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The finite jet determination problem for CR maps of positive codimension into Nash manifolds

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Abstract

We prove the first general finite jet determination result in positive codimension for CR maps from real-analytic minimal submanifolds $M \subset \mathbb{C}^N$ into Nash (real) submanifolds $M' \subset \mathbb{C}^{N'}$. For a sheaf $\mathcal{S}$ of C^∞-smooth CR maps from M into M', we show that the non-existence of so-called *2-approximate CR $\mathcal{S}$-deformations* from M into M' implies the following strong finite jet determination property: There exists a map $\ell : M \to \mathbb{Z}_+$, bounded on compact subsets of M, such that for every point $p \in M$, whenever f, g are two elements of $\mathcal{S}_p$ with $j_p^{\ell(p)} f = j_p^{\ell(p)} g$, then $f = g$. Applying the deformation point of view allows a unified treatment of a number of classes of target manifolds, which includes, among others, strictly pseudoconvex, Levi–non-degenerate, but also some particularly important Levi-degenerate targets, such as boundaries of classical domains.

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1 | INTRODUCTION

If one considers a germ of a holomorphic map $H : (\mathbb{C}^N, 0) \to \mathbb{C}^{N'}$, then one can of course think about it as being given by an N'-tuple of power series $H(Z) = (H^1(Z), \dots, H^{N'}(Z)) \in \mathbb{C}\{Z\}^{N'}$. The coefficients of those power series are completely free (besides the fact that their moduli are only allowed to increase at most geometrically). If one looks at holomorphic maps sending real submanifolds into one another, the simplest possible example $N = N' = 1$, and $\mathbb{R} \subset \mathbb{C}$, does not put too many restrictions on the coefficients of any map $H(z) = \sum_j H_j z^j$ sending the real line into itself either: There exists a neighbourhood $U \subset \mathbb{C}$ of the origin such that $H(U \cap \mathbb{R}) \subset \mathbb{R}$ if (and only if) $H_j = \overline{H_j}$, for all j. However, as a consequence of Cartan's work in [8, 9] if $N = N' = 2$, and one considers the submanifold M to be a piece of the unit sphere (or more generally, any strictly pseudoconvex hypersurface), then the derivatives of order 2 at any fixed point $p \in M$ of any local biholomorphism H sending M into itself, uniquely determine H. This is, historically speaking, the first example of *finite jet determination* for holomorphic maps between real submanifolds $M \subset \mathbb{C}^N$ and $M' \subset \mathbb{C}^{N'}$. Our main focus in this paper is the positive codimensional case $N' > N$; in order to formulate and put our results in perspective, let us first recall some standard notation to be used throughout the paper.

Given real-analytic submanifolds $M \subset \mathbb{C}^N$ and $M' \subset \mathbb{C}^{N'}$, with $N, N' \geqslant 2$, we denote by $\mathcal{A}^{\infty}_{\mathrm{CR}}(M, M')$ and $\mathcal{A}^{\omega}_{\mathrm{CR}}(M, M')$ the sheaf of germs of C^∞-smooth and real-analytic CR maps from M into M'. Given a subsheaf $\mathcal{S} \subset \mathcal{A}^{\infty}_{\mathrm{CR}}(M, M')$, we say that $(\mathcal{S}, M, M')$ has the *strong finite jet determination property* if for every compact $K \subset M$, there exists an $\ell = \ell(K)$ such that the ℓ-jet mapping j^ℓ_p is injective on $\mathcal{S}_p$, for all $p \in K$ (see, for example, [15] for the standard notion of ℓ-jet map.)

There is an abundant literature on the finite jet determination property in the setting $N = N'$; we refer the reader to Juhlin's paper [18] and the references therein for real-analytic CR submanifolds, where the problem is by now pretty well understood. For smooth CR manifolds, an account of recent progress can be found in Bertrand's paper [7]. In contrast, for mappings of positive codimension $N' - N > 0$, not much is known so far about the (strong) finite jet determination property. This is mostly due to the fact there are a number of challenges that need to be overcome compared to the equidimensional case, such as the unavailability of the jet parametrization technique or the existence of several 'degeneracy classes' for CR maps. Quite restrictive results were obtained in [12, 19] and it is only very recently that a way around this problem has been found for the model case of sphere targets in [27]. We refer the reader to this paper for a thorough discussion.

Sphere targets are interesting because they are in some sense the 'flat' model manifolds for CR geometry of strictly pseudoconvex hypersurfaces. However, most (even strictly pseudoconvex) manifolds cannot be embedded into spheres; for those, one needs more general target models. In the present paper, we provide the first general finite jet determination result in positive codimension for CR maps from real-analytic minimal submanifolds $M \subset \mathbb{C}^N$ into Nash (real) submanifolds $M' \subset \mathbb{C}^{N'}$, that is, semi-algebraic real-analytic submanifolds (see [1]). This is done by considering, for every subsheaf $\mathcal{S} \subset \mathcal{A}^{\infty}_{\mathrm{CR}}(M, M')$, what we call *2-approximate CR $\mathcal{S}$-deformations*, introduced properly in Definition 3.1. These objects are closely related to deformations recently appearing in the study of regularity properties of CR maps [20–23, 26]. Applying the deformation point of view to the jet determination problem allows a unified treatment of a number of classes of target manifolds, which includes, among others, strictly pseudoconvex, Levi–non-degenerate, but also some particularly important Levi-degenerate targets, such as boundaries of classical domains. Our main result can be stated as follows:

Theorem 1.1. *Let $M \subset \mathbb{C}^N$ be a real-analytic CR submanifold, $M' \subset \mathbb{C}^{N'}$ a Nash submanifold and $\mathcal{S}$ a subsheaf of $\mathcal{A}^{\infty}_{\mathrm{CR}}(M, M')$. Assume that M is minimal and there is no germ of a 2-approximate*

CR $\mathcal{S}$-deformation from M into M'. Then $(M, M', \mathcal{S})$ satisfies the strong finite jet determination property.

Recall that M is minimal if each of its connected component $\hat{M}$ does not contain any proper CR submanifold with the same CR dimension as that of $\hat{M}$ (see [3]). Hence, in the setting of Theorem 1.1, the finite jet determination problem reduces to checking the non-existence of 2-approximate CR $\mathcal{S}$-deformations from M into M'. In order to illustrate the wide range of applications of this approach, we now provide examples of subsheaves $\mathcal{S}$ and submanifolds M, M' satisfying the conditions of Theorem 1.1. The first important situation is when M' is CR and strictly pseudoconvex (that is, locally contained in a strictly pseudoconvex hypersurface), a case in which $\mathcal{S}$ is allowed to be the full sheaf $\mathcal{A}^\infty_{\mathrm{CR}}(M, M')$:

Corollary 1.2. *Let $M \subset \mathbb{C}^N$ be a real-analytic minimal CR submanifold and $M' \subset \mathbb{C}^{N'}$ a strictly pseudoconvex Nash CR submanifold. Then $(M, M', \mathcal{A}^\infty_{\mathrm{CR}}(M, M'))$ satisfies the strong finite jet determination property.*

In particular, if the source is a compact real-analytic hypersurface, then, by [3, 11], the assumptions of Corollary 1.2 are satisfied, and the integer-valued map ℓ can be chosen to be constant on M.

Corollary 1.3. *For every compact real-analytic hypersurface $M \subset \mathbb{C}^N$ and every strictly pseudoconvex Nash real hypersurface $M' \subset \mathbb{C}^{N'}$, there exists an integer $\ell = \ell(M, M')$ such that if $f, g\colon (M, p) \to M'$ are two germs of C^∞-smooth CR maps at some point $p \in M$ with $j^\ell_p f = j^\ell_p g$, it follows that $f = g$.*

Corollary 1.3 can be applied to prove the following boundary uniqueness result for proper holomorphic maps.

Corollary 1.4. *Let $\Omega \subset \mathbb{C}^N$ be a bounded domain with smooth real-analytic boundary and $\Omega' \subset \mathbb{C}^{N'}$ a strictly pseudoconvex domain with smooth Nash boundary. Then there exists an integer ℓ, depending only on $\partial\Omega$ and $\partial\Omega'$, such that if $F, G\colon \Omega \to \Omega'$ are two proper holomorphic mappings extending smoothly up to the boundary near some point $p \in \partial\Omega$ with $j^\ell_p F = j^\ell_p G$, it follows that $F = G$.*

If one allows complex curves inside the target manifold, the finite jet determination property clearly fails to hold for the full sheaf $\mathcal{A}^\infty_{\mathrm{CR}}(M, M')$. Hence, for the next results, we will need to restrict the sheaf $\mathcal{S}$ of maps under consideration, albeit in a 'natural' way: The assumptions are going to be designed to exclude classes of maps which need to be excluded because the targets will be allowed to contain complex subvarieties.

The first examples of this are Levi–non-degenerate hypersurfaces for sources and targets. In particular, this includes hyperquadrics (of possibly positive signature) as targets. Also these are 'flat' manifolds from the CR geometry point of view, and have the distinct advantage that every algebraic CR manifold actually allows for an embedding into a hyperquadric [28] (but not necessarily into a sphere, as already pointed out above). Let us recall that for a real-analytic (connected) Levi–non-degenerate hypersurface $M \subset \mathbb{C}^N$ (and similarly for $M' \subset \mathbb{C}^{N'}$), the minimum of the numbers of the positive and negative eigenvalues of its Levi form is the same at each point and is called the *signature* of M. For such M and M' and for $p \in M$, a CR map $H\colon (\mathbb{C}^N, p) \to \mathbb{C}^{N'}$ sending M into M' is called *CR transversal (at p)* if

$$T^{1,0}_{H(p)}M' + dH(T^{1,0}_p(\mathbb{C}^N)) = T^{1,0}_{H(p)}\mathbb{C}^{N'}.$$

The *signature difference* plays a crucial role in understanding the finite jet determination property of CR transversal maps between Levi–non-degenerate hypersurfaces; this is already observed in the simple case of mappings between hyperquadrics. If one tries to map

$$\mathbb{C}^N_{z,w} \supset \mathbb{H}^N_\ell : \operatorname{Im} w = \sum_{j=1}^{\ell} |z_j|^2 - \sum_{j=\ell+1}^{N-1} |z_j|^2$$

into $\mathbb{H}^{N+2}_{\ell+1}$, then, for any germ of a holomorphic function φ, the map $(z,w) \mapsto (\varphi(z,w), z, \varphi(z,w), w)$ is a CR transversal immersion, but clearly those maps are not determined by any finite jet (at the origin). As an application of Theorem 1.1, we obtain the following result that is optimal in terms of the signatures:

Corollary 1.5. *Let $M \subset \mathbb{C}^N$ be a real-analytic Levi–non-degenerate hypersurface of signature ℓ and $M' \subset \mathbb{C}^{N'}$ be a Nash Levi–non-degenerate hypersurface of signature ℓ', both connected. Let $\mathcal{S}$ denote the subsheaf of $\mathcal{A}^\infty_{\mathrm{CR}}(M,M')$ consisting of CR transversal maps. If the signature difference $\ell' - \ell \in \{0, N'-N\}$, then $(M,M',\mathcal{S})$ satisfies the strong finite jet determination property.*

One further important class of targets we are considering are weakly pseudoconvex hypersurfaces. We start with the following consequence of Theorem 1.1.

