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An algebraic characterization of holomorphic nondegeneracy for real algebraic hypersurfaces and its application to CR mappings

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Abstract. We give a new algebraic characterization of holomorphic nondegeneracy for embedded real algebraic hypersurfaces in $\mathbb{C}^{N+1}$, $N \geq 1$. We then use this criterion to prove the following result about real analyticity of smooth CR mappings : any smooth CR mapping H between a real analytic hypersurface and a rigid polynomial holomorphically nondegenerate hypersurface is real analytic, provided the map H is not totally degenerate in the sense of Baouendi and Rothschild.

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1 Introduction and statement of results

In [23] and [24], N. Stanton introduced a new geometric invariant for real analytic hypersurfaces in complex spaces. More precisely, an embedded real analytic hypersurface M of $\mathbb{C}^{N+1}$, $N \geq 1$ is called *holomorphically nondegenerate* at $p_0 \in M$ if there is no germ at p_0 of a non trivial holomorphic vector field tangent to M (see Sect. 2 for more details). By a real algebraic hypersurface in $\mathbb{C}^{N+1}$, we shall mean a real hypersurface contained in the zero set of a non-zero real polynomial. One of the goals of this paper is to give a new algebraic criterion of holomorphic nondegeneracy for all real algebraic hypersurfaces. With such a criterion, we shall be able to give new results about real analyticity of smooth CR mappings between real analytic hypersurfaces of the same dimension. We introduce now the main results of this paper.

Let (M, p_0) be a germ of a real algebraic hypersurface in $\mathbb{C}^{N+1}$. Let ρ be a real analytic defining function for M near p_0 such that $\rho(p_0, \bar{p_0}) = 0$ and $\mathrm{d}\rho(p_0, \bar{p_0}) \neq 0$. According to [8] and [2], we can find holomorphic local coordinates (z, w), a neighborhood Ω of 0 in $\mathbb{C}^{N+1}$ and a real analytic function φ defined in a neighborhood of 0 in $\mathbb{R}^{2N+1}$ such that p_0 is sent to 0 and such that (M, p_0) is given by

$$\Im\mathrm{m}\, w = \varphi(\Re\mathrm{e}\, w, z, \bar{z}), \quad (z, w) \in \Omega \tag{1}$$

with $\varphi(0) = \mathrm{d}\varphi(0) = 0$, $\varphi(s, 0, \bar{z}) = \varphi(s, z, 0) \equiv 0$, where $s = \Re\mathrm{e}\, w$. Such a choice of coordinates is called *normal*. Using the implicit function theorem and shrinking Ω if necessary, equation (1) is then equivalent to:

$$\bar{w} = Q(z, w, \bar{z}), \quad (z, w) \in \Omega \tag{2}$$

where $Q = Q(z, w, \xi)$ is a holomorphic function defined in a neighborhood of 0 in $\mathbb{C}^{2N+1}$ such that $Q(0) = 0$. Normality of coordinates also gives $Q(z, w, 0) = Q(0, w, \xi) = w$.

Write $Q(z, w, \xi) = \sum_{\beta \in \mathbb{N}^N} \rho_\beta(z, w)\xi^\beta$, where the functions ρ_β are holomorphic in a neighborhood of 0 in $\mathbb{C}^{N+1}$. Such a decomposition was introduced in [11], [10] and [5]. In the sequel, we will use the following notations: by $\mathcal{O}_{N+1}$, we denote the ring of germs at 0 in $\mathbb{C}^{N+1}$ of holomorphic functions; by $\mathcal{A}_{N+1}$, we denote the subring of $\mathcal{O}_{N+1}$ consisting of those germs at 0 in $\mathbb{C}^{N+1}$ of holomorphic functions which are algebraic over the field of rational functions $\mathbb{C}(Z_1, \ldots, Z_{N+1})$ and by $\mathcal{F}_{N+1}$ we mean the quotient field of $\mathcal{A}_{N+1}$, which can be identified with the field of germs at 0 in $\mathbb{C}^{N+1}$ of meromorphic functions algebraic over $\mathbb{C}(Z_1, \ldots, Z_{N+1})$. Furthermore, we shall put $w = z_{N+1}$, but we will use both notations for the same variable. To establish our first result, we will assume that (M, p_0) is algebraic. From this, we deduce that we can choose normal coordinates (z, w) such that Q is in $\mathcal{A}_{2n+1}$ [5]. As a consequence, the family $(\rho_\beta)_{\beta \in \mathbb{N}^N}$ is contained in $\mathcal{A}_{N+1}$. Let $\mathcal{K}(M)$ and $\mathcal{K}(M)\Big(Z_1, \ldots, Z_{N+1}\Big)$ denote respectively the smallest field contained in $\mathcal{F}_{N+1}$ containing $\mathbb{C}$ and the family $(\rho_\beta)_{\beta \in \mathbb{N}^N}$, and the smallest field contained in $\mathcal{F}_{N+1}$ and containing $\mathcal{K}(M)$ and the family $(Z_1, \ldots, Z_{N+1})$. We thus have the following canonical field extensions:

$$\mathbb{C} \subseteq \mathcal{K}(M) \subseteq \mathcal{K}(M)\Big(Z_1, \ldots, Z_{N+1}\Big) \subseteq \mathcal{F}_{N+1}. \tag{3}$$

