by $\mathbb{K}^H$ the field of fractions of $\mathbb{A}^H$.

Proposition 2.4. *Let M be a connected real-analytic generic submanifold through the origin and $H\colon (\mathbb{C}^N, 0) \to \mathbb{C}^{N'}$ be a formal holomorphic map. Assume that the CR orbit of M at 0 is of maximal dimension and choose normal coordinates $Z = (z, w, u)$ for M near 0 as in Lemma 2.1. Let $\mathcal{M}$ be the complexification of M as defined in (2.3) and assume the mapping H splits as follows $H = (F, G) \in \mathbb{C}^{N'-b} \times \mathbb{C}^b$ for some integer $b \in \{1, \dots, N'\}$. Assume that there exists an integer k_0 and a polynomial $R \in \mathbb{C}\{Z, \zeta\}[\Lambda^{k_0}, \Gamma^{k_0}, \xi', \varpi']$ where $(\xi', \varpi') \in \mathbb{C}^b \times \mathbb{C}^b$ such that:*

(i) $R(Z, \zeta, (j^{k_0} H)(0, u), (j^{k_0} \overline{H})(0, u), \xi', \varpi')\big|_{\mathcal{M} \times \mathbb{C}^b \times \mathbb{C}^b} \not\equiv 0,$

(ii) $R(Z, \zeta, (j^{k_0} H)(0, u), (j^{k_0} \overline{H})(0, u), G(Z), \overline{G}(\zeta))\big|_{\mathcal{M}} = 0.$

Then the components of the mapping G are algebraically dependent over the field $\mathbb{K}^H$.

In order to prove Proposition 2.4, we further need to introduce a number of intermediary fields.

Definition 2.5. *Let b be a fixed positive integer and denote by Υ^k holomorphic coordinates in $J_0^k(\mathbb{C}^N, \mathbb{C}^b)$. Using the notation defined in Sections 2.1 and 2.2, for every integer j and ℓ, define $\mathbb{A}^H_{\ell,j}$ to be the subring of $\mathbb{C}[[t^{[\ell]}, u]]$ of those power series D for which there exists an integer k and $E \in \mathbb{C}\{t^{[\ell]}, u\}[\Lambda^k, \Gamma^k, \Upsilon^k]$ such that*

$$D(t^{[\ell]}, u) = E(t^{[\ell]}, u, (j^k H)(0, u), (j^k \overline{H})(0, u), (j^k G) \circ V_j(t^{[j]}, u)).$$

Similarly, let $\overline{\mathbb{A}}_{\ell,j}^{H}$ *be the subring of* $\mathbb{C}[[t^{[\ell]},u]]$ *of those power series* D *that can be written as*

$$D(t^{[\ell]},u)=E(t^{[\ell]},u,(j^kH)(0,u),(j^k\overline{H})(0,u),(j^k\overline{G})\circ\overline{V}_j(t^{[j]},u))$$

for some integer k *and some* $E\in\mathbb{C}\{t^{[\ell]},u\}[\Lambda^k,\Gamma^k,\Upsilon^k]$.

Denote by $\mathbb{X}_j^H$ *the subring of* $\mathbb{C}[[t^{[j]},u]]$ *of those power series* D *for which there exists an integer* k *and* $E\in\mathbb{C}\{t^{[j]},u\}[\Lambda^k,\Gamma^k]$ *such that*

$$D(t^{[j]},u)=E(t^{[j]},u,(j^kH)(0,u),(j^k\overline{H})(0,u)).$$

The fields of fractions of the rings $\mathbb{A}_{\ell,j}^H$, $\overline{\mathbb{A}}_{\ell,j}^{H}$ *and* $\mathbb{X}_j^H$ *will be respectively denoted by* $\mathbb{K}_{\ell,j}^H$, $\overline{\mathbb{K}}_{\ell,j}^{H}$ *and* $\mathbb{Y}_j^H$.

The following lemma is one of the two ingredients of the proof of Proposition 2.4.

Lemma 2.6. *Using the previously defined notation and under the assumptions of Proposition 2.4, the components of the formal mapping* $G\circ V_{2\ell_0}$ *are algebraically dependent over* $\mathbb{Y}_{2\ell_0}^H$ *where* ℓ_0 *is given by Proposition 2.2.*

Proof of Lemma 2.6. By our choice of ℓ_0 (see Proposition 2.2), the holomorphic map

$$(t^{[2\ell_0+1]},u)\mapsto(V_{2\ell_0+1}(t^{[2\ell_0+1]},u),\overline{V}_{2\ell_0}(t^{[2\ell_0]},u))$$

is generically submersive onto $\mathcal{M}$ near the origin. Hence, using assumptions (i) and (ii), we get the following formal power series identities

$$\begin{aligned} R(V_{2\ell_0+1},\overline{V}_{2\ell_0},(j^{k_0}H)(0,u),(j^k\overline{H})(0,u),G\circ V_{2\ell_0+1},\overline{G}\circ\overline{V}_{2\ell_0})&=0,\\ R(V_{2\ell_0+1},\overline{V}_{2\ell_0},(j^{k_0}H)(0,u),(j^{k_0}\overline{H})(0,u),\xi',\varpi')&\not\equiv 0, \end{aligned}\tag{2.7}$$

respectively in the rings $\mathbb{C}[[t^{[2\ell_0+1]},u]]$ and $\mathbb{C}[[t^{[2\ell_0+1]},u]][\xi',\varpi']$. Set $\Xi\in\mathbb{C}[[t^{[2\ell_0+1]},u]][\xi']$ as follows:

$$\Xi:=R(V_{2\ell_0+1},\overline{V}_{2\ell_0},(j^{k_0}H)(0,u),(j^{k_0}\overline{H})(0,u),\xi',\overline{G}\circ\overline{V}_{2\ell_0}).$$

There are two cases to consider:

First case : $\Xi=0$.

It follows from (2.7) that there exists $W \in \mathbb{C}\{t^{[2\ell_0]}, u\}[\Lambda^{k_0}, \Gamma^{k_0}, \varpi']$ such that

$$
(2.8)\qquad \begin{aligned} W(t^{[2\ell_0]}, u, (j^{k_0}H)(0,u), (j^{k_0}\overline{H})(0,u), \varpi') \not\equiv 0 \\ W(t^{[2\ell_0]}, u, (j^{k_0}H)(0,u), (j^{k_0}\overline{H})(0,u), \overline{G} \circ \overline{V}_{2\ell_0}(t^{[2\ell_0]}, u)) = 0, \end{aligned}
$$

which means that the components of $\overline{G} \circ \overline{V}_{2\ell_0}$ are algebraically dependent over $\mathbb{Y}^H_{2\ell_0}$, and so are the components of $G \circ V_{2\ell_0}$.

Second case : $\Xi \not\equiv 0$.