Corollary 1.6. *Let $M \subset \mathbb{C}^N$ be a real-analytic minimal CR submanifold and let $M' \subset \mathbb{C}^{N'}$ be a weakly pseudoconvex Nash hypersurface. Let $\mathcal{S}$ be the subsheaf of $\mathcal{A}^\infty_{\mathrm{CR}}(M,M')$ consisting of the C^∞-smooth CR maps mapping no open subset of M into the Levi-degenerate set of M'. Then $(M,M',\mathcal{S})$ satisfies the strong finite jet determination property.*

For targets that are *everywhere Levi-degenerate*, such as homogeneous Levi-degenerate CR manifolds, Corollary 1.6 does not lead to any conclusion, while Theorem 1.1 happens to provide some of its most interesting applications in this setting. Such class of target manifolds carry a foliation by complex manifolds η, called the *Levi-foliation*, and the natural homogeneous models are the boundaries of *bounded symmetric domains*, see, for example, [29]. Our applications of Theorem 1.1 in this context utilize an invariant called ν, associated to M', which was introduced in [16] and which we are going to discuss in Section 5; this section also includes further results for everywhere Levi-degenerate targets. We shall only mention below one consequence of such results for boundaries of classical domains.

Irreducible bounded symmetric domains are classified in four series, called the classical domains of types I–IV, as well as two exceptional cases, according to Cartan's classification [10] ; we recall that the type I domain $D_I^{m,n}$ consists of the $m \times n$ matrices M satisfying that $I - M^*M$ is positive definite, type II and III domains, D_{II}^m and D_{III}^m, are anti-symmetric and symmetric $m \times m$ matrices satisfying the same condition, and that the type IV domains, D_{IV}^m, are (biholomorphically equivalent to) the tube over the light cone. We will denote the regular parts of their boundaries, respectively, by $M_I^{m,n}$, M_{II}^m, M_{III}^m and M_{IV}^m, see Section 5 for more details. Applying Theorem 1.1 to such targets yields the following:

Corollary 1.7. *Let $M \subset \mathbb{C}^N$ be a connected real-analytic, pseudoconvex, minimal hypersurface, with generically n_+ positive Levi eigenvalues. Let M' be the regular part of the boundary of a classical domain and denote by $\mathcal{S}$ the subsheaf of $\mathcal{A}^\infty_{\mathrm{CR}}(M, M')$ of generically CR transversal maps. Then $(M, M', \mathcal{S})$ has the strong finite jet determination property if*

i) $M' = M_I^{m,n}$, *$m, n \geqslant 2$, and $m + n - 4 < n_+ \leqslant m + n - 2$;*
ii) $M' = M_{II}^m$, *$m \geqslant 4$, and $2m - 8 < n_+ \leqslant 2m - 4$;*
iii) $M' = M_{III}^m$, *$m \geqslant 2$, and $n_+ = m - 1$;*
iv) $M' = M_{IV}^m$, *$m \geqslant N$.*

We mention the following noteworthy application of Corollary 1.7 to proper holomorphic mappings.

Corollary 1.8. *Let $\Omega \subset \mathbb{C}^N$ be a pseudoconvex domain, and $M \subset \partial\Omega$ be a connected real-analytic minimal hypersurface of $\mathbb{C}^N$, with generically n_+ positive Levi eigenvalues. Let Ω' be one of the four types of classical domains $D_I^{m,n}, D_{II}^m, D_{III}^m, D_{IV}^m$ satisfying:*

i) *if $\Omega' = D_I^{m,n}$, $m, n \geqslant 2$, and $m + n - 4 < n_+ \leqslant m + n - 2$;*
ii) *if $\Omega' = D_{II}^m$, $m \geqslant 4$, and $2m - 8 < n_+ \leqslant 2m - 4$;*
iii) *if $\Omega' = D_{III}^m$, $m \geqslant 2$, and $n_+ = m - 1$;*
iv) *if $\Omega' = D_{IV}^m$, $m \geqslant N$.*

There exists a locally bounded map $\ell : M \to \mathbb{Z}_+$, such that given two proper holomorphic mappings $H, G : \Omega \to \Omega'$, extending smoothly up to some point $p \in M$ with $H(p)$ in the regular part of $\partial\Omega'$, if $j_p^{\ell(p)} H = j_p^{\ell(p)} G$, then it follows that $H = G$.

Corollary 1.8 follows immediately from Corollary 1.7 and the well-known fact that boundary values of proper holomorphic maps in that setting are automatically CR transversal maps (see [3, Proposition 9.10.5] or [16]).

It is interesting to note that the range of dimensions given in Corollary 1.7 is also sharp in order for the finite jet determination property to hold; examples are given and discussed in Section 5.

The paper is organized as follows. First, in Section 2, we discuss, for any real-analytic generic minimal submanifold $M \subset \mathbb{C}^N$, and any subsheaf $\mathcal{S} \subset \mathcal{A}^\omega_{\mathrm{CR}}(M, \mathbb{C}^{N'})$, a universal parametrization property we call property (*) and show how it implies the strong finite jet determination property (for the sheaf $\mathcal{S}$). In Section 3, we deal with the second independent part of the proof of Theorem 1.1. It boils down to showing that, for any real-analytic generic submanifold $M \subset \mathbb{C}^N$ (not necessarily minimal), and any Nash submanifold $M' \subset \mathbb{C}^{N'}$ and any subsheaf $\mathcal{S} \subset \mathcal{A}^\omega_{\mathrm{CR}}(M, M')$, the non-existence of 2-approximate CR $\mathcal{S}$-deformations implies that property (*) holds. In Sections 4 and 5, we finalize the proofs of Theorem 1.1 as well as of all its corollaries, by showing how each different geometric setting discussed in those results systematically exclude the existence of 2-approximate deformations. In Section 5, we focus on discussing how our results may be applied for targets that are everywhere Levi-degenerate real hypersurfaces, and highlight the specific case of boundaries of classical domains. The proof of property (*) heavily relies on some universality properties of polynomial relations satisfied by power series which we include at the end of the paper in Section 6.

2 | UNIVERSAL ALGEBRAIC PARAMETRIZATION AND UNIQUE JET DETERMINATION

In this section, we prove that if a given sheaf of maps from a real-analytic generic submanifold $M \subset \mathbb{C}^N$ into $\mathbb{C}^{N'}$ satisfy a certain universal algebraic parametrization property, then such a sheaf of maps must satisfy the strong finite jet determination property (Theorem 2.2). This is one of the main two steps of the proof of Theorem 1.1. The key guideline in the proof of this result is to keep track of the universal algebraic equations satisfied by the sheaf of maps on the so-called iterated complexifications of M. This is achieved by combining the use of the iterated Segre mappings technique introduced in [2] following the strategy developed in [27] (in the sphere case), and several universal properties of polynomial equations satisfied by power series, proved at the end of the paper in Section 6.

2.1 | Iterated complexifications

Let $M \subset \mathbb{C}^N$ be a real-analytic generic submanifold, of CR dimension n and codimension d, with $p_0 \in M$. Shrinking M near p_0 if necessary, we may choose some polydisc neighbourhood U of p_0, and a real-valued real-analytic map $r = (r_1, \dots, r_d)$ defined on U such that M is given by

$$M = \{Z \in U : r(Z, \bar{Z}) = 0\}, \tag{2.1}$$

with $\partial r_1 \wedge \dots \partial r_d \neq 0$ on U. Define the complexification of M by

$$\mathcal{M} := \{(Z, \zeta) \in U \times U^* : r(Z, \zeta) = 0\},$$

where

$$U^* = \{Z : \bar{Z} \in U\},$$

which (for small enough U) is a complex submanifold of complex dimension $2n + d$ of $U \times U^*$. Furthermore, as in [30], we shall consider the iterated complexifications $\mathcal{M}^j$, for $j \geqslant 1$, as follows. For $j = 2\ell - 1$ odd, we set

$$\begin{aligned}\mathcal{M}^{2\ell-1} := \{&(Z, \zeta^1, Z^1, \dots, Z^{\ell-1}, \zeta^\ell) \in U \times U^* \times \dots \times U^* :\\ &(Z, \zeta^1) \in \mathcal{M}, (Z^1, \zeta^1) \in \mathcal{M}, (Z^1, \zeta^2) \in \mathcal{M}, \dots, (Z^{\ell-1}, \zeta^\ell) \in \mathcal{M}\},\end{aligned}$$

and for $j = 2\ell$ even we set

$$\begin{aligned}\mathcal{M}^{2\ell} := \{&(Z, \zeta^1, \dots, Z^{\ell-1}, \zeta^\ell, Z^\ell) \in U \times U^* \times \dots \times U :\\ &(Z, \zeta^1) \in \mathcal{M}, (Z^1, \zeta^1) \in \mathcal{M}, (Z^1, \zeta^2) \in \mathcal{M}, \dots, (Z^\ell, \zeta^\ell) \in \mathcal{M}\}.\end{aligned}$$

We note that $\mathcal{M}^j$ is a complex submanifold of $U \times U^* \times \cdots \times U^* \subset \mathbb{C}^{(j+1)N}$ for j odd and of $U \times U^* \times \cdots \times U \subset \mathbb{C}^{(j+1)N}$ for j even. We shall only consider the iterated complexifications $\mathcal{M}^j$ for $j \leqslant 2N+1$.