To finish with these notations, define $\delta(M, p_0)$ by the formula:

$$\begin{aligned} \delta(M, p_0) &= [\mathcal{K}(M)\Big(Z_1, \ldots, Z_{N+1}\Big) : \mathcal{K}(M)] \\ &= \dim_{\mathcal{K}(M)} \mathcal{K}(M)\Big(Z_1, \ldots, Z_{N+1}\Big). \end{aligned}$$

With all this in mind, we have the following

Theorem 1.1 *Under the preceding notations, the following conditions are equivalent:*
i) (M, p_0) *is holomorphically nondegenerate;*
ii) $\mathcal{F}_{N+1}$ *is an algebraic extension over* $\mathcal{K}(M)$*;*
iii) $\mathcal{K}(M)\Big(Z_1, \dots, Z_{N+1}\Big)$ *is an algebraic extension over* $\mathcal{K}(M)$*;*
iv) $\mathcal{K}(M)\Big(Z_1, \dots, Z_{N+1}\Big)$ *is a finite extension over* $\mathcal{K}(M)$ *i.e.* $\delta(M, p_0) < \infty$.

This new characterization will be useful to prove holomorphic extendability of smooth CR mappings between some real analytic hypersurfaces in $\mathbb{C}^{N+1}$. In fact, we will consider CR mappings which are *not totally degenerate* in the sense of [4](see Sect. 2 for precise statements). We will prove the following

Theorem 1.2 *Let* (M, p_0) *and* (M', p'_0) *be two germs of real analytic hypersurfaces in* $\mathbb{C}^{N+1}$*,* $N \geq 1$*. Suppose that* (M', p'_0) *is given locally by*

$$\Im\mathrm{m}\, w' = p(z', \bar{z}'), \qquad (z', w') \in \mathbb{C}^{N+1} \tag{4}$$

where p *is a real polynomial such that* $p(0) = 0$*. Let* H *be a germ of a* C^∞ *smooth CR mapping between* M *and* M' *which is not totally degenerate at* p_0 *and such that* $H(p_0) = p'_0$*. If* (M', p'_0) *is holomorphically nondegenerate, then* H *extends holomorphically to a neighborhood of* p_0*.*

Theorem 1.2 would be a consequence of a recent result of S. Baouendi, X. Huang and L. Rothschild [1] if M was furthermore assumed to be algebraic. For an extensive survey about holomorphic extendability of CR mappings, we refer the reader to the excellent paper of Forstneric [13]. In the case where H is assumed to be a CR diffeomorphism, we obtain the following corollary:

Corollary 1 *Let* (M, p_0) *and* (M', p'_0) *be two germs of real analytic hypersurfaces in* $\mathbb{C}^{N+1}$*,* $N \geq 1$*. Suppose that* (M', p'_0) *is given locally by (4). If* (M', p'_0) *is holomorphically nondegenerate, then every germ of a* C^∞ *smooth CR diffeomorphism between* M *and* M' *which sends* p_0 *to* p'_0 *is in fact real analytic.*

2 Notations and definitions

2.1 Essential finiteness and holomorphic nondegeneracy

In this section, we recall briefly some basic definitions about the geometric concepts of essential finiteness and holomorphic nondegeneracy. For further details, see [11], [2], [10] and [23] [24].
Let M be a real analytic hypersurface in $\mathbb{C}^{N+1}$ and $p_0 \in M$. Let ρ be a

real analytic defining function for M near p_0. For each point v near p_0, we define a complex hypersurface called the *Segre variety* associated to v by

$$Q_v = \{p \in U \ / \ \rho(p, \bar{v}) = 0\},$$

where U is a small neighborhood of p_0 in $\mathbb{C}^{N+1}$. These invariant complex hypersurfaces appeared in different works concerning mapping problems such as in [26], [11], [2], [10], to name a few. M is then called *essentially finite* at p_0 (according to the terminology of [2]) if the map $v \to Q_v$ has finite fibers near p_0. One can show that it suffices to check this last condition only for the fiber concerning p_0 ([11]). Moreover, if M is given near p_0 by (2), it is easily seen that M is essentially finite at p_0 if the ideal generated by the ρ_β in $\mathcal{O}_{N+1}$ is of finite codimension [2] [3]. If this is so, we then define

$$esstype_{p_0}(M) = \dim_{\mathbb{C}} \mathcal{O}_{N+1} / \Big(\rho_\beta \Big).$$

This number is known to be an invariant of M (see [3] or [11] for a more geometric approach).
Now, recall that by a holomorphic vector field (defined in an open set Ω in $\mathbb{C}^{N+1}$), we mean a vector field of type $(1,0)$ with holomorphic coefficients in Ω. An embedded real hypersurface M in $\mathbb{C}^{N+1}$ is called *holomorphically nondegenerate* at $p_0 \in M$ if there is no germ at p_0 of a nontrivial holomorphic vector field tangent to M. It is known that if M is essentially finite at $p_0 \in M$, then M is holomorphically nondegenerate at this point. Moreover, if a connected real analytic hypersurface is holomorphically nondegenerate at one point, it is holomorphically nondegenerate at each of its point and the set of essentially finite points is open and dense in M. For more details, see [23], [24], [5].