It follows from (2.7) that there exists $P \in \mathbb{C}\{t^{[2\ell_0+1]}, u\}[\Lambda^{k_0}, \Gamma^{k_0}, \xi', \varpi']$ such that

$$
(2.9)\qquad \begin{aligned} P(t^{[2\ell_0+1]}, u, (j^{k_0}H)(0,u), (j^{k_0}\overline{H})(0,u), \xi', \overline{G} \circ \overline{V}_{2\ell_0}) \not\equiv 0 \\ P\big(t^{[2\ell_0+1]}, u, (j^{k_0}H)(0,u), (j^{k_0}\overline{H})(0,u), G \circ V_{2\ell_0+1}, \overline{G} \circ \overline{V}_{2\ell_0}\big) = 0. \end{aligned}
$$

Note that (2.9) means that the components of $G \circ V_{2\ell_0+1}$ are algebraically dependent over the field $\overline{\mathbb{K}}^H_{2\ell_0+1,2\ell_0}$. From (2.9), we first claim that the components of $G \circ V_{2\ell_0+1}$ are algebraically dependent over $\mathbb{K}^H_{2\ell_0+1,2\ell_0-1}$. For this, consider the finitely generated field extension $\mathbb{K}^H_{2\ell_0,2\ell_0-1} \hookrightarrow \mathbb{K}^H_{2\ell_0,2\ell_0-1}(\overline{G} \circ \overline{V}_{2\ell_0})$. By standard commutative algebra (see e.g. [ZS58]) and interchanging the components of G if necessary, we may write $\overline{G} = (\overline{G}^*, \overline{G}^\sharp) \in \mathbb{C}^{b^*} \times \mathbb{C}^{b-b^*}$, where in the above splitting, the components of $\overline{G}^* \circ \overline{V}_{2\ell_0}$ are algebraically independent over $\mathbb{K}^H_{2\ell_0,2\ell_0-1}$ and all components of $\overline{G}^\sharp \circ \overline{V}_{2\ell_0}$ are algebraically dependent over the field $\mathbb{K}^H_{2\ell_0,2\ell_0-1}(\overline{G}^* \circ \overline{V}_{2\ell_0})$. (Note that if all components of $\overline{G} \circ \overline{V}_{2\ell_0}$ are algebraically dependent over $\mathbb{K}^H_{2\ell_0,2\ell_0-1}$, then it follows from (2.9) and the transitivity of the property of being algebraic over any field, that the components of $G \circ V_{2\ell_0+1}$ are algebraically dependent over $\mathbb{K}^H_{2\ell_0+1,2\ell_0-1}$, the desired claim.) By (2.9), the components of $G \circ V_{2\ell_0+1}$ are algebraically dependent over the field $\mathbb{K}_{2\ell_0+1,2\ell_0-1}(\overline{G} \circ \overline{V}_{2\ell_0})$, and therefore are also algebraically dependent over the field $\mathbb{K}_{2\ell_0+1,2\ell_0-1}(\overline{G}^* \circ \overline{V}_{2\ell_0})$ (since $\mathbb{K}_{2\ell_0,2\ell_0-1} \subset \mathbb{K}_{2\ell_0+1,2\ell_0-1}$). Hence there exist an integer k_1 and a polynomial $P_1 \in \mathbb{C}\{t^{[2\ell_0+1]}, u\}[\Lambda^{k_1}, \Gamma^{k_1}, \Upsilon^{k_1}, \xi', \tau^*]$, $\tau^* \in \mathbb{C}^{b^*}$, such that

$$
P_1\big(t^{[2\ell_0+1]}, u, (j^{k_1}H)(0,u), (j^{k_1}\overline{H})(0,u), (j^{k_1}G) \circ V_{2\ell_0-1}, \xi', \overline{G}^* \circ \overline{V}_{2\ell_0}\big) \not\equiv 0
$$

and

$$
(2.10)\qquad \begin{aligned} P_1\big(t^{[2\ell_0+1]}, u, (j^{k_1}H)(0,u), (j^{k_1}\overline{H})(0,u), (j^{k_1}G) \circ V_{2\ell_0-1}, \\ G \circ V_{2\ell_0+1}, \overline{G}^* \circ \overline{V}_{2\ell_0}\big) = 0. \end{aligned}
$$

Define $A \in \mathbb{C}[[t^{[2\ell_0+1]}, u, \tau^*]]$ by

$$A := P_1(t^{[2\ell_0+1]}, u, (j^{k_1}H)(0, u, (j^{k_1}\overline{H})(0, u), (j^{k_1}G) \circ V_{2\ell_0-1}, G \circ V_{2\ell_0+1}, \tau^*),$$

and let us prove that

$$A = A(t^{[2\ell_0+1]}, u, \tau^*) = 0. \tag{2.11}$$

By contradiction, suppose it is not the case. Then there exists a multiindex $\alpha \in \mathbb{N}^n$ such that $(\partial^\alpha_{t^{2\ell_0+1}} A)|_{t^{2\ell_0+1}=t^{2\ell_0-1}} \not= 0$. Note that by using (2.6), we may write

$$\begin{aligned}(\partial^\alpha_{t^{2\ell_0+1}} A)|_{t^{2\ell_0+1}=t^{2\ell_0-1}} = \\ \widetilde{P}_1(t^{[2\ell_0]}, u, (j^{k_2}H(0, u), (j^{k_2}\overline{H})(0, u), (j^{k_2}G \circ V_{2\ell_0-1}), \tau^*),\end{aligned} \tag{2.12}$$

for some $\widetilde{P}_1 \in \mathbb{C}\{t^{[2\ell_0]}, u\}[\Lambda^{k_2}, \Gamma^{k_2}, \Upsilon^{k_2}, \tau^*]$ where $k_2 = \max(|\alpha|, k_1)$. Using (2.10), (2.12) and the definition of A given in (2.11), we see that

$$\widetilde{P}_1(t^{[2\ell_0]}, u, (j^{k_2}H)(0, u), (j^{k_2}\overline{H})(0, u), (j^{k_2}G) \circ V_{2\ell_0-1}, \overline{G}^* \circ \overline{V}_{2\ell_0}) = 0,$$

which proves that the components of $\overline{G}^* \circ \overline{V}_{2\ell_0}$ are algebraically dependent over $\mathbb{K}^H_{2\ell_0,2\ell_0-1}$, a contradiction. This proves that (2.11) holds. It follows from the first condition in (2.10) that

$$P_1(t^{[2\ell_0+1]}, u, (j^{k_1}H)(0, u), (j^{k_1}\overline{H})(0, u), (j^{k_1}G) \circ V_{2\ell_0-1}, \xi', \tau^*) \not\equiv 0$$

and therefore (2.11) proves that the components of $G \circ V_{2\ell_0+1}$ are algebraically dependent over $\mathbb{K}^H_{2\ell_0+1,2\ell_0-1}$ which proves the claim.