Next, we also define, for every integer $k \geqslant 1$ the following real-analytic submanifold of $\mathcal{M}^{2k+1}$:

$$\begin{aligned}\mathcal{X}^k &= \{(Z, \zeta^1, \dots, \zeta^k, p, \bar{p}) \in U \times \cdots \times U^* : (Z, \zeta^1, \dots, \zeta^k, p) \in \mathcal{M}^{2k},\ p \in M\} \\ &= \mathcal{M}^{2k+1} \cap \left\{ Z^k = \overline{\zeta^{k+1}} \right\}. \end{aligned} \tag{2.2}$$

2.2 | Universal algebraic parametrization and jet determination

For a real-analytic generic submanifold $M \subset \mathbb{C}^N$, recall that $\mathcal{A}^{\omega}_{\mathrm{CR}}(M, \mathbb{C}^{N'})$ denotes the sheaf over M of $\mathbb{C}^{N'}$-valued real-analytic CR maps over M. As is customary (see, for example, [3]), since M is generic, we can identify a section of $\mathcal{A}^{\omega}_{\mathrm{CR}}(M, \mathbb{C}^{N'})$, that is, a real-analytic CR map h defined on some open subset $U \subset M$ valued in $\mathbb{C}^{N'}$, with the germ of a holomorphic map $H : \tilde{U} \to \mathbb{C}^{N'}$ along U, for some open neighbourhood $\tilde{U}$ of U in $\mathbb{C}^N$.

Throughout the paper, for any holomorphic function ψ defined on some open subset O of $\mathbb{C}^r$, we denote by $\bar{\psi}$ the holomorphic function defined on O^* by $\bar{\psi}(\xi) = \overline{\psi(\bar{\xi})}$.

The next notion introduces the precise type of parametrization property we will be studying in this paper.

Definition 2.1. Let $M \subset \mathbb{C}^N$ be a real-analytic generic submanifold, $\mathcal{S}$ a subsheaf of $\mathcal{A}^{\omega}_{CR}(M, \mathbb{C}^{N'})$, and $p_0 \in M$. We say that $\mathcal{S}$ satisfies property $(*)_{p_0}$ if there exist a sufficiently small neighbourhood Ω_0 of p_0 in $\mathbb{C}^N$, a positive integer r, a finite family of $\mathbb{C}^{N'}$-valued polynomial maps $\mathcal{P}^{(1)}, \dots, \mathcal{P}^{(L)}$, universal in the sense that they are independent of p_0, M and $\mathcal{S}$, and a holomorphic map $A(Z, \zeta^1, Z^1)$, defined on $\Omega_0 \times \Omega_0^* \times \Omega_0$, universal in the sense that it only depends on M and p_0, such that for every $q \in M_0 := M \cap \Omega_0$ and every $f \in \mathcal{S}_q$, there exists $\ell \in \{1, \dots L\}$ such that

$$\mathcal{P}_j^{(\ell)}\Big(A(Z, \zeta^1, Z^1), \big(\partial^\mu f(Z^1), \partial^\mu \bar{f}(\zeta^1)\big)_{|\mu| \leqslant r}, f_j(Z) \Big) = 0,\ j = 1, \dots, N', \text{ and} \tag{2.3}$$

$$\frac{\partial \mathcal{P}_j^{(\ell)}}{\partial T}\Big(A(Z, \zeta^1, Z^1), \big(\partial^\mu f(Z^1), \partial^\mu \bar{f}(\zeta^1)\big)_{|\mu| \leqslant r}, f_j(Z) \Big) \not\equiv 0,\ j = 1, \dots, N', \tag{2.4}$$

for $(Z, \zeta^1, Z^1) \in \mathcal{M}^2$ sufficiently close to $(q, \bar{q}, q)$, and where we write $\mathcal{P}^{(\ell)} = (\mathcal{P}_1^{(\ell)}, \dots, \mathcal{P}_{N'}^{(\ell)})$, with T denoting its last argument.

We further say that $\mathcal{S}$ satisfies property $(*)$ if $\mathcal{S}$ satisfies property $(*)_{p_0}$ for every $p_0 \in M$.

Our goal now is to prove the following finite jet determination result.

Theorem 2.2. *Let $M \subset \mathbb{C}^N$ be a real-analytic generic submanifold, $p_0 \in M$ and $\mathcal{S}$ be a subsheaf of $\mathcal{A}^{\omega}_{CR}(M, \mathbb{C}^{N'})$ satisfying $(*)_{p_0}$. If M is minimal at p_0, there exists a neighbourhood M_0 of p_0 in M and an integer $K > 0$, such that for every $q \in M_0$, if f, g are two elements of $\mathcal{S}_q$ satisfying $j_q^K f = j_q^K g$, then $f = g$.*

In order to prove Theorem 2.2, we will show that maps satisfying property $(*)_{p_0}$ satisfy a more useful (other) universal algebraic parametrization property, given in Proposition 2.7, from which we will establish the desired finite jet determination result in Section 2.3.4.

2.3 | Proof of Theorem 2.2

The proof will be divided into several steps. Without loss of generality, we may assume that $p_0 = 0$.

2.3.1 | Parametrizing derivatives along iterated complexifications

We start with the following:

Proposition 2.3. *Let $M \subset \mathbb{C}^N$ be a real-analytic generic submanifold through the origin and $\mathcal{S}$ be a subsheaf of $\mathcal{A}^{\omega}_{CR}(M, \mathbb{C}^{N'})$ satisfying $(*)_0$. Then there exist a sufficiently small neighbourhood Ω_0 of 0 in $\mathbb{C}^N$, and for every multiindex $\gamma \in \mathbb{N}^N$, a finite family of $\mathbb{C}^{N'}$-valued universal polynomial maps $\mathcal{P}^{(\gamma,1)}, \dots \mathcal{P}^{(\gamma,\sigma_\gamma)}$ (independent of M and $\mathcal{S}$), and a holomorphic map $A_\gamma(Z, \zeta^1, Z^1)$, defined on Ω_0^3, depending only on M and γ, with the following property: For every $q \in M_0 := M \cap \Omega_0$ and every $f \in \mathcal{S}_q$, there exists $\delta \in \{1, \dots \sigma_\gamma\}$ such that*

$$\mathcal{P}_j^{(\gamma,\delta)}\Big(A_\gamma(Z, \zeta^1, Z^1), \big(\partial^\mu f(Z^1), \partial^\mu \bar{f}(\zeta^1)\big)_{|\mu| \leqslant r+|\gamma|}, \partial^\gamma f_j(Z)\Big) = 0, \; j = 1, \dots, N', \tag{2.5}$$

$$\frac{\partial \mathcal{P}_j^{(\gamma,\delta)}}{\partial T}\Big(A_\gamma(Z, \zeta^1, Z^1), \big(\partial^\mu f(Z^1), \partial^\mu \bar{f}(\zeta^1)\big)_{|\mu| \leqslant r+|\gamma|}, \partial^\gamma f_j(Z)\Big) \not\equiv 0, \; j = 1, \dots, N', \tag{2.6}$$

for $(Z, \zeta^1, Z^1) \in \mathcal{M}^2$ sufficiently close to $(q, \bar{q}, q)$.

Proof. Let Ω_0 be given by Definition 2.1. Shrinking Ω_0 if necessary, we may assume that $\Omega_0 \subset U$, where U is given in Section 2.1 and choose holomorphic coordinates $x \in \mathbb{C}^{N+2n}$ for $\mathcal{M}^2 \cap (\Omega_0 \times \Omega_0^* \times \Omega_0)$. For $q \in \Omega_0 \cap M$, we write x_q for the coordinates of $(q, \bar{q}, q)$. Fix $j \in \{1, \dots, N'\}$ and set

$$g(x) = \Big(A(Z, \zeta^1, Z^1), \big(\partial^\mu f(Z^1), \partial^\mu \bar{f}(\zeta^1)\big)_{|\mu| \leqslant r}\Big)\Big|_{\mathcal{M}^2}, \quad h(x) = f_j(Z)|_{\mathcal{M}^2}.$$

Then every component of $g(x)$ and $h(x)$ belongs to $\mathbb{C}\{x - x_q\}$. By assumption, $\mathcal{P}_j^{(\ell)}(g(x), h(x)) = 0$ and $(\mathcal{P}_j^{(\ell)})_T(g(x), h(x)) \not\equiv 0$ for some ℓ. An application of Lemma 6.3 yields finitely many polynomials $\mathrm{Y}_1, \dots, \mathrm{Y}_{e_\gamma}$, depending only on $\mathcal{P}_j^{(\ell)}$ (and not on h, g), such that there exists $d \in \{1, \dots, e_\gamma\}$ satisfying

$$\mathrm{Y}_d((\partial^\alpha g(x))_{|\alpha| \leqslant |\gamma|}, \partial^\gamma h(x)) = 0, \quad \frac{\partial \mathrm{Y}_d}{\partial T}((\partial^\alpha g(x))_{|\alpha| \leqslant |\gamma|}, \partial^\gamma h(x)) \not\equiv 0. \tag{2.7}$$

The conclusion of the lemma now follows from the observation that for every multi-index α, we have that $\partial_x^\alpha g(x) = P^\alpha(A_\alpha(Z, \zeta^1, Z^1), (\partial^\mu f(Z^1), \partial^\mu \bar{f}(\zeta^1))_{|\mu| \leqslant r+|\alpha|})|_{\mathcal{M}^2}$ for some universal polynomial P^α and some holomorphic map A_α depending only on A. We leave the details to the reader. $\square$

We now iterate the previous result along iterated complexifications of M to reach the following statement.