2.2 Totally degenerate CR mappings

In theorem 1.2, we will deal with not totally degenerate CR mappings in the sense of [4]. We recall this nondegeneracy condition introduced by Baouendi and Rothschild. Let (M, p_0) and (M', p_0') two germs of real analytic embedded hypersurfaces in $\mathbb{C}^{N+1}$ and H a C^∞ smooth CR mapping between M and M' such that $H(p_0) = p_0'$. Suppose that M (resp. M') is given in the so-called *normal* form $\Im\mathrm{m}\, w = \varphi(z, \bar{z}, \Re\mathrm{e}\, w)$ with $\varphi(0) = \mathrm{d}\varphi(0) = 0$ and $\varphi(z, 0, \Re\mathrm{e}\, w) \equiv 0$ (we add the prime for M'). Write $H = (f_1, \ldots, f_N, g)$ in the (z', w') coordinates and consider $(z, \bar{z}, s)$ ($s = \Re\mathrm{e}\, w$) as local coordinates for M near p_0. For $j = 1, \ldots, N$, let $\Sigma_j(Z, \bar{Z}, S) \in \mathbb{C}[[Z, \bar{Z}, S]]$ the (formal) Taylor series associated to f_j at 0. It is proved in [3] that there exists a unique formal power series $F_j \in \mathbb{C}[[Z_1, \ldots, Z_N, W]]$ such that

$\Sigma_j(Z,\bar{Z},S) = F_j(Z,S+i\varphi(Z,\bar{Z},S))$ in $\mathbb{C}[[Z,\bar{Z},S]]$. Then H is *totally degenerate* at p_0 if $\det\left(\left(\frac{\partial F_i}{\partial Z_j}\right)_{i,j=1,\dots,N}\right)(Z,0) \equiv 0$ in $\mathbb{C}[[Z]]$. In [4], it is proved that the last condition is independent of the choice of normal coordinates.

3 Proof of Theorem 1.1

The proof of theorem 1.1 is based on the following proposition 1 whose proof can be found in [5] (see remarks after lemma 4.5). It gives us a criterion to know whether or not a real analytic hypersurface is holomorphically nondegenerate.

Proposition 1 *With the previous notations, the following conditions are equivalent:*
i) (M,p_0) *is holomorphically nondegenerate;*
ii) there exists $(\beta_1,\dots,\beta_{N+1}) \in (\mathbb{N}^N)^{N+1}$ *such that*

$$\det\Big((\frac{\partial \rho_{\beta_i}}{\partial z_j})_{i,j=1,\dots,N+1}\Big) \not\equiv 0.$$

Before proving theorem 1.1, we need to recall some basic facts from field theory which can be found in [20] [27] or [14]. We assume that K and k are two fields and that K is a field extension of k. A *finite* subset $S = \{s_1,\dots,s_p\}$ $(p \in \mathbb{N}^*)$ of K is called *algebraically independent over* k if $(\ P \in k[X_1,\dots,X_p]\ and\ P(s_1,\dots,s_p) = 0 \Rightarrow P \equiv 0)$ is true. A subset S of K is called *algebraically independent over* k if every finite subset of S is algebraically independent in the preceding sense. A subset $S \subseteq K$ is called a *transcendence basis of* K/k if
i) S is algebraically independent over k
ii) K is an algebraic extension over $k(S)$(here $k(S)$ denotes the smallest field contained in K and containing k and S).

Proof of Theorem 1.1:
$(iii) \Rightarrow (iv)$: Since $\mathcal{K}(M)\Big(Z_1,\dots,Z_{N+1}\Big)$ is a finitely generated extension over $\mathcal{K}(M)$, if $\mathcal{K}(M)\Big(Z_1,\dots,Z_{N+1}\Big)$ is an algebraic extension over $\mathcal{K}(M)$, then a standard result from field theory ([20] prop. 1.21 p.10) asserts that $\mathcal{K}(M)\Big(Z_1,\dots,Z_{N+1}\Big)$ is a finite extension over $\mathcal{K}(M)$, i.e. $\dim_{\mathcal{K}(M)}\mathcal{K}(M)\Big(Z_1,\dots,Z_{N+1}\Big) < \infty$ (as a $\mathcal{K}(M)$-vector space).
$(iv) \Rightarrow (iii)$: In commutative algebra, it is well known that a finite extension is algebraic ([20] lemma 1.19 p.9).

It is clear that (ii) implies (iii). Let us show the converse. For this, it suffices to note that from (iii), we know that all elements of $\mathbb{C}(Z_1, \dots, Z_{N+1})$ (considered as elements of $\mathcal{F}_{N+1}$) are algebraic over $\mathcal{K}(M)$; since all elements of $\mathcal{F}_{N+1}$ are algebraic over $\mathbb{C}(Z_1, \dots, Z_{N+1})$, we deduce (ii) thanks to the transitivity of the property of being algebraic ([27] p.61).
$(i) \iff (ii)$: According to [27] (p.99 corollary 2 and p.96 after definition 1), $\mathcal{F}_{N+1}$ is an algebraic extension over $\mathcal{K}(M)$ if and only if the family $(\rho_\alpha)_{\alpha \in \mathbb{N}^N}$ contains a transcendence basis of $\mathcal{F}_{N+1}/\mathbb{C}$. Since all transcendence basis have the same cardinal ([20] p.178 and [27] p.99) and since $(Z_1, \dots, Z_{N+1})$ is a transcendence basis of $\mathcal{F}_{N+1}/\mathbb{C}$ (recall that all elements of $\mathcal{F}_{N+1}$ are algebraic over $\mathbb{C}(Z_1, \dots, Z_{N+1})$), we obtain that $\mathcal{F}_{N+1}$ is an algebraic extension over $\mathcal{K}(M)$ if and only if there exists $(\alpha_1, \dots, \alpha_{N+1}) \in (\mathbb{N}^N)^{N+1}$ with $\alpha_i \in \mathbb{N}^N$ for $i = 1, \dots, N$, such that $(\rho_{\alpha_1}, \dots, \rho_{\alpha_{N+1}})$ is a transcendence basis of $\mathcal{F}_{N+1}/\mathbb{C}$. Since $\mathbb{C}$ is of characteristic zero, using corollary 23.16 p.216 and proposition 23.17 p.217 of [20] (or theorem II p.134 of [14], volume 1), we obtain that the last assertion is equivalent to the existence of a $N+1$-uple $(\alpha_1, \dots, \alpha_{N+1}) \in (\mathbb{N}^N)^{N+1}$ such that:

$$\det\Big((\frac{\partial \rho_{\alpha_i}}{\partial Z_j})_{i,j=1,\dots,N+1} \Big)(Z) \not\equiv 0.$$

It suffices now to apply proposition 1 to obtain the desired result.

We conclude this section with some examples, but first of all we would like to point out the fact that when (M, p_0) is a germ of a holomorphically nondegenerate real algebraic hypersurface, $\mathcal{F}_{N+1}$ is far away from being a finite extension over $\mathcal{K}(M)$. In most of the following examples, the field $\mathcal{K}(M)$ will be contained in the field of rational functions over $\mathbb{C}$. The fact that $\mathcal{F}_{N+1}$ is not a finite extension over $\mathbb{C}(Z_1, \dots, Z_{N+1})$ justifies the above remark.

$i)$ $M := \Im\mathrm{m}\, w = |z_1|^{2k_1} + |z_1 z_2|^{2k_2}, (z, w) \in \mathbb{C}^3, k_i \in \mathbb{N}^*, i = 1, 2.$
Here, it is easy to see M is not essentially finite at 0 in the sense of [2] (see also [11] [10]). However, since $P(z_1) = 0$ and $Q(z_2) = 0$ where $P(T) = T^{k_1} - z_1^{k_1}$ and $Q(T) = (z_1^{k_1})^{k_2} T^{k_1 k_2} - ((z_1 z_2)^{k_2})^{k_1}$, according to theorem 1.1, M is holomorphically nondegenerate at 0. Furthermore, to obtain the number $\delta(M, 0)$ previously defined, it suffices to use the following algebraic formula ([27]) :

$$\begin{aligned} \delta(M,0) = & [\mathbb{C}(Z_1, Z_2) : \mathbb{C}\Big(Z_1^{k_1}, (Z_1 Z_2)^{k_2}\Big)(Z_1)] \\ & \times [\mathbb{C}\Big(Z_1^{k_1}, (Z_1 Z_2)^{k_2}\Big)(Z_1) : \mathbb{C}(Z_1^{k_1}, (Z_1 Z_2)^{k_2})], \end{aligned}$$

and the last number is for example given by the degree of the minimal polynomial of Z_1 with coefficients in $\mathbb{C}(Z_1^{k_1}, (Z_1 Z_2)^{k_2})$. We thus obtain $\delta(M, 0) = k_1 k_2$.

ii) $M := \Im\mathrm{m}\, w = |z_1|^2 + |z_2|^2,\ (z_1, z_2, w) \in \mathbb{C}^3$. M is strictly pseudoconvex near 0 and $esstype_0(M) = \delta(M, 0) = 1$.

iii) $M := \Im\mathrm{m}\, w = |z_1|^4 + |z_2|^4 + (\Re\mathrm{e}\, w)|z_1|^2,\ (z_1, z_2, w) \in \mathbb{C}^3$. In normal form, M is also given by : $\bar{w} = \frac{w(1 - i|z_1|^2) - 2i(|z_1|^4 + |z_2|^4)}{1 + i|z_1|^2}$. It is clear that M is essentially finite at 0 (hence holomorphically nondegenerate at 0, see [23]), but it can easily be shown that $esstype_0(M) = 4$ and $\delta(M, 0) = 2$.

iv) $M := \Im\mathrm{m}\, w = (\Re\mathrm{e}\, w)|z|^6$, $(z, w) \in \mathbb{C}^2$. Here, $esstype_0(M) = \infty$ since M is not of finite type in the sense of Kohn [15] (recall that the notions of finite type in the sense of Kohn and essential finiteness are the same in $\mathbb{C}^2$). Nevertheless, M is also given by $\bar{w} = w(1 + 2\sum_{n=1}^{\infty}(-i|z|^6)^n$, and consequently M is holomorphically nondegenerate at 0 and $\delta(M, 0) = 3$.

v) An example of a rigid essentially finite algebraic hypersurface with $esstype_0(M) \neq \delta(M, 0)$. $M := \Im\mathrm{m}\, w = |z_1|^8 + |z_2|^2 + |z_1 z_2|^2, (z_1, z_2, w) \in \mathbb{C}^3$. One can check that $esstype_0(M) = 4$ whereas $\delta(M, 0) = 1$.