Now, quite similarly, we claim that the components of $G \circ V_{2\ell_0+1}$ are algebraically dependent over $\mathbb{K}^H_{2\ell_0+1,2\ell_0-3}$. Indeed, by the previous conclusion, we get the existence of an integer k_3 and of a polynomial $P_2 \in \mathbb{C}\{t^{[2\ell_0+1]}, u\}[\Lambda^{k_3}, \Gamma^{k_3}, \Upsilon^{k_3}, \xi']$ such that

$$\begin{aligned}P_2(t^{[2\ell_0+1]}, u, (j^{k_3}H)(0, u), (j^{k_3}\overline{H})(0, u), (j^{k_3}G) \circ V_{2\ell_0-1}, \xi') \not\equiv 0 \\ P_2(t^{[2\ell_0+1]}, u, (j^{k_3}H)(0, u), (j^{k_3}\overline{H})(0, u), (j^{k_3}G) \circ V_{2\ell_0-1}, G \circ V_{2\ell_0+1}) = 0.\end{aligned} \tag{2.13}$$

By considering the finitely generated field extension

$$\mathbb{K}^H_{2\ell_0-1,2\ell_0-3} \hookrightarrow \mathbb{K}^H_{2\ell_0-1,2\ell_0-3}((j^{k_3}G) \circ V_{2\ell_0-1}),$$

we may, without loss of generality, split the formal map $j^{k_3}G =: (\widetilde{G}, \widehat{G})$ where $\widetilde{G}$ is $\mathbb{C}^a$-valued formal map satisfying the following properties : all components of the map $\widetilde{G} \circ V_{2\ell_0-1}$ are algebraically independent over $\mathbb{K}^H_{2\ell_0-1,2\ell_0-3}$ and every component of the map $\widehat{G} \circ V_{2\ell_0-1}$ is algebraically dependent over $\mathbb{K}^H_{2\ell_0-1,2\ell_0-3}(\widetilde{G} \circ V_{2\ell_0-1})$. Hence, by (2.13) and the transivity of the property of being algebraic, the components of $G \circ V_{2\ell_0+1}$ are algebraically dependent over $\mathbb{K}^H_{2\ell_0+1,2\ell_0-3}(\widetilde{G} \circ V_{2\ell_0-1})$ (since $\mathbb{K}^H_{2\ell_0-1,2\ell_0-3} \subset \mathbb{K}^H_{2\ell_0+1,2\ell_0-3}$). We may therefore find an integer k_4 and a polynomial $P_3 \in \mathbb{C}\{t^{[2\ell_0+1]}, u\}[\Lambda^{k_4}, \Gamma^{k_4}, \Upsilon^{k_4}, \xi', \widetilde{\tau}]$, $\widetilde{\tau} \in \mathbb{C}^a$, such that

$$P_3\big(t^{[2\ell_0+1]}, u, (j^{k_4}H)(0,u), (j^{k_4}\overline{H})(0,u), (j^{k_4}G) \circ V_{2\ell_0-3}, \xi', \widetilde{G} \circ V_{2\ell_0-1}\big) \not\equiv 0$$

and

$$\begin{aligned} P_3\big(t^{[2\ell_0+1]}, u, (j^{k_4}H)(0,u), (j^{k_4}\overline{H})(0,u), (j^{k_4}G) \circ V_{2\ell_0-3},& \\ G \circ V_{2\ell_0+1}, \widetilde{G} \circ V_{2\ell_0-1}\big) = 0.& \end{aligned} \tag{2.14}$$

Setting

$$B := P_3(t^{[2\ell_0+1]}, u, (j^{k_4}H)(0,u), (j^{k_4}\overline{H})(0,u), (j^{k_4}G \circ V_{2\ell_0-3}), G \circ V_{2\ell_0+1}, \widetilde{\tau}),$$

we claim that

$$B = B(t^{[2\ell_0+1]}, u, \widetilde{\tau}) := 0. \tag{2.15}$$

Suppose, by contradiction, that (2.15) does not hold. Then there exists a multiindex $\beta \in \mathbb{N}^{2n}$ such that

$$\frac{\partial^\beta B}{\partial t^{2\ell_0+1} \partial t^{2\ell_0}}\bigg|_{(t^{2\ell_0+1}, t^{2\ell_0}) = (t^{2\ell_0-3}, t^{2\ell_0-2})} \not\equiv 0.$$

Observe that in view of (2.6),

$$\frac{\partial^\beta B}{\partial t^{2\ell_0+1} \partial t^{2\ell_0}}\bigg|_{(t^{2\ell_0+1}, t^{2\ell_0}) = (t^{2\ell_0-3}, t^{2\ell_0-2})}$$

may be written in the form

$$\widetilde{P}_3(t^{[2\ell_0-1]}, u, (j^{k_5}H)(0,u), (j^{k_5}\overline{H})(0,u), (j^{k_5}G) \circ V_{2\ell_0-3}, \widetilde{\tau}),$$

for some $\widetilde{P}_3 \in \mathbb{C}\{t^{[2\ell_0-1]}, u\}[\Lambda^{k_5}, \Gamma^{k_5}, \Upsilon^{k_5}, \widetilde{\tau}]$ where $k_5 = \max(|\alpha|, k_4)$. By using (2.15), (2.14) and the above mentioned observation, we see that

$$\widetilde{P}_3(t^{[2\ell_0-1]}, u, (j^{k_5}H(0,u), (j^{k_5}\overline{H})(0,u), (j^{k_5}G) \circ V_{2\ell_0-3}, \widetilde{G} \circ V_{2\ell_0-1}) = 0,$$

which proves that the components of $\widetilde{G}\circ V_{2\ell_0-1}$ are algebraically dependent over $\mathbb{K}^H_{2\ell_0-1,2\ell_0-3}$, which is a contradiction. Hence (2.15) holds. From the first condition in (2.14) we get that

$$P_3(t^{[2\ell_0+1]},u,(j^{k_4}H)(0,u),(j^{k_4}\overline{H})(0,u),(j^{k_4}G)\circ V_{2\ell_0-3},\xi',\widetilde{\tau})\not\equiv 0,$$

which together with (2.15) imply that the components of $G\circ V_{2\ell_0+1}$ are algebraically dependent over $\mathbb{K}^H_{2\ell_0+1,2\ell_0-3}$ which proves the claim.