Proposition 2.4. *Let $M \subset \mathbb{C}^N$ be a real-analytic generic submanifold through the origin and $\mathcal{S}$ be a subsheaf of $\mathcal{A}^{\omega}_{CR}(M,\mathbb{C}^{N'})$ satisfying $(*)_0$. Then for every integer $\ell \geqslant 1$, there exists a neighbourhood Ω_ℓ of 0 in $\mathbb{C}^N$, a finite family of $\mathbb{C}^{N'}$-valued universal polynomial maps $\mathcal{R}^{(\ell,1)},\dots \mathcal{R}^{(\ell,e_\ell)}$ (independent of M and $\mathcal{S}$) and a holomorphic map $A^{(\ell)}(Z,\zeta^1,Z^1,\dots,Z^\ell,\zeta^{\ell+1})$, defined on $\Omega_\ell^{2\ell+2}$ and depending only on M, such that for every $q \in M_\ell := M\cap\Omega_\ell$ and every $f\in\mathcal{S}_q$, there exists $\nu\in\{1,\dots e_\ell\}$*

$$\mathcal{R}_j^{(\ell,\nu)}\Big(A^{(\ell)}(Z,\zeta^1,\dots,\zeta^{\ell+1}),\big(\partial^\mu f(Z^\ell),\partial^\mu\bar f(\zeta^{\ell+1})\big)_{|\mu|\leqslant 2\ell r},f_j(Z)\Big)=0,\ j=1,\dots,N', \tag{2.8}$$

$$\frac{\partial\mathcal{R}_j^{(\ell,\nu)}}{\partial T}\Big(A^{(\ell)}(Z,\zeta^1,\dots,\zeta^{\ell+1}),\big(\partial^\mu f(Z^\ell),\partial^\mu\bar f(\zeta^{\ell+1})\big)_{|\mu|\leqslant 2\ell r},f_j(Z)\Big)\not\equiv 0,\ j=1,\dots,N', \tag{2.9}$$

for $(Z,\zeta^1,Z^1,\dots,Z^\ell,\zeta^{\ell+1})\in\mathcal{M}^{2\ell+1}$ sufficiently close to $(q,\bar q,\dots,q,\bar q)$. In particular, we have

$$\mathcal{R}_j^{(\ell,\nu)}\Big(A^{(\ell)}(Z,\zeta^1,\dots,\zeta^\ell,p,\bar p),\big(\partial^\mu f(p),\partial^\mu\bar f(\bar p)\big)_{|\mu|\leqslant 2\ell r},f_j(Z)\Big)=0,\ j=1,\dots,N', \tag{2.10}$$

$$\frac{\partial\mathcal{R}_j^{(\ell,\nu)}}{\partial T}\Big(A^{(\ell)}(Z,\zeta^1,\dots,\zeta^\ell,p,\bar p),\big(\partial^\mu f(p),\partial^\mu\bar f(\bar p)\big)_{|\mu|\leqslant 2\ell r},f_j(Z)\Big)\not\equiv 0,\ j=1,\dots,N', \tag{2.11}$$

for all $(Z,\zeta^1,Z^1,\dots,\zeta^\ell,p)\in\mathcal{M}^{2\ell}$ and $p\in M$, sufficiently close to $(q,\bar q,q,\dots,\bar q,q)$ and q, respectively.

Proof. We choose a neighbourhood Ω_0 satisfying $\Omega_0^*=\Omega_0$ and such that Proposition 2.3 holds.

We show how the proposition is proven for $\ell=1$. We first apply Proposition 2.3 for $\gamma=0$ (which in that case is just the defining property $(*)_0$) and obtain a (universal) family $\mathcal{P}^{(0,\delta)}$, where $\delta\in\{1,\dots,\sigma_0\}$, such that, for every $q\in M\cap\Omega_0$ and every $f\in\mathcal{S}_q$, we have, as a consequence of (2.5) and (2.6), that for some $\delta\in\{1,\dots,\sigma_0\}$ and for all $j=1,\dots,N'$,

$$\begin{cases}\mathcal{P}_j^{(0,\delta)}\Big(A_0(Z,\zeta^1,Z^1),\big(\partial^\mu f(Z^1),\partial^\mu\bar f(\zeta^1)\big)_{|\mu|\leqslant r},f_j(Z)\Big)=0,\\ \frac{\partial\mathcal{P}_j^{(0,\delta)}}{\partial T}\Big(A_0(Z,\zeta^1,Z^1),\big(\partial^\mu f(Z^1),\partial^\mu\bar f(\zeta^1)\big)_{|\mu|\leqslant r},f_j(Z)\Big)\not\equiv 0,\end{cases} \tag{2.12}$$

for $(Z,\zeta^1,Z^1)\in\mathcal{M}^2$ sufficiently close to $(q,\bar q,q)$. Fix j in what follows and $\mu\in\mathbb{N}^N$ with $|\mu|\leqslant r$. Another application of Proposition 2.3 this time with $\gamma=\mu$ yields a family $\mathcal{P}_i^{(\mu,\eta)}$ where $\eta\in\{1,\dots,\sigma_\mu\}$ such that after conjugating the identities (2.5) and (2.6) with γ replaced by μ, we have for every $i = 1, \dots, N'$ and some η,

$$\begin{cases} \overline{\mathcal{P}_i^{(\mu,\eta)}}\Big(\bar{A}_\mu(\zeta^1, Z^1, \zeta^2), \big(\partial^\alpha \bar{f}(\zeta^2), \partial^\alpha f(Z^1)\big)_{|\alpha|\leqslant r+|\mu|}, \partial^\mu \bar{f}_i(\zeta^1)\Big) = 0, \\ \overline{\frac{\partial \mathcal{P}_i^{(\mu,\eta)}}{\partial T}}\Big(\bar{A}_\mu(\zeta^1, Z^1, \zeta^2), \big(\partial^\alpha \bar{f}(\zeta^2), \partial^\alpha f(Z^1)\big)_{|\alpha|\leqslant r+|\mu|}, \partial^\mu \bar{f}_i(\zeta^1)\Big) \not\equiv 0, \end{cases} \tag{2.13}$$

for (ζ^1, Z^1, ζ^2) near $(\bar{q}, q, \bar{q})$ with $(Z^1, \zeta^1) \in \mathcal{M}$ and $(Z^1, \zeta^2) \in \mathcal{M}$. We choose $\Omega_1 \subset \Omega_0$ so that $\mathcal{M}^3 \cap (\Omega_1)^4$ is covered by one holomorphic coordinate chart, denoted by x. We apply Lemma 6.5 to

$$\begin{cases} P = \mathcal{P}_j^{(0,\delta)}, \quad \Theta = \left(\overline{\mathcal{P}_i^{(\mu,\eta)}}\right)_{\substack{|\mu|\leqslant r \\ i=1,\dots,N'}}, \quad v(x) = \left(\partial^\mu \bar{f}_i(\zeta^1)|_{\mathcal{M}^3}\right)_{\substack{|\mu|\leqslant r \\ i=1,\dots,N'}}, \\ \quad h(x) = f_j(Z)|_{\mathcal{M}^3}, \quad u(x) = \Big(A_0(Z, \zeta^1, Z^1), \big(\partial^\mu f(Z^1)\big)_{|\mu|\leqslant r}\Big)\Big|_{\mathcal{M}^3}, \\ \quad w(x) = \Big(\big(\bar{A}_\beta(\zeta^1, Z^1, \zeta^2)\big)_{|\beta|\leqslant r}, \big(\partial^\alpha \bar{f}(\zeta^2), \partial^\alpha f(Z^1)\big)_{|\alpha|\leqslant 2r}\Big)\Big|_{\mathcal{M}^3} \end{cases}$$

and get that there are finitely many polynomials, $\Psi_1, \dots, \Psi_K$, where $K = K(j)$, depending only on $\mathcal{P}_j^{(0,\delta)}$ and the $\mathcal{P}_i^{(\mu,\eta)}$ for $1 \leqslant \eta \leqslant \sigma_\mu$, $|\mu| \leqslant r$ and $i = 1, \dots, N'$ such that, for some $k \in \{1, \dots, K\}$

$$\begin{cases} \Psi_k\Big(A_0(Z, \zeta^1, Z^1), \big(\bar{A}_\beta(\zeta^1, Z^1, \zeta^2)\big)_{|\beta|\leqslant r}, \big(\partial^\alpha \bar{f}(\zeta^2), \partial^\alpha f(Z^1)\big)_{|\alpha|\leqslant 2r}, f_j(Z)\Big) = 0, \\ \frac{\partial \Psi_k}{\partial T}\Big(A_0(Z, \zeta^1, Z^1), \big(\bar{A}_\beta(\zeta^1, Z^1, \zeta^2)\big)_{|\beta|\leqslant r}, \big(\partial^\alpha \bar{f}(\zeta^2), \partial^\alpha f(Z^1)\big)_{|\alpha|\leqslant 2r}, f_j(Z)\Big) \not\equiv 0, \end{cases} \tag{2.14}$$

for $(Z, \zeta^1, Z^1, \zeta^2) \in \mathcal{M}^3$ sufficiently close to $(q, \bar{q}, q, \bar{q})$. This proves the proposition for $\ell = 1$. For arbitrary ℓ, the general case follows along the same line of arguments as those explained for the case $\ell = 1$. The proof is complete. $\square$

2.3.2 | Iterated Segre maps

Let M be as above. We may choose normal coordinates $Z = (z, w) \in \mathbb{C}^n \times \mathbb{C}^d$ for M near 0 (see, for example, [3]), meaning that (the germ of) M (at 0) is given by

$$w = Q(z, \bar{z}, \bar{w}), \tag{2.15}$$

where $Q = (Q^1, \dots, Q^d)$ is a $\mathbb{C}^d$-valued holomorphic map defined in some fixed neighbourhood of the origin satisfying

$$Q(z, \bar{z}, \bar{Q}(\bar{z}, z, w)) = w, \ Q(z, 0, \bar{w}) = Q(0, \bar{z}, \bar{w}) = \bar{w}.$$

We now make use of the Segre maps associated to M, as introduced in [2, 3]. For $p \in U$ (which later will furthermore lie on M), let us recall how the Segre map v^k of order $k \in \mathbb{Z}_+$ is defined. Following the notation of [5, 27], we first set $v^0(p) := p$ and for $k \geqslant 0$ we inductively define

$$v^{k+1}(t^0, t^1, \dots, t^k; p) = (t^0, Q(t^0, \overline{v^k}(t^1, \dots, t^k; p)). \tag{2.16}$$

The Segre maps are defined and holomorphic over $U_1 \times \dots \times U_1 \times U$ provided U_1 and U are sufficiently small neighbourhoods of the origin in $\mathbb{C}^n$ and $\mathbb{C}^N$, respectively. As we will need only finitely many of those Segre maps, we choose and fix neighbourhoods U_1 and U as above so that all the maps v^ks we are going to need are well defined and holomorphic on $U_1^{k+1} \times U$. For every integer $k \geqslant 1$, the real-analytic map $\Xi : U_1^{2k} \times (M \cap U) \to \mathcal{M}^{2k+1}$ given by

$$\Xi(t^0, \dots, t^{2k-1}, p) := (v^{2k}(t^0, \dots, t^{2k-1}; p), \overline{v^{2k-1}}(t^1, \dots, t^{2k-1}; p), \dots, \overline{v^1}(t^{2k-1}; p), p, \bar{p}) \qquad (2.17)$$

parametrizes the submanifold $\mathcal{X}^k$ given by (2.2) in a neighbourhood of the origin.