vi) (the light cone [12]) Let $M := (\Im\mathrm{m}\, Z_1)^2 + (\Im\mathrm{m}\, Z_2)^2 - (\Im\mathrm{m}\, Z_3)^2 = 0$ near the point $p_0 = (0, i, i)$ in $\mathbb{C}^3$. We can find local holomorphic coordinates (z, w) such that p_0 is sent to 0 and such that M is given near p_0 by $w - \bar{w} = -2i + \Big((z_1 - \bar{z}_1)^2 + (z_2 - \bar{z}_2 + 2i)^2\Big)^{\frac{1}{2}}$ ([12]). Taking new coordinates so that M is in normal form, one can show that M is given by $\bar{w} = Q(z, w, \bar{z})$ with $Q(z, w, \xi) \in \mathcal{A}_5$ and $\rho_1(z, w) = \dfrac{\partial Q}{\partial \xi_1}(z, w, 0) = \dfrac{z_1}{(z_1{}^2 + (z_2 + 2i)^2)^{\frac{1}{2}}}$, $\rho_2(z, w) = Q_{\xi_2}(z, w, 0) = -1 + \dfrac{z_2 + 2i}{(z_1{}^2 + (z_2 + 2i)^2)^{\frac{1}{2}}}$, and $\rho_{21}(z, w) = Q_{\xi_2 \xi_1}(z, w, 0) = \dfrac{z_1(z_2 + 2i)}{(z_1{}^2 + (z_2 + 2i)^2)^{\frac{3}{2}}}$.
To see that M is holomorphically nondegenerate at p_0, it suffices to note the fact that the following identities $\rho_{21} Z_1 + (\rho_1)^2(1 + \rho_2) \equiv 0$, $\rho_{21} Z_2 + 2i\rho_{21} - (\rho_2 + 1)^2 \rho_1 \equiv 0$ hold and then to apply theorem 1.1. The last equalities also give the fact that $\delta(M, p_0) = 1$.

4 Proof of Theorem 1.2

We first introduce some notations.
If M' is given locally by (4), it is easily seen that after a holomorphic change of variables, one can assume that M' is given by $\Im\mathrm{m}\, w_1 = q(z_1, \bar{z_1})$ where q is a real polynomial with no pure terms. Thus, we can suppose

that in equation (4), p is a real polynomial with no pluriharmonic terms. We can then write $p(z',\bar{z}') = \sum_{1\le|\alpha|\le r} a_\alpha(z')\bar{z}'^\alpha$, where $r\in\mathbb{N}^*$, a_α is a holomorphic polynomial and $a_\alpha(0)=0$, for every α such that $1\le|\alpha|\le r$. Now, combining theorem 1.1 and the normal form of M', it is easy to see that M' is holomorphically nondegenerate at 0 if and only if the field of rational functions over $\mathbb{C}$ is a finite algebraic extension over $\mathcal{K}(M')$, where $\mathcal{K}(M')$ denotes the smallest field contained in $\mathbb{C}(Z_1,\dots,Z_N)$ and containing $\mathbb{C}$ and the family $(a_\alpha)_{1\le|\alpha|\le r}$. We now turn to the proof of theorem 1.2. Let H be a germ of a C^∞ smooth CR mapping which is not totally degenerate at p_0. Write $H=(f,g)=(f_1,\dots,f_N,g)$ in the (z',w') coordinates. The component g is called the transversal component of H. To prove theorem 1.2, we will first prove that the functions g and $a_\alpha(f)$ extend holomorphically to a neighborhood of p_0. We will then use the algebraic criterion obtained in theorem 1.1 to obtain the desired result. Note that a similar procedure has been used by M. Derridj in [9].

Proposition 2 *With the previous notations and hypothesis, the functions g and $a_\alpha(f)$ for $1\le|\alpha|\le r$ extend holomorphically to a (common) neighborhood of p_0 in $\mathbb{C}^{N+1}$.*

We will suppose that (M,p_0) is given by the real analytic parameterization (1), and we will use $(z,\bar{z},s)$ ($s=\Re\mathrm{e}\, w$) as local coordinates. Since $H(M)\subseteq M'$, there exists a neighborhood O sufficiently small of p_0 in M such that the following identity holds on O :

$$\bar{g}-g=-2i\sum_{1\le|\alpha|\le r} a_\alpha(f)\bar{f}^\alpha \tag{5}$$

Let ψ be the entire function defined by $\psi(z,\xi)=2i\sum_{1\le|\alpha|\le r} a_\alpha(z)\xi^\alpha$, $(z,\xi)\in\mathbb{C}^{2N}$. Define also the following basis of CR vector fields for M near p_0:

$$\bar{L}_j=\frac{\partial}{\partial\bar{z}_j}-2i\frac{\varphi_{\bar{z}_j}}{1+i\varphi_s}\frac{\partial}{\partial\bar{w}}\qquad j=1,\dots,N.$$

Put $d=\det((\bar{L}_j\bar{f}_k)_{j,k=1,\dots,N})$ and $\bar{L}^\beta=\bar{L}_1^{\beta_1},\dots,\bar{L}_N^{\beta_N}$ for each multi-index $\beta=(\beta_1,\dots,\beta_N)\in\mathbb{N}^N$. To prove proposition 2, we will need the following two lemmas. The first one can be found in [3].