Proceeding inductively (i.e. reproducing the above procedure ℓ_0-2 more times), one obtains that the components of $G\circ V_{2\ell_0+1}$ are algebraically dependent over $\mathbb{K}^H_{2\ell_0+1,1}$. Therefore, there exist an integer k_* and $P_*\in\mathbb{C}\{t^{[2\ell_0+1]},u\}[\Lambda^{k_*},\Gamma^{k_*},\Upsilon^{k_*},\xi']$ such that

(2.16)
$$\begin{aligned}P_*(t^{[2\ell_0+1]},u,(j^{k_*}H)(0,u),(j^{k_*}\overline{H})(0,u),(j^{k_*}G)\circ V_1,\xi')\not\equiv 0\\ P_*(t^{[2\ell_0+1]},u,(j^{k_*}H)(0,u),(j^{k_*}\overline{H})(0,u),(j^{k_*}G)\circ V_1,G\circ V_{2\ell_0+1})=0.\end{aligned}$$

By considering the field extension $\mathbb{K}^H_{1,0}\hookrightarrow\mathbb{K}^H_{1,0}((j^{k_*}G)\circ V_1)$ and mimicking a procedure similar to the one above (the only difference being that the evaluation of some power series should be done along the subspace $t^{2\ell_0+1}=t^{2\ell_0}=\ldots=t^2=0$), one may show that the components of the mapping $G\circ V_{2\ell_0+1}$ are algebraically depending over the field $\mathbb{K}^H_{2\ell_0+1,0}$, which obviously coincides with $\mathbb{Y}_{2\ell_0+1}$. We now claim that this statement implies the desired conclusion of the lemma. Indeed, by the above, there exist an integer $\widehat{k}$ and of a polynomial $\widehat{P}\in\mathbb{C}\{t^{[2\ell_0+1]},u\}[\Lambda^{\widehat{k}},\Gamma^{\widehat{k}},\xi']$ such that

$$\begin{aligned}\widehat{P}(t^{[2\ell_0+1]},u,(j^{\widehat{k}}H)(0,u),(j^{\widehat{k}}\overline{H})(0,u),\xi')\not\equiv 0\\ \widehat{P}(t^{[2\ell_0+1]},u,(j^{\widehat{k}}H)(0,u),(j^{\widehat{k}}\overline{H})(0,u),G\circ V_{2\ell_0+1})=0.\end{aligned}\tag{2.17}$$

Choose $\gamma\in\mathbb{N}^n$ of minimal length such that

$$(\partial^{\gamma}_{t^1}\widehat{P})(0,t^2,\ldots,t^{2\ell_0+1},u,(j^{\widehat{k}}H)(0,u),(j^{\widehat{k}}\overline{H})(0,u),\xi')\not\equiv 0.$$

Since $V_{2\ell_0+1}(0,t^2,\ldots,t^{2\ell_0+1})=V_{2\ell_0}(t^2,\ldots,t^{2\ell_0+1})$, (2.17) implies

$$\begin{aligned}(\partial^{\gamma}_{t^1}\widehat{P})(0,t^2,\ldots,t^{2\ell_0+1},u,&(j^{\widehat{k}}H)(0,u),\\ &(j^{\widehat{k}}\overline{H})(0,u),(G\circ V_{2\ell_0})(t^2,\ldots,t^{2\ell_0+1}))=0,\end{aligned}\tag{2.18}$$

which provides the desired conclusion of the lemma. □

The second ingredient in the proof of Proposition 2.4 is the following lemma which follows from the statement and an inspection of the proof of [BER99b, Proposition 4.1.18].

Lemma 2.7. *Let* $\Psi\colon (\mathbb{C}^{r_1}_x \times \mathbb{C}^{r_2}_s \times \mathbb{C}^{r_3}_y, 0) \to (\mathbb{C}^q, 0)$ *be a germ of a holomorphic map such that* $\Psi(x,s,0)=0$ *with* $r_3 \geq q$*. Assume the* $q \times r_3$ *matrix* $\left[\left(\frac{\partial \Psi}{\partial y}\right)(x,0,0)\right]$ *has rank* q *at a generic point* $x \in \mathbb{C}^{r_1}$ *sufficiently close to* 0*. Then there exists germs of holomorphic maps*

$$\Delta\colon (\mathbb{C}^{r_1},0) \to \mathbb{C}, \quad \Delta(x) \not\equiv 0, \quad \varphi\colon (\mathbb{C}^{r_1} \times \mathbb{C}^{r_2} \times \mathbb{C}^q, 0) \to (\mathbb{C}^{r_3}, 0)$$

satisfying

$$\Psi\left(x, s, \varphi\left(x, \frac{s}{\Delta(x)}, \frac{\omega}{\Delta(x)}\right)\right) = \omega, \quad \varphi(x,s,0)=0,$$

for all $(x,s,\omega) \in \mathbb{C}^{r_1} \times \mathbb{C}^{r_2} \times \mathbb{C}^q$ *with* x *sufficiently small,* $\Delta(x) \neq 0$ *and* $\left|\frac{s}{\Delta(x)}\right| + \left|\frac{\omega}{\Delta(x)}\right|$ *sufficiently small.*

Remark 2.8. The first identity in (2.21) also holds in the ring

$$(\operatorname{Frac} \mathbb{C}\{x\})[[s,\omega]]$$

where $\operatorname{Frac} \mathbb{C}\{x\}$ is the field of fractions of $\mathbb{C}\{x\}$.

Proof of Proposition 2.4. By Lemma 2.6, there exist an integer k and of a polynomial $P \in \mathbb{C}\{t^{[2\ell_0]}, u\}[\Lambda^k, \Gamma^k, \xi']$ such that

$$\begin{aligned} P(t^{[2\ell_0]}, u, (j^k H)(0,u), (j^k \overline{H})(0,u), \xi') &\not\equiv 0 \\ P(t^{[2\ell_0]}, u, (j^k H)(0,u), (j^k \overline{H})(0,u), G \circ V_{2\ell_0}(t^{[2\ell_0]}, u)) &= 0. \end{aligned} \tag{2.19}$$

As in [BRZ01], consider the following linear automorphism L of $\mathbb{C}^{2n\ell_0 + c}$ defined by

$$L(x,y,u) := (y^0, x^1, \ldots, x^{\ell_0}, x^{\ell_0 - 1} + y^{\ell_0 - 1}, \ldots, x^1 + y^1, u),$$

where $x = (x^1, \ldots, x^{\ell_0})$ and $y = (y^0, y^1, \ldots, y^{\ell_0 - 1})$. Set

$$(V_{2\ell_0} \circ L)(x,y,u) =: (\Psi(x,y,u), u).$$

It follows from (2.19) that , there exists a polynomial $\mathcal{P} \in \mathbb{C}\{x, y, u\}[\Lambda^k, \Gamma^k, \xi']$ such that

$$\begin{gathered}\mathcal{P}(x, y, u, (j^k H)(0, u), (j^k \overline{H})(0, u), \xi') \not\equiv 0 \\ \mathcal{P}(x, y, u, (j^k H)(0, u), (j^k \overline{H})(0, u), G(\Psi(x, y, u), u)) = 0.\end{gathered} \tag{2.20}$$