Recall now the following minimality criterion from [4]:

Theorem 2.5. *Let $M \subset \mathbb{C}^N$ be a germ at the origin of a real-analytic generic minimal submanifold. With the above notation, there exists an integer $s \leqslant N$ such that the following holds:*

$$\max\left\{ \mathrm{rk} \frac{\partial v^{2s}}{\partial(t^0, t^{s+1}, t^{s+2}, \dots, t^{2s-1})} \left(0, x^1, \dots, x^{s-1}, x^s, x^{s-1}, \dots, x^1; 0\right) \, : \, x^1, \dots, x^s \in U_1 \right\} = N, \qquad (2.18)$$

$$v^{2s}(0, x^1, \dots, x^{s-1}, x^s, x^{s-1}, \dots, x^1; 0) = 0. \qquad (2.19)$$

2.3.3 | Lifting

Using the Segre mappings as parametrizations of the iterated complexifications, we obtain, as a consequence of (2.17), (2.2) and Proposition 2.4, the following:

Proposition 2.6. *Let $M \subset \mathbb{C}^N$ be a real-analytic generic submanifold through the origin given in normal coordinates as above and $\mathcal{S}$ be a subsheaf of $\mathcal{A}^{\omega}_{CR}(M, \mathbb{C}^{N'})$ satisfying $(*)_0$. Let $s \in \mathbb{Z}_+$ be given by Theorem 2.5. Then there exists a finite family of $\mathbb{C}^{N'}$-valued universal polynomial maps $\mathcal{R}^{(1)}, \dots \mathcal{R}^{(\Delta)}$ (independent of M and the subsheaf), a holomorphic map $\Phi(t^0, \dots, t^{2s-1}, \lambda, \omega)$, depending only on M and defined in a neighbourhood of the origin in $\mathbb{C}^{(2s)n+2N}$ such that if $q = (z_q, w_q) \in M$ is sufficiently close to the origin and $f \in \mathcal{S}_q$, there exists $\nu \in \{1, \dots, \Delta\}$ such that for every $j = 1, \dots, N'$,*

$$\begin{cases} \mathcal{R}_j^{(\nu)}\Big(\Phi(t^0, \dots, t^{2s-1}, p, \bar{p}), \big(\partial^\mu \bar{f}(\bar{p}), \partial^\mu f(p)\big)_{|\mu| \leqslant 2sr}, (f_j \circ v^{2s}(t^0, \dots, t^{2s-1}; p))\Big) = 0, \\ \dfrac{\partial \mathcal{R}_j^{(\nu)}}{\partial T}\Big(\Phi(t^0, \dots, t^{2s-1}, p, \bar{p}), \big(\partial^\mu \bar{f}(\bar{p}), \partial^\mu f(p)\big)_{|\mu| \leqslant 2sr}, (f_j \circ v^{2s}(t^0, \dots, t^{2s-1}; p))\Big) \not\equiv 0, \end{cases} \qquad (2.20)$$

for all $(t^0, \dots, t^{2s-1}) \in \mathbb{C}^{2sn}$ and $p \in M$ sufficiently close to $(z_q, \bar{z}_q, \dots, z_q, \bar{z}_q)$ and q, respectively.

Using similar arguments as those of [27, §3.3] together with Proposition 2.6, we obtain the following result which may be viewed as a universal algebraic parametrization property satisfied by the sheaf $\mathcal{S}$.

Proposition 2.7. *Let $M \subset \mathbb{C}^N$ be a real-analytic generic minimal submanifold through the origin. Then there exists a real-analytic map $\Psi(\xi, p, q, Z)$ defined on some open polydisc $V \times W^3 \subset \mathbb{C}^a \times$*

$\mathbb{C}^{3N}$ for some integer $a \geqslant 1$, depending only on M, holomorphic with respect to (ξ, Z), with $0 \in W$, and a finite collection of universal $\mathbb{C}^{N'}$-valued polynomial maps $\mathcal{H}^{(1)}, \dots, \mathcal{H}^{(b)}$ independent of M, with the following property. If $\mathcal{S}$ is a subsheaf of $\mathcal{A}^{\omega}_{CR}(M, \mathbb{C}^{N'})$ satisfying $(*)_0$ then for every $q \in M \cap W$ and $f \in \mathcal{S}_q$, there exists $\nu \in \{1, \dots, b\}$ such that for every $j = 1, \dots, N'$,

$$\begin{cases} \mathcal{H}_j^{(\nu)}\Big(\Psi(\xi, p, q, Z), \big(\partial^\mu \bar{f}(\bar{p}), \partial^\mu f(p)\big)_{|\mu| \leqslant 2sr}, f_j(Z)\Big) = 0, \\ \dfrac{\partial \mathcal{H}_j^{(\nu)}}{\partial T}\Big(\Psi(\xi, p, q, Z), \big(\partial^\mu \bar{f}(\bar{p}), \partial^\mu f(p)\big)_{|\mu| \leqslant 2sr}, f_j(Z)\Big) \not\equiv 0, \end{cases} \tag{2.21}$$

for every $p \in M$ and $Z \in \mathbb{C}^N$ sufficiently close to q, and every $\xi \in V$.

Remark 2.8. Enlarging the collections of polynomials $\mathcal{H}^{(1)}, \dots, \mathcal{H}^{(b)}$ if necessary, we may assume that for every map $f \in \mathcal{S}_q$, we can choose a polynomial $\mathcal{H}^{(\nu)}$ satisfying (2.21) in Proposition 2.7 which further satisfies

$$\deg_T \mathcal{H}_j^{(\nu)}(X, \Lambda, T) = \deg_T \mathcal{H}_j^{(\nu)}(\Psi(\xi, p, q, Z), \big(\partial^\mu \bar{f}(\bar{p}), \partial^\mu f(p)\big)_{|\mu| \leqslant 2sr}, T), \ j = 1, \dots, N',$$

where p, q, Z and ξ are as in the proposition and where X and Λ denote, respectively, the first and second arguments in the polynomials $\mathcal{H}^{(1)}, \dots, \mathcal{H}^{(b)}$.

2.3.4 | Completion of the proof of Theorem 2.2

Let $\mathcal{H}^{(k)} = \mathcal{H}^{(k)}(X, \Lambda, T)$, $k = 1, \dots, b$, the polynomial maps and Ψ be given by Proposition 2.7. Shrinking V and W if necessary, we may assume that Ψ is real-analytic in a neighbourhood of $\overline{V \times W^3}$. In what follows, for any open subset O in real euclidean space, we denote by $\mathcal{C}^\omega(\overline{O})$ the ring of real-analytic functions in a neighbourhood of $\overline{O}$.

For every $k \in \{1, \dots, b\}$ and $j \in \{1, \dots, N'\}$, set $\delta^{k,j} := \deg_T \mathcal{H}_j^{(k)}$ and write

$$\mathcal{H}_j^{(k)}(X, \Lambda, T) = \Delta_j^k(X, \Lambda) T^{\delta^{k,j}} + \cdots$$

For (X, Λ) outside the zero locus of Δ_j^k, denote by $\sigma_\ell^{k,j} = \sigma_\ell^{k,j}(X, \Lambda)$ for $\ell = 1, \dots, \delta^{k,j}$, the $\delta^{k,j}$ roots (counted with multiplicity) of $\mathcal{H}_j^k(X, \Lambda, T)$, and denote by $\widehat{\Lambda}$ another variable (of the same dimension as that of Λ). Now, for every $k_1, k_2 \in \{1, \dots, b\}$ and every $j = 1, \dots, N'$, set

$$\Upsilon_j^{k_1,k_2}(X, \Lambda, \widehat{\Lambda}, T) = \prod_{\ell_1=1}^{\delta^{k_1,j}} \prod_{\ell_2=1}^{\delta^{k_2,j}} \Big(T - \Big(\sigma_{\ell_1}^{k_1,j}(X, \Lambda) - \sigma_{\ell_2}^{k_2,j}(X, \widehat{\Lambda})\Big)\Big).$$

By Newton's theorem on symmetric polynomials, there exist positive integers e_1, e_2 such that

$$\Omega_j^{k_1,k_2}(X, \Lambda, \widehat{\Lambda}, T) := (\Delta_j^{k_1}(X, \Lambda))^{e_1} (\Delta_j^{k_2}(X, \widehat{\Lambda}))^{e_2} \Upsilon_j^{k_1,k_2}(X, \Lambda, \widehat{\Lambda}, T) \in \mathbb{C}[X, \Lambda, \widehat{\Lambda}, T].$$

Next, we write

$$\Omega_j^{k_1,k_2}(X,\Lambda,\widehat{\Lambda},T) = \sum_{\nu=0}^{\delta^{k_1,j}\delta^{k_2,j}} \Omega_{j,\nu}^{k_1,k_2}(X,\Lambda,\widehat{\Lambda})T^{\nu},$$

and set, for $\beta \in \mathbb{N}^N$, $\xi \in V$, $(p,q) \in W^2$,

$$\Theta_{j,\nu,\beta}^{k_1,k_2}(\xi,p,q,\Lambda,\widehat{\Lambda}) := \frac{\partial^{|\beta|}}{\partial Z^{\beta}}\left[\Omega_{j,\nu}^{k_1,k_2}(\Psi(\xi,p,q,Z),\Lambda,\widehat{\Lambda})\right]\Big|_{Z=q}.$$

Note that each above function belongs to the ring $C^{\omega}(\overline{V\times W^2})[\Lambda,\widehat{\Lambda}]$. According to [14], the ring $C^{\omega}(\overline{V\times W^2})$ is noetherian and therefore so is $C^{\omega}(\overline{V\times W^2})[\Lambda,\widehat{\Lambda}]$.