Lemma 1 *There exists a family $(R_\alpha)_{1\le|\alpha|\le r}$ of holomorphic polynomials such that :*
i) $R_\alpha\in\mathbb{C}[T_1,\dots,T_{r(\alpha)}]$ where $r(\alpha)=m(\alpha)N+m(\alpha)$ and $m(\alpha)$ is defined by $\mathrm{card}\{\beta,\ 1\le|\beta|\le|\alpha|\}$
ii) The following identity holds on O:

$$d^{2|\alpha|-1}\psi_{\xi_\alpha}(f,\bar{f})=R_\alpha\Big((\bar{L}^\beta\bar{f})_{1\le|\beta|\le|\alpha|},(\bar{L}^\beta\bar{g})_{1\le|\beta|\le|\alpha|}\Big).$$

Lemma 2 *There exists a neighborhood O' of p_0 in M, a family of positive integers $(n_\alpha)_{1\leq|\alpha|\leq r}$ and two families of holomorphic polynomials $(S_\alpha)_{1\leq|\alpha|\leq r}$ and $(W_\alpha)_{1\leq|\alpha|\leq r}$ such that:*
i) $\forall\alpha\; S_\alpha \in \mathbb{C}[T_1,\ldots,T_{N+Nq(\alpha)+q(\alpha)}]$, *where* $q(\alpha)=\operatorname{card}\{\beta,\ 1\leq|\beta|\leq n(\alpha)\}$*;*
ii) $\forall\alpha\; W_\alpha \in \mathbb{C}[T_1,\ldots,T_{Nq(\alpha)}]$*;*
iii) $\forall\alpha\; W_\alpha\Big((\bar{L}^\beta\bar{f})_{1\leq|\beta|\leq n(\alpha)}\Big)$ *does not vanish on* O'*;*
iv) $a_\alpha(f)=\dfrac{S_\alpha\Big((\bar{L}^\beta\bar{f})_{0\leq|\beta|\leq n(\alpha)},(\bar{L}^\beta\bar{g})_{1\leq|\beta|\leq n(\alpha)}\Big)}{W_\alpha\Big((\bar{L}^\beta\bar{f})_{1\leq|\beta|\leq n(\alpha)}\Big)}$ on O'.

Proof of Lemma 2: In fact, we will show (iv) in a neighborhood O'_α; it will then suffices to take for O' the neighborhood $\cap_{1\leq|\alpha|\leq r}O'_\alpha$. We show this lemma by induction and we begin with $|\alpha|=r$. For α such that $|\alpha|=r$, according to lemma 1 and since $\psi_{\xi_\alpha}(f,\bar{f})=2i\ \alpha!\ a_\alpha(f)$, we have on O :

$$2i\ d^{2r-1}\alpha!\ a_\alpha(f)=R_\alpha\Big((\bar{L}^\beta\bar{f})_{1\leq|\beta|\leq r},(\bar{L}^\beta\bar{g})_{1\leq|\beta|\leq r}\Big). \tag{6}$$

Since the map H is not totally degenerate at p_0, according to lemmas 3.18 and 3.19 of [3], there exists a multi-index γ such that $\bar{L}^\gamma(d)(p_0)\neq 0$. Consequently, there exists a multi-index γ' such that $\bar{L}^{\gamma'}(d^{2r-1})(p_0)\neq 0$. Let $O'_r=O'_{|\alpha|}$ be a neighborhood of p_0 in M such that $\bar{L}^{\gamma'}(d^{2r-1})(p)\neq 0,\ \forall p\in O'_r$. So, for $|\alpha|=r$, taking $n(\alpha)$ large enough and putting

$$\begin{aligned}&S_\alpha\Big((\bar{L}^\beta\bar{f})_{0\leq|\beta|\leq n(\alpha)},(\bar{L}^\beta\bar{g})_{1\leq|\beta|\leq n(\alpha)}\Big)=\\&\bar{L}^{\gamma'}\Big(R_\alpha\Big((\bar{L}^\beta\bar{f})_{1\leq|\beta|\leq r},(\bar{L}^\beta\bar{g})_{1\leq|\beta|\leq r}\Big)\Big)\\&W_\alpha\Big((\bar{L}^\beta\bar{f})_{1\leq|\beta|\leq n(\alpha)}\Big)=2i\ \alpha!\ \bar{L}^{\gamma'}(d^{2r-1}),\end{aligned}$$

we obtain the desired result. Suppose the result established for $p+1\leq|\gamma|\leq r$, with $1\leq p\leq r-1$. Let α such that $|\alpha|=p$. Using lemma 1 for α, we get on O

$$d^{2|\alpha|-1}\psi_{\xi_\alpha}(f,\bar{f})=R_\alpha\Big((\bar{L}^\beta\bar{f})_{1\leq|\beta|\leq|\alpha|},(\bar{L}^\beta\bar{g})_{1\leq|\beta|\leq|\alpha|}\Big).$$

But we have $\psi_{\xi_\alpha}(f,\bar{f})=2i\ \alpha!\ a_\alpha(f)+C_\alpha\Big((a_\beta)_{p+1\leq|\beta|\leq r},\bar{f}\Big)$, where C_α is a holomorphic polynomial. We thus obtain on O:

$$\begin{aligned}d^{2|\alpha|-1}2i\ \alpha!\ a_\alpha(f)=&R_\alpha\Big((\bar{L}^\beta\bar{f})_{1\leq|\beta|\leq|\alpha|},(\bar{L}^\beta\bar{g})_{1\leq|\beta|\leq|\alpha|}\Big)\\&-d^{2|\alpha|-1}\ C_\alpha\Big((a_\beta(f))_{p+1\leq|\beta|\leq r},\bar{f}\Big)\end{aligned} \tag{7}$$