It follows from (2.6) that

$$(V_{2\ell_0} \circ L)(x, 0, u) = V_{2\ell_0}(0, x^1, \ldots, x^{\ell_0}, x^{\ell_0 - 1}, \ldots, x^1, u) = (0, u)$$

and hence $\Psi(x, 0, u) = 0$. Since the rank of the matrix $\left[\left(\frac{\partial \Psi}{\partial y}\right)(x, 0, 0)\right]$ is the same as the rank of the matrix $\left[\left(\frac{\partial (v_{2\ell_0} \circ L)}{\partial y}\right)(x, 0, 0)\right]$, it follows from Proposition 2.2 that the rank of the matrix $\left[\left(\frac{\partial \Psi}{\partial y}\right)(x, 0, 0)\right]$ is equal to $N - c$ at generic points $x \in \mathbb{C}^{n\ell_0}$ sufficiently close to 0. Lemma 2.7 together with Remark 2.8 therefore implies that there exist germs of holomorphic maps

$$\Delta\colon (\mathbb{C}^{n\ell_0}, 0) \to \mathbb{C}, \quad \Delta(x) \not\equiv 0, \quad \varphi\colon (\mathbb{C}^{n\ell_0} \times \mathbb{C}^N, 0) \to (\mathbb{C}^{n\ell_0}, 0)$$

satisfying

$$\Psi\left(x, \varphi\left(x, \frac{Z}{\Delta(x)}\right), u\right) = (z, w), \quad \varphi(x, (0, u)) = 0, \tag{2.21}$$

as a formal power series identity in the ring $(\operatorname{Frac}\mathbb{C}\{x\})[[Z]]$ (where $Z = (z, w, u)$). One would like to set $y = \varphi(x, \frac{Z}{\Delta(x)})$ in the second identity in (2.20), but a priori, this might lead to a trivial relation. To overcome this, we claim that there exists $\beta \in \mathbb{N}^{n\ell_0}$ such that

$$\mathcal{P}_{y^\beta}\left(x, \varphi\left(x, \frac{Z}{\Delta(x)}\right), u, (j^k H)(0, u), (j^k \overline{H})(0, u), \xi'\right) \not\equiv 0, \tag{2.22}$$

as a power series identity in the ring $(\operatorname{Frac}\mathbb{C}\{x\})[[Z]][\xi']$. Indeed, if (2.22) does not hold for every $\beta \in \mathbb{N}^{n\ell_0}$, then setting $(z, w) = (0, 0)$ and using the second identity in (2.21), we would get

$$\mathcal{P}_{y^\beta}\left(x, 0, u, (j^k H)(0, u), (j^k \overline{H})(0, u), \xi'\right) = 0$$

for every $\beta \in \mathbb{N}^{n\ell_0}$ which would contradict the first identity in (2.20). This proves the claim. Choose $\beta \in \mathbb{N}^{n\ell_0}$ of minimal length such that (2.22) holds.

From (2.20) and (2.21), we get

$$\mathcal{P}_{y^\beta}\left(x, \varphi\left(x, \frac{Z}{\Delta(x)}\right), u, (j^k H)(0,u), (j^k \overline{H})(0,u), G(Z)\right) = 0 \tag{2.23}$$

as a power series identity in the ring $(\operatorname{Frac}\mathbb{C}\{x\})[[Z]]$. From (2.22) and (2.23), it follows that for a generic point x_0 sufficiently close to 0 in $\mathbb{C}^{n\ell_0}$ (satisfying in particular $\Delta(x_0) \neq 0$), the following identities hold in the ring $\mathbb{C}[[Z]]$

$$\begin{aligned} &\mathcal{P}_{y^\beta}\left(x_0, \varphi\left(x_0, \frac{Z}{\Delta(x_0)}\right), u, (j^k H)(0,u), (j^k \overline{H})(0,u), G(Z)\right) = 0 \\ &\mathcal{P}_{y^\beta}\left(x_0, \varphi\left(x_0, \frac{Z}{\Delta(x_0)}\right), u, (j^k H)(0,u), (j^k \overline{H})(0,u), \xi'\right) \not\equiv 0. \end{aligned} \tag{2.24}$$

Fix one such point x_0. Since $\varphi\left(x_0, \frac{Z}{\Delta(x_0)}\right)$ is holomorphic in some neighborhood of the origin in $\mathbb{C}^N$, it follows from (2.24) that the components of the mapping G are algebraically dependent over the field $\mathbb{K}^H$. The proof of Proposition 2.4 is complete. □

3. Reducing the Artin approximation property to nested approximation

We show in this section how to reduce the Artin approximation property to the nested approximation property for a certain system of real-analytic equations. The precise statement is given by Proposition 3.3 below. We first construct some system of real-analytic equations associated to any formal holomorphic mapping and any germ of a generic submanifold whose CR orbits are all of the same dimension. The tools used here are adapted from the ones developed in an algebraic context in [M12].

Let M be a germ of a real-analytic generic submanifold through the origin in $\mathbb{C}^N$, $N \geq 2$ and $H\colon (\mathbb{C}^N, 0) \to \mathbb{C}^{N'}$ be a fixed formal holomorphic mapping. The CR orbit of M passing through the origin is assumed to be of maximal dimension and we choose and fix normal coordinates $Z = (z, w, u)$ as in Lemma 2.1.

Consider the following field extensions $\mathbb{K}^H \hookrightarrow \mathbb{K}^H(H) \hookrightarrow \operatorname{Frac}\mathbb{C}[[Z]]$, where $\mathbb{K}^H$ is given as in Definition 2.3. As a finitely generated field extension $\mathbb{K}^H \hookrightarrow \mathbb{K}^H(H)$, we may choose r components of the mapping H, denoted by G, such that these components form a transcendence basis of the above field extension, with $r \in \{0, \ldots, N'\}$. Hence, without loss of generality, we may split the map $H = (F, G)$ where $F = (F_1, \ldots, F_{N'-r})$ and G are respectively $\mathbb{C}^{N'-r}_{\tau'}$ and $\mathbb{C}^r_{\xi'}$ valued formal power series mappings. We write $\tau' = (\tau'_1, \ldots, \tau'_{N'-r})$, $Z' = (\xi', \tau') \in \mathbb{C}^r \times \mathbb{C}^{N'-r}$ and introduce ϖ' as another variable in $\mathbb{C}^r$. Recall also, that every integer k, Λ^k and Γ^k denote coordinates in $J_0^k(\mathbb{C}^N, \mathbb{C}^{N'})$. In the rest of this section, we shall assume that $r < N'$.