For every k_1,k_2 as above, $j\in\{1,\dots,N'\}$ and $\nu\in\{0,\dots,\delta^{k_1,j}\delta^{k_2,j}\}$, let $E(k_1,k_2,j,\nu)\in\mathbb{Z}_+$ be such that the ideal generated by the $\Theta_{j,\nu,\beta}^{k_1,k_2}$ for all $\beta\in\mathbb{N}^N$ coincides that of generated by the $\Theta_{j,\nu,\beta}^{k_1,k_2}$ for $|\beta|\leqslant E(k_1,k_2,j,\nu)$ (in the above mentioned ring). Set

$$K = \max\left\{E(k_1,k_2,j,\nu) : k_1,k_2\in\{1,\dots,b\},\ j=1,\dots,N',\ \nu\in\{0,\dots,\delta^{k_1,j}\delta^{k_2,j}\}\right\}.$$

We now claim that the conclusion of the theorem holds with the above choice of K and with $M_0 = M\cap W$. Indeed, pick $q\in M\cap W$ and let f,g both belong to $\mathcal{S}_q$. Assume that $j_q^K f = j_q^K f$, that is, that $f(Z)-g(Z) = O(|Z-q|^{K+1})$. It follows from Proposition 2.7 that there exist $k_1,k_2\in\{1,\dots,b\}$ such that, for every $j=1,\dots,N'$,

$$\begin{cases} \mathcal{H}_j^{(k_1)}\Big(\Psi(\xi,p,q,Z),\big(\partial^{\mu}\bar{f}(\bar{p}),\partial^{\mu}f(p)\big)_{|\mu|\leqslant 2sr}, f_j(Z)\Big) = 0, \\ \dfrac{\partial\mathcal{H}_j^{(k_1)}}{\partial T}\Big(\Psi(\xi,p,q,Z),\big(\partial^{\mu}\bar{f}(\bar{p}),\partial^{\mu}f(p)\big)_{|\mu|\leqslant 2sr}, f_j(Z)\Big) \not\equiv 0, \end{cases} \tag{2.22}$$

$$\begin{cases} \mathcal{H}_j^{(k_2)}\big(\Psi(\xi,p,q,Z),(\partial^{\mu}\bar{g}(\bar{p}),\partial^{\mu}g(p))_{|\mu|\leqslant 2sr}, g_j(Z)\big) = 0, \\ \dfrac{\partial\mathcal{H}_j^{(k_2)}}{\partial T}\big(\Psi(\xi,p,q,Z),(\partial^{\mu}\bar{g}(\bar{p}),\partial^{\mu}g(p))_{|\mu|\leqslant 2sr}, g_j(Z)\big) \not\equiv 0, \end{cases} \tag{2.23}$$

for every $p\in M$ and $Z\in\mathbb{C}^N$ sufficiently close to q, and every $\xi\in V$. From the above construction, we have for every j

$$\Omega_j^{k_1,k_2}\Big(\Psi(\xi,p,q,Z),\big(\partial^{\mu}\bar{f}(\bar{p}),\partial^{\mu}f(p)\big)_{|\mu|\leqslant 2sr},(\partial^{\mu}\bar{g}(\bar{p}),\partial^{\mu}g(p))_{|\mu|\leqslant 2sr}, f_j(Z)-g_j(Z)\Big) = 0, \tag{2.24}$$

for all ξ,p,Z as above. Because of Remark 2.8 and our construction, we can assume that for every j

$$\Omega_{j,\nu}^{k_1,k_2}\Big(\Psi(\xi,p,q,Z),\big(\partial^{\mu}\bar{f}(\bar{p}),\partial^{\mu}f(p)\big)_{|\mu|\leqslant 2sr},(\partial^{\mu}\bar{g}(\bar{p}),\partial^{\mu}g(p))_{|\mu|\leqslant 2sr}\Big) \not\equiv 0, \text{ for } \nu = \delta^{k_1,j}\delta^{k_2,j}. \tag{2.25}$$

Suppose, by contradiction that $f \neq g$, that is, that there exists some j such that $f_j \neq g_j$. For this choice of j, denote by $\hat{\nu}$ the smallest $\nu \in \{0, \dots, \delta^{k_1,j}\delta^{k_2,j}\}$ such that

$$\Omega_{j,\nu}^{k_1,k_2}\left(\Psi(\xi, p, q, Z), \left(\partial^\mu \bar{f}(\bar{p}), \partial^\mu f(p)\right)_{|\mu|\leqslant 2sr}, (\partial^\mu \bar{g}(\bar{p}), \partial^\mu g(p))_{|\mu|\leqslant 2sr}\right) \not\equiv 0,$$

which exists in view of (2.25). Rewriting (2.24) and using that $f_j \neq g_j$, we reach that

$$\begin{cases} \Omega_{j,\hat{\nu}}^{k_1,k_2}\left(\Psi(\xi, p, q, Z), \left(\partial^\mu \bar{f}(\bar{p}), \partial^\mu f(p)\right)_{|\mu|\leqslant 2sr}, (\partial^\mu \bar{g}(\bar{p}), \partial^\mu g(p))_{|\mu|\leqslant 2sr}\right) = \\ -\sum_{\nu>\hat{\nu}} \Omega_{j,\hat{\nu}}^{k_1,k_2}\left(\Psi(\xi, p, q, Z), \left(\partial^\mu \bar{f}(\bar{p}), \partial^\mu f(p)\right)_{|\mu|\leqslant 2sr}, (\partial^\mu \bar{g}(\bar{p}), \partial^\mu g(p))_{|\mu|\leqslant 2sr}\right)\left(f_j(Z) - g_j(Z)\right)^\nu. \end{cases} \tag{2.26}$$

Hence (2.26) implies that for all ξ, Z, p as above,

$$\Omega_{j,\hat{\nu}}^{k_1,k_2}\left(\Psi(\xi, p, q, Z), \left(\partial^\mu \bar{f}(\bar{p}), \partial^\mu f(p)\right)_{|\mu|\leqslant 2sr}, (\partial^\mu \bar{g}(\bar{p}), \partial^\mu g(p))_{|\mu|\leqslant 2sr}\right) = O(|Z - q|^{K+1}), \tag{2.27}$$

that is, that for all ξ, p as above

$$\Theta_{j,\hat{\nu},\beta}^{k_1,k_2}\left(\xi, p, q, \left(\partial^\mu \bar{f}(\bar{p}), \partial^\mu f(p)\right)_{|\mu|\leqslant 2sr}, (\partial^\mu \bar{g}(\bar{p}), \partial^\mu g(p))_{|\mu|\leqslant 2sr}\right) = 0, \ \forall \beta, \text{ with } |\beta| \leqslant K.$$

From our choice of K, it follows that

$$\Theta_{j,\hat{\nu},\beta}^{k_1,k_2}\left(\xi, p, q, \left(\partial^\mu \bar{f}(\bar{p}), \partial^\mu f(p)\right)_{|\mu|\leqslant 2sr}, (\partial^\mu \bar{g}(\bar{p}), \partial^\mu g(p))_{|\mu|\leqslant 2sr}\right) = 0, \ \forall \beta \in \mathbb{N}^N,$$

and hence that for all $\xi \in V$, $p \in M$ and $Z \in \mathbb{C}^N$ sufficiently close to q

$$\Omega_{j,\hat{\nu}}^{k_1,k_2}\left(\Psi(\xi, p, q, Z), \left(\partial^\mu \bar{f}(\bar{p}), \partial^\mu f(p)\right)_{|\mu|\leqslant 2sr}, (\partial^\mu \bar{g}(\bar{p}), \partial^\mu g(p))_{|\mu|\leqslant 2sr}\right) = 0,$$

which contradicts the choice of $\hat{\nu}$. Hence $f = g$ and the theorem is proven.

3 | PROOF OF THE UNIVERSAL ALGEBRAIC PARAMETRIZATION PROPERTY

In this section, we prove the second main step of the proof of Theorem 1.1, which consists essentially of proving that under the assumptions of Theorem 1.1, the universal algebraic parametrization $(*)$ studied in the previous section is satisfied, see Theorem 3.3.

3.1 | 2-approximate CR deformations

We first define the important notion used in Theorem 1.1 and essentially going back to [21].

Definition 3.1. Let $M \subset \mathbb{C}^N$ and $M' \subset \mathbb{C}^{N'}$ be, respectively, a real-analytic CR submanifold and a real-analytic submanifold, $\mathcal{S}$ a subsheaf of $\mathcal{A}_{CR}^\infty(M, M')$. We say that a germ of a C^∞-smooth

CR map $B\colon (M\times\mathbb{C}^k,(p,0))\to\mathbb{C}^{N'}$, for some point $p\in M$ and some integer $k\geqslant 1$, is a germ of a 2-approximate CR $\mathcal{S}$-deformation from M into M' if it satisfies the following properties :

(i) $B|_{t=0}\in\mathcal{S}_p$;
(ii) $\mathrm{rk}\frac{\partial B}{\partial t}(p,0)=k$;
(iii) for every germ of a real-analytic function $\rho\colon (M',B(p,0))\to\mathbb{R}$, vanishing on M', we have

$$\rho(B(\xi,t),\overline{B(\xi,t)})=O(|t|^3),$$

for $\xi\in M$ near p and $t\in\mathbb{C}^k$ close to 0.

If a germ of a $\mathcal{C}^\infty$-smooth CR map $B\colon (M\times\mathbb{C}^k,(p,0))\to\mathbb{C}^{N'}$ only satisfies (ii) and (iii), we simply say that B is a (germ of a) 2-approximation CR deformation from M into M'.

Definition 3.1 has a nice geometric interpretation when the target M' is CR. First note that by (i), $B|_{t=0}\in\mathcal{S}_p$ maps (M,p) into $M'\subset\mathbb{C}^{N'}_{Z'}$. If we expand

$$B(\xi,t)=B_0(\xi)+\sum_{\alpha\in\mathbb{N}^k}B_\alpha(\xi)t^\alpha,\ B_\alpha=(B_\alpha^1,\dots,B_\alpha^{N'}),$$

we see that for every $\alpha\in\mathbb{N}^k$ with $|\alpha|=1$ the holomorphic vector field

$$X_\alpha=\sum_{j=1}^{N'}B_\alpha^j(\xi)\frac{\partial}{\partial Z'_j}$$

is *tangent to* M' along (the germ at $B_0(p)$ of) $B_0(M)$. We can think of X_α as a CR section of the pullback bundle $B_0^*T^{(1,0)}M'$. Furthermore, the k-dimensional space $E\subset B_0^*T^{(1,0)}M'$ spanned by the first-order vector fields X_α for $|\alpha|=1$ near p is *Levi-null*. This condition can be expressed in two ways. First, directly from (iii) we see that for *every* real-analytic function ρ vanishing on M' near $B_0(p)$, the Levi form $\mathcal{L}^\rho$ satisfies:

$$\begin{aligned}\mathcal{L}^\rho(X_\alpha,X_\beta)=\sum_{j,\ell=1}^{N'}\frac{\partial^2\rho(B_0(\xi),\overline{B_0(\xi)})}{\partial Z'_j\bar Z'_\ell}X_\alpha^j(\xi)\overline{X_\beta^\ell(\xi)}=0,\\ \sum_{j=1}^{N'}X_\alpha^j(\xi)\frac{\partial\rho}{\partial Z'_j}(B_0(\xi),\overline{B_0(\xi)})=0,\end{aligned}\tag{3.1}$$

for $|\alpha|,|\beta|=1$ and $\xi\in M$ near p. Equivalently, if we define the Levi form of M' abstractly as

$$\mathcal{L}\colon (T^{(1,0)}M)^2\to {}^{\mathbb{C}TM}\!\big/_{T^{(1,0)}M'\oplus T^{(0,1)}M'},\quad \mathcal{L}(X_p,Y_p)=[X,\bar Y]_p\mod T^{(1,0)}M'\oplus T^{(0,1)}M',$$

then $\mathcal{L}$ pulls back to $B_0^*T^{(1,0)}M'$, and the condition is then expressed as $\mathcal{L}(X_\alpha,X_\beta)=0$ for every α,β as above. Hence, a 2-approximate CR $\mathcal{S}$-deformation from M into M' gives rise to a CR family of Levi-null vectors along the image of M under B_0. In particular, if M' is a (weakly) pseudoconvex real hypersurface, there is a neighbourhood ω of p in M such that $B_0(\omega)$ is contained in the Levi-degenerate set of M'. We thus have shown the following:

Lemma 3.2. *Let $M \subset \mathbb{C}^N$ be a real-analytic CR submanifold, $M' \subset \mathbb{C}^{N'}$ a real-analytic pseudoconvex hypersurface, and $\mathcal{S}$ be the subsheaf of $\mathcal{A}^{\omega}_{\mathrm{CR}}(M, M')$ consisting of those real-analytic CR maps mapping no open subset of M into the Levi-degenerate set of M'. Then there is no germ of a 2-approximate CR $\mathcal{S}$-deformation from M into M'.*

3.2 | Universal algebraic parametrization

The main result of Section 3 is that the non-existence of 2-approximate CR deformations is sufficient for the property $(*)$ to hold for a given sheaf of maps $\mathcal{S}$:

Theorem 3.3. *Let $M \subset \mathbb{C}^N$ be a real-analytic generic submanifold, $M' \subset \mathbb{C}^{N'}$ be a Nash submanifold, and $\mathcal{S}$ be a subsheaf of $\mathcal{A}^{\omega}_{CR}(M, M')$. Assume that there is no germ of a 2-approximate CR $\mathcal{S}$-deformation from M into M'. Then $\mathcal{S}$ satisfies property $(*)$ from Definition 2.1.*

The proof of Theorem 3.3 is partly built on some arguments used in [20, 27] where special types of target manifolds were considered (spheres or strongly pseudoconvex manifolds) in the finite jet determination and regularity problem for CR maps. We show here how the ideas developed there can be generalized to deal with the more general targets considered in Theorem 3.3.

Denote by d' the codimension of M' in $\mathbb{C}^{N'} \simeq \mathbb{R}^{2N'}$. Since M' is a Nash submanifold, we may find, see, for example, [1], a finite open (semi-algebraic) covering of $M' = \cup_{j=1}^m M'_j$ such that each M'_j is Nash diffeomorphic to $\mathbb{R}^{2N'-d'}$. Hence we may assume that there exist open subsets $\Omega_1, \dots, \Omega_m$ of $\mathbb{C}^{N'} \simeq \mathbb{R}^{2N'}$ and real-algebraic maps $\rho_j : \Omega_j \to \mathbb{R}^{d'}$ of rank d', such that $M'_j = \{Z' \in \Omega_j : \rho_j(Z', \bar{Z}') = 0\}$. We fix such a choice of real-algebraic maps for the remainder of the proof of Theorem 3.3.

Note that in what follows, we will consider the real-algebraic maps ρ_j as real-algebraic maps defined over Ω_j, but also complexify them and view them as complex algebraic maps defined over some fixed open neighbourhood of $\Omega_j^{\mathbb{C}} \subset \mathbb{C}^{2N'}$ of $\{(Z', \bar{Z}') \in \mathbb{C}^{2N'} : Z' \in \Omega_j\}$ and keep the same notation for the complexified map $\rho_j(Z', \zeta')$. As a consequence, for any open subset $O \subset \mathbb{R}^k$, we shall denote by $\mathcal{C}^{\mathrm{alg}}(O)$ the ring of real-algebraic functions over O, and for any open subset $\mathcal{O} \subset \mathbb{C}^r$, the ring of complex-algebraic (or algebraic holomorphic) functions over $\mathcal{H}^{\mathrm{alg}}(\mathcal{O})$.

3.3 | Degeneracy of local holomorphic maps

We shall prove Theorem 3.3 by showing that property $(*)_p$ holds for every $p \in M$. We therefore fix in what follows a point $p \in M$ which we may assume to be the origin and use normal coordinates and the notation for defining equations and coordinates for M near 0 from Section 2.3.2.

We are going to use the basis of real-analytic CR vector fields $\bar{L}_1, \dots, \bar{L}_n$ defined in a (sufficiently small) neighbourhood U of 0 in $\mathbb{C}^N$ by

$$\bar{L}_j = \frac{\partial}{\partial \bar{z}_j} + \sum_{\nu=1}^{d} \bar{Q}^{\nu}_{\bar{z}_j}(\bar{z}, z, w) \frac{\partial}{\partial \bar{w}_\nu}, \quad j = 1, \dots, n.$$

For $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, we write as usual $\bar{L}^\alpha := \bar{L}_1^{\alpha_1}, \dots, \bar{L}_n^{\alpha_n}$. Setting $\zeta^1 = (\chi^1, \tau^1) \in \mathbb{C}^n \times \mathbb{C}^d$, the complexifications of these CR fields yield holomorphic vector fields

$$\mathcal{L}_j := \frac{\partial}{\partial \chi_j^1} + \sum_{\nu=1}^{d} \bar{Q}^\nu_{\chi_j^1}(\chi^1, z, w) \frac{\partial}{\partial \tau_\nu^1}, \quad j = 1, \dots, n, \tag{3.2}$$

tangent to $\mathcal{M}$. We shall also consider the holomorphic vector fields

$$\mathcal{T}_j := \frac{\partial}{\partial z_j^1} + \sum_{\nu=1}^{d} Q^\nu_{z_j^1}(z^1, \chi^1, \tau^1) \frac{\partial}{\partial w_\nu^1}, \quad j = 1, \dots, n, \tag{3.3}$$

where similarly $Z^1 = (z^1, w^1)$. Note that each vector field $\mathcal{T}_j$ is tangent to $\mathcal{M}^2$.

For every $q \in M_0 := M \cap U$, and for every germ of a real-analytic CR map $f : (M, q) \to M'$, there exists $j \in \{1, \dots, m\}$ such that $f : (M, q) \to M'_j$. In what follows, we drop the subscript notation for ρ_j and denote it by ρ. We also do the same for the associated open subset Ω_j that we write simply Ω. We note that this will not have any consequence, since it can be shown that all invariants associated to the map f that we will consider below are independent of the choice of j, and, most importantly, that for the whole collection of germs of maps under consideration, we always have only *finitely* many choices of open subsets M_νs in which the images of the germs lie. Having mentioned this, we now write $\rho = (\rho^{(1)}, \dots, \rho^{(d')})$ and define:

$$\mu^f := \mathrm{Rk} \left\{ \bar{L}^\alpha \rho^{(i)}_{Z'}(f, \bar{f}) : |\alpha| \leqslant N', \ 1 \leqslant i \leqslant d' \right\}, \tag{3.4}$$

where Rk stands for the generic rank (of a germ of a real-analytic map at q). It is easy to show, and also well-known, that μ^f does not depend of the choice of basis of CR vector fields. It can also be shown that it is independent of the choice of the real-algebraic defining function ρ of M' chosen near $f(q)$ (see [19, 20]).

As in [20], the (generic) *degeneracy* of f is defined by $\kappa^f := N' - \mu^f$. If $\kappa^f > 0$ we say that f is a *holomorphically degenerate* map, and if not, a *holomorphically non-degenerate* map.

The proof of Theorem 3.3 will be split into whether maps under consideration are holomorphically degenerate or not.

3.4 | Proof of Theorem 3.3 for holomorphically non-degenerate maps

We consider a germ f as above, at a point $q \in M_0 = U \cap M$, satisfying $\kappa^f = 0$. Hence, there exist $\alpha^{(1)}, \dots, \alpha^{(N')} \in \mathbb{N}^n$ with each $|\alpha^{(\ell)}| \leqslant N'$, and $i_1, \dots, i_{N'} \in \{1, \dots, d'\}$ such that we have on (M, q), that is, on M near q:

$$\det\left(\bar{L}^{\alpha^{(\ell)}} \rho^{(i_\ell)}_{Z'_k}(f, \bar{f}) \right)_{1 \leqslant k, \ell \leqslant N'} \not\equiv 0. \tag{3.5}$$

Note furthermore that we have on (M, q)

$$\bar{L}^{\alpha^{(\ell)}} \rho^{(i_\ell)}(f, \bar{f}) = 0, \ 1 \leqslant \ell \leqslant N'. \tag{3.6}$$

It is clear that (3.6) may be written as an identity near $q \in M$ in the form

$$\mathcal{B}^{\ell}\left(\left(\bar{L}^{\alpha}\bar{f}\right)_{|\alpha|\leqslant N'}, \bar{f}, f\right) = 0,\ 1 \leqslant \ell \leqslant N', \tag{3.7}$$

for some universal map $\mathcal{B} = (\mathcal{B}^1, \dots, \mathcal{B}^{N'})$ where each $\mathcal{B}^{\ell} = \mathcal{B}^{\ell}((\Lambda_\alpha)_{|\alpha|\leqslant N'}, \zeta', Z') \in \mathcal{H}^{\mathrm{alg}}(\Omega^{\mathbb{C}})[(\Lambda_\alpha)_{|\alpha|\leqslant N'}]$ depends only on ρ and the choice of the $\alpha^{(\ell)}$s and i_ℓs. Complexifying (3.6), and using (3.7), (3.5) and Lemma 6.1, we obtain the existence of finitely many $\mathbb{C}^{N'}$-valued polynomial maps $\mathcal{D}^1, \dots, \mathcal{D}^e \in (\mathbb{C}[(\Lambda_\alpha)_{|\alpha|\leqslant N'}, T])^{N'}$, depending only on $\mathcal{B}$, such that for some $\nu \in \{1, \dots, e\}$, for $(Z, \zeta^1) \in \mathcal{M}$ near $(q, \bar{q})$ and every $j = 1, \dots, N'$, we have

$$\begin{cases} \mathcal{D}_j^{\nu}\left(\left(\mathcal{L}^{\alpha}\bar{f}(\zeta^1)\right)_{|\alpha|\leqslant N'}, f_j(Z)\right) & = 0, \\ \dfrac{\partial \mathcal{D}_j^{\nu}}{\partial T}\left(\left(\mathcal{L}^{\alpha}\bar{f}(\zeta^1)\right)_{|\alpha|\leqslant N'}, f_j(Z)\right) & \not\equiv 0. \end{cases} \tag{3.8}$$

Hence, since we have only finitely many choices for the map $\mathcal{B}$, we have proved that property $(*)_0$ holds for every germ of a holomorphically degenerate map $f \in \mathcal{S}_q$ with $q \in M_0$. (Note that we have not used anywhere above the assumption about non-existence of 2-approximate CR $\mathcal{S}$-deformations.) The proof of Theorem 3.3, in this first case, is therefore complete.