We use the same procedure as the one used just before. Let γ be a multi-index such that $\bar{L}^{\gamma}(d^{2|\alpha|+1})(p_0) \neq 0$. Applying $\bar{L}^{\gamma}$ to equation (7) and using the induction hypothesis, we then have on the neighborhood $U = \cap_{|\beta|=p+1}^{r} O'_{\beta}$:

$$\bar{L}^{\gamma}(d^{2|\alpha|-1})\, 2i\, \alpha!\, a_{\alpha}(f) = \bar{L}^{\gamma}\Big(R_{\alpha}\Big((\bar{L}^{\beta}\bar{f})_{1\leq|\beta|\leq|\alpha|}, (\bar{L}^{\beta}\bar{g})_{1\leq|\beta|\leq|\alpha|}\Big)\Big) \\ -\bar{L}^{\gamma}\Big(d^{2|\alpha|-1}C_{\alpha}(u,\bar{f})\Big) \qquad (8)$$

where $u = \left(\frac{S_{\beta}\Big((\bar{L}^{\delta}\bar{f})_{0\leq|\delta|\leq n(\beta)},(\bar{L}^{\delta}\bar{g})_{1\leq|\delta|\leq n(\beta)}\Big)}{W_{\beta}\Big((\bar{L}^{\delta}\bar{f})_{1\leq|\delta|\leq n(\beta)}\Big)}\right)_{p+1\leq|\beta|\leq r}$.

It becomes now clear that taking $n(\alpha)$ large enough and a neighborhood of p_0 $O'_{\alpha} \subseteq U$ such that $\bar{L}^{\gamma}(d^{2|\alpha|-1})(p) \neq 0$ for $p \in O'_{\alpha}$, $a_{\alpha}(f)$ will satisfy (iv) with all the properties (i),(ii) and (iii).

Proof of Proposition 2: Since (M', p'_0) is holomorphically nondegenerate, M' is of finite type in the sense of Kohn ([15]) and Bloom-Graham ([6]). Indeed, if it was not the case, we would have $p(z', \bar{z}') \equiv 0$ and then clearly $M' := \Im\mathrm{m}\, w' = 0$ would be holomorphically degenerate at 0. Consequently, M' is of finite type at p'_0 and H is not totally degenerate. We then use proposition 3.28 of [4], to assert that M is of finite type (in the sense of Bloom-Graham) at p_0. Recall that this last property is equivalent to the non existence of a germ of a complex hypersurface included in M through p_0, since M is real analytic([6]). Now, we can apply Trépreau's theorem [25] to H which is C^{∞} on M. Consequently, there exists a neighborhood Ω of 0 in $\mathbb{C}^{N+1}$ such that $\Omega \cap M \subseteq O \cap O'$ and such that if we put $\Omega^{-} = \{\, (z,w) \in \Omega \,/\, \Im\mathrm{m}\, w < \varphi(\Re\mathrm{e}\, w, z, \bar{z}) \,\}$ and $\bar{\Omega}^{-} = \{\, (z,w) \in \Omega \,/\, \Im\mathrm{m}\, w \leq \varphi(\Re\mathrm{e}\, w, z, \bar{z}) \,\}$, the following holds:
there exists a map $\mathcal{H} = (\mathcal{F}_1, \ldots, \mathcal{F}_N, \mathcal{G})$ holomorphic on Ω^{-}, $C^{\infty}(\bar{\Omega}^{-})$ such that $\mathcal{H} = H = (f,g)$ on $M \cap \Omega$. Note that the side of extension of H has no importance in the sequel of the proof. Now, it follows easily from part $iv)$ of lemma 2 and the original Lewy-Pinchuk reflection principle (involving the real analyticity of M and Morera's theorem, see [22] [16] [21] for more details) that $a_{\alpha}(f)$ extends holomorphically to a neighborhood of p_0, for every α such that $1 \leq |\alpha| \leq r$. Replacing in equation (5) the functions $a_{\alpha}(f)$ by their values given by lemma 2, a similar procedure such as the one done just before gives the holomorphic extendability of the transversal component g near p_0, which is equivalent to its real analyticity (see [7]).

We now turn to the end of the proof of theorem 1.2. For this, we will need the following lemma, whose proof can be found in [1] and which is a consequence of a theorem of Malgrange.

Lemma 3 *Let $G = G(z, w)$ be a holomorphic function defined in a neighborhood of 0 in $\mathbb{C}^{p+1}$ $(p \in \mathbb{N})$ such that $G \not\equiv 0$. If f is a C^∞ smooth function in a neighborhood of 0 in $\mathbb{R}^p$ and satisfies the following identity in a neighborhood of 0 in $\mathbb{R}^p$*

$$G(x, f(x)) \equiv 0,$$

then f is real analytic near 0.

End of the proof of Theorem 1.2: Since M' is holomorphically nondegenerate at p_0', according to part (ii) of theorem 1.1 and the remarks beginning this section, the fact that the field of rational functions is an algebraic extension over $\mathcal{K}(M)$ is equivalent to the following fact:
For every $i = 1, \dots, N$, there exists positive integers k_i and two families of holomorphic polynomials $\Big((A_i^q)_{q=0,\dots,k_i-1}\Big)_{i=1,\dots,N}$ and $\Big((B_i^q)_{q=0,\dots,k_i-1}\Big)_{i=1,\dots,N}$, elements of $\mathbb{C}[T_1, \dots, T_m]$ where $m = \mathrm{card}\{\alpha,\ 1 \le |\alpha| \le r\}$ such that :
$i)$ $B_i^q\Big((a_\alpha(Z))_{1\le|\alpha|\le r}\Big) \not\equiv 0$,
$ii)$ $Z^{k_i} + \sum_{j=0}^{k_i-1} \frac{A_i^j\left((a_\alpha(Z))_{1\le|\alpha|\le r}\right)}{B_i^j\left((a_\alpha(Z))_{1\le|\alpha|\le r}\right)} Z_i^j \equiv 0,\ \forall i = 1, \dots, N.$
Consequently, there exists holomorphic polynomials $P_i^j \in C[T_1, \dots, T_m]$ with $0 \le j \le k_i,\ i = 1, \dots, N$ such that $\sum_{j=0}^{k_i} P_i^j\bigg(\Big((a_\alpha)_{1\le|\alpha|\le r}\Big)(Z)\bigg) Z_i^j$ $\equiv 0$, with $P_i^{k_i}\Big((a_\alpha)_{|\alpha|\le r})(Z)\Big) \not\equiv 0$ for all $i = 1, \dots, N$. Considering $(z, \bar{z}, s)$ $(s = \Re\mathrm{e}\, w)$ as local coordinates for M, we obtain