From the fact that G is a transcendence basis of the field extension $\mathbb{K}^H \hookrightarrow \mathbb{K}^H(H)$, there exist an integer k and for every integer $j \in \{1, \ldots, N' - r\}$ a polynomial $\mathscr{A}_j \in \mathbb{C}\{Z\}[\Lambda^k, \Gamma^k, \xi'][\tau'_j]$ such that

$$\mathscr{A}_j(Z, (j^k H)(0, u), (j^k \overline{H})(0, u), G(Z), F_j(Z)) = 0. \tag{3.1}$$

We may also assume that if we write

$$\mathscr{A}_j(Z, \Lambda^k, \Gamma^k, \xi', \tau'_j) = \sum_{\nu=0}^{m_j} \mathscr{A}_{j,\nu}(Z, \Lambda^k, \Gamma^k, \xi')(\tau'_j)^\nu,$$

then

$$\mathscr{A}_{j,m_j}(Z, (j^k H)(0, u), (j^k \overline{H})(0, u), G(Z)) \not\equiv 0. \tag{3.2}$$

The following lemma is a slight variation of [M12, Lemma 4.1]. Its proof, being analogous to that of [M12, Lemma 4.1], is therefore left to the reader.

Lemma 3.1. *With the above notation, the following holds. For every real-valued real-analytic function $\rho(Z, \overline{Z}, Z', \overline{Z}') \in \mathbb{R}\{Z, \overline{Z}\}[Z', \overline{Z}']$ in $\mathbb{C}^N \times \mathbb{C}^{N'}$, there exists a non-trivial polynomial $\mathscr{Q}^\rho \in \mathbb{C}\{Z, \zeta\}[\Lambda^k, \Gamma^k, \xi', \varpi'][X]$, $X \in \mathbb{C}$, such that for every formal holomorphic map $f\colon (\mathbb{C}^N, 0) \to \mathbb{C}^{N'-r}$, $g\colon (\mathbb{C}^N, 0) \to \mathbb{C}^r$, $T\colon (\mathbb{C}^c, 0) \to \mathbb{C}^{\ell_k}$, where $\ell_k := \operatorname{card}\{\alpha \in \mathbb{N}^N : |\alpha| \leq k\}$, satisfying*

$$\mathscr{A}_j(Z, T(u), \overline{T}(u), g(Z), f_j(Z)) = 0, \quad \text{and} \quad \mathscr{A}_{j,m_j}(Z, T(u), \overline{T}(u), g(Z)) \not\equiv 0,$$

for $j = 1, \ldots, N' - r$, then

$$\mathscr{Q}^\rho(Z, \zeta, T(u), \overline{T}(u), g(Z), \bar{g}(\zeta), \rho(Z, \zeta, f(Z), g(Z), \bar{f}(\zeta), \bar{g}(\zeta))) = 0,$$

as a power series identity in the ring $\mathbb{C}[[Z, \zeta]]$. In addition, writing

$$\mathscr{Q}^\rho(Z, \zeta, \Lambda^k, \Gamma^k, \xi', \varpi', X) = \sum_{\nu=0}^{\delta} \mathscr{Q}^\rho_\nu(Z, \zeta, \Lambda^k, \Gamma^k, \xi', \varpi') X^\nu,$$

δ is independent of ρ and $\mathscr{Q}^\rho_\delta(Z, \zeta, T(u), \overline{T}(u), g(Z), \bar{g}(\zeta)\big|_{\mathcal{M}} \not\equiv 0$.

For every real-valued real-analytic function

$$\rho(Z,\overline{Z},Z',\overline{Z}') \in \mathbb{R}\{Z,\overline{Z}\}[Z',\overline{Z}'],$$

consider the polynomial $\mathscr{Q}^\rho$ given by Lemma 3.1. Set

$$\begin{aligned}(3.3)\qquad p_\rho := \inf\{\nu \in \{0,\ldots,\delta\} :\\ \mathscr{Q}^\rho_\nu(Z,\zeta,(j^kH)(0,u),(j^k\overline{H})(0,u),\xi',\varpi')\big|_{\mathcal{M}\times\mathbb{C}^{2r}} \not\equiv 0\},\end{aligned}$$

that is well defined according to (3.1), (3.2) and Lemma 3.1. The next lemma is quite analogous to [M12, Lemma 4.2] and we therefore also omit its proof.

Lemma 3.2. *With the above notation, there exist $\mathscr{S}_1,\ldots,\mathscr{S}_b \in \mathbb{C}\{u\}[\Lambda^k,\Gamma^k]$ such that for every formal power series mapping $T \in (\mathbb{C}[[u]])^{\ell_k}$, T satisfies the system of equations*

$$\mathscr{Q}^\rho_\nu(Z,\zeta,T(u),\overline{T}(u),\xi',\varpi')\big|_{\mathcal{M}\times\mathbb{C}^{2r}} = 0$$

for every real-valued real-analytic function $\rho \in \mathbb{R}\{Z,\overline{Z}\}[Z',\overline{Z}']$ and every $\nu \leq p_\rho - 1$ if and only if

$$\mathscr{S}_q(u,T(u),\overline{T(u)})\big|_{\mathbb{R}^c} = 0,\ q = 1,\ldots,b.$$

We now have all the tools to prove the following:

Proposition 3.3. *Let M and H be as above. Denote by r the transcendence degree of the field extension $\mathbb{K}^H \hookrightarrow \mathbb{K}^H(H)$ and k the integer given by* (3.1). *Assume that $r < N'$. For $x \in \mathbb{R}^{N-c}$, $u \in \mathbb{R}^c$, $\Lambda^k \in J^k_0(\mathbb{C}^N,\mathbb{C}^{N'})$, consider the following system of complex-valued real-analytic equations*

$$\begin{aligned}(3.4)\qquad &\mathscr{A}_j(x,u,\Lambda^k,\overline{\Lambda^k},\xi',\tau'_j) = 0, \quad \mathscr{S}_q(u,\Lambda^k,\overline{\Lambda^k}) = 0,\\ &j \in \{1,\ldots,N'-r\},\ q \in \{1,\ldots,b\},\end{aligned}$$

where $\mathscr{A}_j$ and $\mathscr{S}_q$ are given by (3.1) *and Lemma 3.2 respectively. Then, the system* (3.4) *has the following properties:*

(i) *The formal mapping $\xi' = G(x,u)$, $\tau' = F(x,u)$, $\Lambda^k = (j^kH)(0,u)$ is a formal solution of* (3.4).