$$\sum_{j=0}^{k_i} P_i^j\bigg(\Big(a_\alpha(f)(z, \bar{z}, s)\Big)_{1\le|\alpha|\le r}\bigg) f_i^j(z, \bar{z}, s) = 0,$$

in a suitable neighborhood Δ of 0 in $\mathbb{R}^{2N+1}$ (neighborhood where the functions $a_\alpha(f)$ are real analytic for every $1 \le |\alpha| \le r$). Define $\Theta_\alpha = a_\alpha(f)$. Then Θ_α is real analytic in Δ; we complexify Θ_α to obtain a holomorphic function defined in a neighborhood of 0 in $\mathbb{C}^{2N+1}$, which will be still denoted by Θ_α. For (τ, σ) in a small neighborhood of 0 in $\mathbb{C}^{2N+1} \times \mathbb{C}$, define the following holomorphic functions:

$$G_i(\tau, \sigma) = \sum_{j=0}^{k_i} P_i^j\bigg(\Big(\Theta_\alpha(\tau)\Big)_{1\le|\alpha|\le r}\bigg)\sigma^j, \quad \text{for} \quad i = 1, \dots, N.$$

To prove that f_i for $i = 1, \dots, N$ is real analytic it suffices to show, according to lemma 3, the following conditions: $G_i \not\equiv 0$ for all $i = 1, \dots, N$. Suppose that there exists i_0 such that $G_{i_0} \equiv 0$. We then would have

$$P_{i_0}^{k_{i_0}}\left(\left(a_\alpha(f)\right)_{1\leq|\alpha|\leq r}\right) \equiv 0,$$

in a neighborhood of p_0 in M. The last equality is also equivalent to $Q_{i_0}(f) \equiv 0$, where Q_{i_0} is a holomorphic polynomial not identically zero according to our hypothesis. Consequently, if $\Sigma(Z,\bar{Z},S) = (\Sigma_1(Z,\bar{Z},S), \dots, \Sigma_N(Z,\bar{Z},S))$ denotes the formal power series at 0 associated to $f = (f_1, \dots, f_N)$, we would obtain $Q_{i_0}(\Sigma(Z,\bar{Z},S)) \equiv 0 \quad \text{in } \mathbb{C}[[Z,\bar{Z},S]]$. But if $F_j(Z,W) \in \mathbb{C}[[Z,W]]$ denotes the unique formal power series associated to $\Sigma_j(Z,\bar{Z},S)$ such that $F_j\left(Z, S+i\varphi(Z,\bar{Z},S)\right) = \Sigma_j(Z,\bar{Z},S)$ in $\mathbb{C}[[Z,\bar{Z},S]]$, the last equation would lead to $Q_{i_0}\left(F(Z,S+i\varphi(Z,\bar{Z},S)\right) \equiv 0$. Replacing 0 to S and $\bar{Z}$, we then would get

$$Q_{i_0}\left(F(Z,0)\right) \equiv 0,$$

which contradicts the fact that H is not totally degenerate according to [4](p.491). Thus, $H = (f,g)$ is real analytic near p_0.

Remarks: 1. After having announced our results, Joël Merker and Francine Meylan in [17] gave independently a proof of theorem 1.2.
2. It is worth noticing that the proof of theorem 1.2 can give also the following result. If one assumes in the statement of theorem 1.2 that M' is of finite type in the sense of Kohn and Bloom-Graham at p'_0 (i.e. p not pluriharmonic) instead of the assumption of holomorphic nondegeneracy, then a similar procedure as the one done for the proof of theorem 1.2 gives the following: for every polynomial Q which is algebraic over $\mathcal{K}(M')$, there exists a neighborhood of p_0 such that $Q(f)$ extends holomorphically to this neighborhood. Note that this is much more than the extendability of the so-called reflection function ([3] [17]), as shown by the following example in $\mathbb{C}^5$. If $M' := \Im\mathrm{m}\, w' = \sum_{j=1}^{3} |\prod_i^j z_i'|^{2k_j}, k_j \in \mathbb{N}^*, (z_1', z_2', z_3', z_4', w') \in \mathbb{C}^5$, then the reflection function gives the extendability of $f_1^{k_1}, (f_1 f_2)^{k_2}, (f_1 f_2 f_3)^{k_3}$ and the normal component g whereas our method gives the extendability of f_1, f_2, f_3 and g. Note that here we do not have any information for the component f_4.
3. The results of the paper were announced in the note [18]. Theorem 1.1 has a obvious generalization to any algebraic generic Cauchy-Riemann manifold. Moreover, a reflection principle such that theorem 1.2 also holds for such manifolds. For further details, see [19].

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