(ii) *For every partially algebraic subset $\Sigma' \subset \mathbb{C}^N \times \mathbb{C}^{N'}$ passing through $(0,H(0))$, and for every sequence of formal mappings $\xi' = g^\ell(x,u)$, $\tau' = f^\ell(x,u)$, $\Lambda^k = T^\ell(u)$ converging as $\ell \to \infty$ in the Krull topology* *to* $G(x,u)$*,* $F(x,u)$ *and* $(j^kH)(0,u)$ *respectively and satisfying* (3.4)*, if* $\mathrm{Graph}\, H \cap (M \times \mathbb{C}^{N'}) \subset \Sigma'$*, then for* ℓ *sufficiently large, one also has* $\mathrm{Graph}\, h^\ell \cap (M \times \mathbb{C}^{N'}) \subset \Sigma'$*, where in this inclusion,* $h^\ell := (f^\ell, g^\ell)$ *denotes the complexification over* $\mathbb{C}^N$ *of the original power series mapping defined over* $\mathbb{R}^N$*.*

Proof. Part (i) of the proposition follows from the construction of the system (3.4) and, more specifically from (3.1), (3.3) and Lemma 3.2.

In order to prove (ii), we will follow the steps of the proof of [M12, Proposition 4.3] and make use of Proposition 2.4 of the present paper. To this end, we fix a partially algebraic subset $\Sigma' \subset \mathbb{C}^N \times \mathbb{C}^{N'}$ passing through $(0, H(0))$ and assume that $\mathrm{Graph}\, H \cap (M \times \mathbb{C}^{N'}) \subset \Sigma'$. Without loss of generality, we may assume that the germ of Σ' at $(0, H(0))$ is different from the germ at $(0, H(0))$ of $\mathbb{C}^N \times \mathbb{C}^{N'}$ (since otherwise the conclusion of the proposition is obvious). We also fix a sequence of formal mappings $\xi' = g^\ell(x,u)$, $\tau' = f^\ell(x,u)$, $\Lambda^k = T^\ell(u)$ converging as $\ell \to \infty$ in the Krull topology to $G(x,u)$, $F(x,u)$ and $(j^kH)(0,u)$ respectively and satisfying (3.4). Suppose, by contradiction, that there is a subsequence $(h^{\ell_i})_i$ such that for every integer i, $\mathrm{Graph}\, h^{\ell_i} \cap (M \times \mathbb{C}^{N'}) \not\subset \Sigma'$, where h^{ℓ_i} denotes the formal mapping from $(\mathbb{C}^N, 0)$ to $\mathbb{C}^{N'}$ obtained by complexifying the original mapping from $(\mathbb{R}^N, 0)$ to $\mathbb{C}^{N'}$. Since Σ' is partially algebraic, we may find a finite number of non-trivial real-valued real-analytic functions $\rho_1, \ldots, \rho_e$ in the ring $\mathbb{R}\{Z, \overline{Z}\}[Z', \overline{Z}']$ such that Σ' is given by the zero set of these e functions near $(0, H(0))$. By the pigeonhole principle, there is a certain subsequence $(h^{\tilde{\ell}_i})_i$ of $(h^\ell)_\ell$ such that for every integer i, $\mathrm{Graph}\, h^{\tilde{\ell}_i} \cap (M \times \mathbb{C}^{N'}) \not\subset \Sigma_1'$ where Σ_1' is the partially algebraic set near $(0, H(0))$ given by the zero set of ρ_1. Without loss of generality, we may assume that $(h^{\tilde{\ell}_i})_i$ is the whole sequence $(h^\ell)_\ell$. This means that for every integer ℓ

$$\rho_1(Z, \zeta, h^\ell(Z), \bar{h}^\ell(\zeta))\big|_{\mathcal{M}} \not\equiv 0. \tag{3.5}$$

Since $\xi' = g^\ell(x,u)$, $\tau' = f^\ell(x,u)$, $\Lambda^k = T^\ell(u)$ satisfy the first set of equations of (3.4), after complexification, we have $\mathscr{A}_j(Z, T^\ell(u), \overline{T}^\ell(u), g^\ell(Z), f^\ell(Z)) = 0$, $j = 1, \ldots, N' - r$. We also note that since $\xi' = g^\ell(x,u)$, $\tau' = f^\ell(x,u)$, $\Lambda^k = T^\ell(u)$ converge as $\ell \to \infty$ in the Krull topology to $G(x,u)$, $F(x,u)$ and $(j^kH)(0,u)$ respectively, we have $\mathscr{A}_{j,m_j}(Z, T^\ell(u), \bar{T}^\ell(u), g^\ell(Z)) \neq 0$, $j = 1, \ldots, N' - r$, for ℓ large enough. We may therefore apply Lemma 3.1 to get that for ℓ large enough

$$\mathscr{Q}^{\rho_1}(Z, \zeta, T^\ell(u), \overline{T}^\ell(u), g^\ell(Z), \bar{g}^\ell(Z), \rho_1(Z, \zeta, h^\ell(Z), \bar{h}^\ell(\zeta))) = 0. \tag{3.6}$$

Using the fact that $\Lambda^k = T^\ell(u)$ satisfy the second set of equations of the system (3.4) and Lemma 3.2, we get that for $\nu = 0, \ldots, p_{\rho_1} - 1$,

$$\mathscr{Q}_\nu^{\rho_1}(Z, \zeta, T^\ell(u), \overline{T}^\ell(u), \xi', \varpi')\big|_{\mathcal{M}\times\mathbb{C}^{2r}} = 0.$$

Hence (3.6) implies that for ℓ large enough

$$\sum_{\nu=p_{\rho_1}}^{\delta} \mathscr{Q}_\nu^{\rho_1}(Z, \zeta, T^\ell(u), \overline{T}^\ell(u), g^\ell(Z), \bar{g}^\ell(\zeta)) \left(\rho_1(Z, \zeta, h^\ell(Z), \bar{h}^\ell(\zeta))\right)^\nu |_{\mathcal{M}} = 0.$$

Therefore, using (3.5) we get

(3.7)

$$\mathscr{Q}_{p_{\rho_1}}^{\rho_1}(Z, \zeta, T^\ell(u), \overline{T}^\ell(u), g^\ell(Z), \bar{g}^\ell(\zeta))\big|_{\mathcal{M}} =$$

$$\sum_{\nu=1+p_{\rho_1}}^{\delta} \mathscr{Q}_\nu^{\rho_1}(Z, \zeta, T^\ell(u), \overline{T}^\ell(u), g^\ell(Z), \bar{g}^\ell(\zeta)) \left(\rho_1(Z, \zeta, h^\ell(Z), \bar{h}^\ell(\zeta))\right)^{\nu - p_{\rho_1}} |_{\mathcal{M}}$$

Observe now that the left-hand side of (3.7) converges as $\ell \to \infty$ (in the Krull topology) to $\mathscr{Q}_{p_{\rho_1}}^{\rho_1}(Z, \zeta, (j^k H)(0, u), (j^k \overline{H})(0, u), g(Z), \bar{g}(\zeta))\big|_{\mathcal{M}}$. On the other hand, since $\mathrm{Graph}\, H \cap (M \times \mathbb{C}^{N'}) \subset \Sigma' \subset \Sigma_1'$, the expression $\rho_1(Z, \zeta, h^\ell(Z), \bar{h}^\ell(\zeta))|_{\mathcal{M}}$ converges as $\ell \to \infty$ to $\rho_1(Z, \zeta, H(Z), \overline{H}(\zeta))|_{\mathcal{M}} = 0$. Hence, as $\ell \to \infty$, (3.7) implies that $\mathscr{Q}_{p_{\rho_1}}^{\rho_1}(Z, \zeta, (j^k H)(0, u), (j^k \overline{H})(0, u), g(Z), \bar{g}(\zeta))\big|_{\mathcal{M}} = 0$. But by definition of p_{ρ_1}, we have $\mathscr{Q}_{p_{\rho_1}}^{\rho_1}(Z, \zeta, (j^k H)(0, u), (j^k \overline{H})(0, u), \xi', \varpi')\big|_{\mathcal{M}\times\mathbb{C}^{2r}} \not\equiv 0$. If $r = 0$, we immediately reach a contradiction. If $r \geq 1$, Proposition 2.4 applies and shows that the components of the mapping G are algebraically dependent over the field $\mathbb{K}^H$. This contradicts the fact that G is a transcendence basis of $\mathbb{K}^H(H)$ over $\mathbb{K}^H$. The proof of Proposition 3.3 is therefore complete. □

4. Proof of Theorem 1.1

In order to complete the proof of Theorem 1.1, we need the following result which follows from the work of Denef-Lipschitz [DL80] providing a positive answer to the so-called "nested approximation property" for solutions of analytic systems whose "nested" part depend only on one variable (see e.g. [R15, Theorem 10.5]). As mentioned in the introduction, such a result does no longer hold if the nested part of the formal solution depends on two variables or more.

Theorem 4.1. *Let* $\Phi \in (\mathbb{R}\{t,x,y\})^q$ *where* $t \in \mathbb{R}$, $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $y = (y_1, \ldots, y_m) \in \mathbb{R}^m$, $n, m, q \geq 1$. *Let* $\hat{y}(t,x) \in (\mathbb{R}[[t,x]])^m$, $\hat{y}(0) = 0$, *satisfying* $\Phi(t, x, \hat{y}(t,x)) = 0$. *Assume that* $\hat{y}(t,x)$ *is of the form* $(\hat{y}_0(t), \hat{y}_1(t,x)) \in \mathbb{R}^k \times \mathbb{R}^{m-k}$, *with* $k \in \{0, \ldots, m\}$. *Then for every integer* ℓ, *there exists* $y^\ell(t,x) \in (\mathbb{R}\{t,x\})^m$ *of the form* $(y_0^\ell(t), y_1^\ell(t,x))$ *satisfying* $\Phi(t, x, \hat{y}^\ell(t,x)) = 0$ *and agreeing with* $\hat{y}(t,x)$ *up to order* ℓ *(at* 0*).*

We now proceed to the proof of Theorem 1.1. First, note that in case $N = 1$, since any real-analytic curve in the complex domain is locally biholomorphically equivalent to a piece of the real line, the desired conclusion from a direct application of Artin's approximation theorem [A68]. From now, we may assume that $N \geq 2$.

Let $M \subset \mathbb{C}^N$ be a real-analytic CR submanifold satisfying the assumptions of Theorem 1.1. If the CR orbits have all the same dimension and are of codimension zero in M (i.e. M is everywhere minimal), the conclusion of Theorem 1.1 has been obtained in [MMZ03]. We will therefore assume that all CR orbits of M are of the same dimension and of codimension one in M. Assume, in addition, that M is generic. Let $p \in M$ and choose normal coordinates $Z = (z, w, u) \in \mathbb{C}^n \times \mathbb{C}^{d-c} \times \mathbb{C}^c$ as in Lemma 2.1, vanishing at p. Let $H \colon (\mathbb{C}^N, 0) \to \mathbb{C}^{N'}_{Z'}$ be a formal holomorphic map and $S' \subset \mathbb{C}^N \times \mathbb{C}^{N'}$ be a partially algebraic subset such that $\operatorname{Graph} H \cap (M \times \mathbb{C}^{N'}) \subset S'$. Using the notation introduced in previous sections, we denote by r the transcendence degree of the field extension $\mathbb{K}^H \hookrightarrow \mathbb{K}^H(H)$.

If $r = N'$, then $M \times \mathbb{C}^{N'} \subset S'$. Indeed, suppose that it is not the case. Then we can find a real-analytic function ψ near $(0, H(0))$ belonging the ring $\mathbb{R}\{Z, \overline{Z}\}[Z', \overline{Z}']$, vanishing on S' near $(0, H(0))$, such that

$$\psi(Z, \overline{Z}, Z', \overline{Z}')|_{M \times \mathbb{C}^{N'}} \not\equiv 0$$

near $(0, H(0))$. Since $\psi(Z, \zeta, H(Z), \overline{H}(\zeta))|_{\mathcal{M}} = 0$, Proposition 2.4 implies that the components of H are algebraically dependent over $\mathbb{K}^H$, which contradicts the fact that $r = N'$. Hence $M \times \mathbb{C}^{N'} \subset S'$, and, therefore, in order to approximate H in the Krull topology by a sequence of holomorphic mappings $h^\ell \colon (\mathbb{C}^N, 0) \to \mathbb{C}^{N'}$ satisfying $\operatorname{Graph} h^\ell \cap (M \times \mathbb{C}^{N'}) \subset S'$, it is enough to truncate the mapping H.

We may assume now that $r < N'$. Consider the system of real-analytic equations given by (3.4). Since all CR orbits of M are of codimension one in M we have $c = 1$. We may therefore use Theorem 4.1 in conjunction with Proposition 3.3 (i) to obtain the existence of a sequence of real-analytic mappings $\xi' = g^\ell(x, u)$, $\tau' = f^\ell(x, u)$, $\Lambda^k = T^\ell(u)$ converging as $\ell \to$

∞ to $G(x,u)$, $F(x,u)$ and $(j^kH)(0,u)$ respectively and satisfying the system (3.4). Complexify the sequence $h^\ell = (f^\ell, g^\ell)$ to get a sequence of germs at 0 of holomorphic mappings from $\mathbb{C}^N$ to $\mathbb{C}^{N'}$. Then h^ℓ converges to H as $\ell \to \infty$ in the Krull topology and by Proposition 3.3 (ii), it satisfies $\mathrm{Graph}\, h^\ell \cap (M \times \mathbb{C}^{N'}) \subset S'$ for ℓ large enough. This proves that M has the Artin approximation property when M is generic.

The general case when M is not necessarily generic can be reduced to the generic case following the same arguments as in [MMZ03]. The proof of Theorem 1.1 is complete.

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