3.5 | Proof of Theorem 3.3 for holomorphically degenerate maps

For germs of holomorphically degenerate maps, their degeneracy belongs to $\{1, \dots, N'\}$. Hence, it is enough to prove that property $(*)_0$ holds for maps of a fixed degeneracy $\kappa > 0$. Let M_0 and U be as before, $q \in M_0$ and $f \in \mathcal{S}_q$ of degeneracy $\kappa^f = \kappa \geqslant 1$.

We choose $\alpha^{(1)}, \dots, \alpha^{(N'-\kappa)} \in \mathbb{N}^n$ with $|\alpha^{(\ell)}| \leqslant N' - \kappa$ and $i_1, \dots, i_{N'-\kappa} \in \{1, \dots, d'\}$ such that near $q \in M$, we have:

$$\mathrm{Rk}\left\{\bar{L}^{\alpha^{(\ell)}}\rho_{Z'}^{(i_\ell)}(f, \bar{f}) : 1 \leqslant \ell \leqslant N' - \kappa\right\} = N' - \kappa. \tag{3.9}$$

In order to prove the proposition, we will supplement the equations

$$\bar{L}^{\alpha^{(\ell)}}\rho(f, \bar{f}) = 0,\ 1 \leqslant \ell \leqslant N' - \kappa, \tag{3.10}$$

which hold on M near q with other ones following [20] (see also [6, 22] for the smooth case).

In the $N' \times (N' - \kappa)$ matrix $(\bar{L}^{\alpha^{(\ell)}}\rho_{Z'_k}^{i_\ell}(f, \bar{f}))_{\substack{1\leqslant k\leqslant N' \\ 1\leqslant \ell\leqslant N'-\kappa}}$, we choose a minor of size $N' - \kappa$ whose (generic) rank (near q) equals $N' - \kappa$. Observe that even though such a choice depends on the map f, there are only *finitely many choices* of such minors for the entire collection of maps f under consideration. We fix in what follows the choice of such a minor and assume, without loss of generality, that the $(N' - \kappa) \times (N' - \kappa)$ matrix $(\bar{L}^{\alpha^{(\ell)}}\rho_{Z'_k}^{i_\ell}(f, \bar{f}))_{1\leqslant k,\ell\leqslant N'-\kappa}$ is of (generic) rank $N' - \kappa$ (near q).

In what follows, we shall call a map *admissible*, if it depends only on ρ, κ and the above choice of minor (and not on q and f). For every $q \in M_0$, we denote by $\mathbb{M}_q$, respectively, $\mathbb{M}_q^{CR}$, the quotient field of the ring of germs at q of real-analytic, respectively real-analytic and CR, functions on M.

An element of $\mathbb{M}_q$ is CR if it belongs to $\mathbb{M}_q^{CR}$. The following lemma follows from inspecting the arguments of [20, section 4].

Lemma 3.4. *With the above notation, for every $q \in M_0$ and every $f \in \mathcal{S}_q$, there exist (unique) $V^1, \dots, V^\kappa \in (\mathbb{M}_q^{CR})^{N'}$ with the following properties:*

(i) *For every $j = 1, \dots, \kappa$,*

$$ {}^tV^j \cdot \rho_{Z'}(f, \bar{f}) = \sum_{k=1}^{N'} V_k^j \rho_{Z'_k}(f, \bar{f}) = 0, \text{ in } (\mathbb{M}_q)^{d'}. $$

(ii) *There exist a universal polynomial $\mathcal{D}$, and, for every $j = 1, \dots, \kappa$, universal $\mathbb{C}^{N'-\kappa}$-valued polynomial maps of their arguments $\mathcal{G}^j$ (independent of $q \in M_0$ and $f \in \mathcal{S}_q$, and depending only on κ and the above-mentioned choice of minor) such that, for every $q \in M_0$ and for every map $f \in \mathcal{S}_q$ as above, $\mathcal{D}((\bar{L}^\alpha \rho_{Z'}(f, \bar{f}))_{|\alpha| \leqslant N'}) \not\equiv 0$, and*

$$ (V_1^j, \dots, V_{N'-\kappa}^j) = \frac{\mathcal{G}^j\Big(\big(\bar{L}^\alpha \rho_{Z'}(f, \bar{f})\big)_{|\alpha| \leqslant N'}\Big)}{\mathcal{D}\Big(\big(\bar{L}^\alpha \rho_{Z'}(f, \bar{f})\big)_{|\alpha| \leqslant N'}\Big)}, \quad V_\nu^j = \delta_{\nu, j+N'-\kappa}, \ \nu = N' - \kappa + 1, \dots, N', $$

where $\delta_{\nu, j+N'-\kappa}$ denotes the usual Kronecker symbol.

Remark 3.5.

(a) Note furthermore that the construction given in [20] also shows that the vectors $V^1, \dots, V^\kappa$ in the above lemma form a basis over $\mathbb{M}_q$ of the vector space consisting of those vectors $X \in (\mathbb{M}_q)^{N'}$ satisfying $X \cdot \bar{L}^\alpha \rho_{Z'}(f, \bar{f}) = 0$ for all $\alpha \in \mathbb{N}^n$ with $|\alpha| \leqslant N'$.
(b) In what follows, we shall consider each vector V^j, $j = 1, \dots, \kappa$, as column vectors and $\rho_{Z'}$ as a $N' \times d'$ matrix.

Writing $\mathbb{V} = (V^1, \dots, V^\kappa)$, for every $q \in M_0$, each $f \in \mathcal{S}_q$ (as above) satisfies the following system in $\mathbb{M}_q$:

$$ \begin{cases} \rho(f, \bar{f}) = 0, \\ \overline{{}^t\mathbb{V}} \cdot \rho_{\bar{Z}'}(f, \bar{f}) = 0 \\ {}^t\mathbb{V} \cdot \rho_{Z'}(f, \bar{f}) = 0 \\ \bar{L}_\nu \mathbb{V} = 0, \ \nu = 1, \dots, n. \end{cases} \tag{3.11} $$

It follows from Lemma 3.4(ii) that we may write

$$ \mathbb{V} = \mathcal{R}\Big(\big(\bar{L}^\alpha \bar{f}\big)_{|\alpha| \leqslant N'}, \bar{f}, f\Big) \tag{3.12} $$

for some universal admissible map $\mathcal{R} = \mathcal{R}((\Lambda_\alpha)_{|\alpha| \leqslant N'}, \zeta', Z')$ whose components belong to the quotient field of $\mathcal{H}^{\mathrm{alg}}(\Omega^{\mathbb{C}})[(\Lambda_\alpha)_{|\alpha| \leqslant N'}]$. Here, we view $\mathbb{V}$ and $\mathcal{R}$ as $N' \times \kappa$ matrices, and we will denote the entries of $\mathcal{R}$ by $(\mathcal{R}_{ij})_{\substack{1 \leqslant i \leqslant N' \\ 1 \leqslant j \leqslant \kappa}}$.

As a consequence, the system (3.11) may be rewritten as a system of equations of the form

$$\mathrm{Y}\Big(\overline{\mathbb{V}}, \big(\bar{L}^{\alpha}\bar{f}\big)_{|\alpha|\leqslant N'+1}\bar{f}, f\Big) = 0, \tag{3.13}$$

satisfied in $\mathbb{M}_q$, by every $f \in \mathcal{S}_q$ as above, for $q \in M_0$. Here, $\mathrm{Y} = \mathrm{Y}(T, (\Lambda_\alpha)_{|\alpha|\leqslant N'+1}, \zeta', Z')$ is a universal admissible map with components in the ring $\mathcal{H}^{\mathrm{alg}}(\Omega^{\mathbb{C}})[T, (\Lambda_\alpha)_{|\alpha|\leqslant N'+1}]$. We now claim the following:

Claim 3.6. Under the assumptions of Theorem 3.3, for every $q \in M_0$ and every $f \in \mathcal{S}_q$ as above,

$$\mathrm{Rk}\left\{\bar{L}^{\beta}\mathrm{Y}_{Z'}\Big(\overline{\mathbb{V}}, \big(\bar{L}^{\alpha}\bar{f}\big)_{|\alpha|\leqslant N'+1}, \bar{f}, f\Big) \,:\, |\beta| \leqslant N'\right\} = N',$$

where the rank is taken with respect to the field $\mathbb{M}_q$.

Claim 3.6 is proven in the case where M' is strictly pseudoconvex in [20]. The proof of Claim 3.6 in the present more general setting is deferred to Section 3.6. We shall now complete the proof of Theorem 3.3 assuming the claim.

For every $\beta \in \mathbb{N}^n$ with $|\beta| \leqslant N'$, thanks to the chain rule, we may write

$$\bar{L}^{\beta}\Big(\mathrm{Y}\Big(\overline{\mathbb{V}}, \big(\bar{L}^{\alpha}\bar{f}\big)_{|\alpha|\leqslant N'+1}, \bar{f}, f\Big)\Big) = \mathrm{Y}^{(\beta)}\Big(\big(\bar{L}^{\gamma}\overline{\mathbb{V}}\big)_{|\gamma|\leqslant N'}, \big(\bar{L}^{\alpha}\bar{f}\big)_{|\alpha|\leqslant 2N'+1}, \bar{f}, f\Big),$$

for some universal admissible map $\mathrm{Y}^{(\beta)} = \mathrm{Y}^{(\beta)}((T_\gamma)_{|\gamma|\leqslant N'}, (\Lambda_\alpha)_{|\alpha|\leqslant 2N'+1}, \zeta', Z')$ with components in $\mathcal{H}^{\mathrm{alg}}(\Omega^{\mathbb{C}})[(T_\gamma)_{|\gamma|\leqslant N'}, (\Lambda_\alpha)_{|\alpha|\leqslant 2N'+1}]$.

For every $q \in M_0$ and every $f \in \mathcal{S}_q$, we may extract, according to Claim 3.6, N' components from the map $(\mathrm{Y}^{(\beta)})_{|\beta|\leqslant N'}$, that we denote in what follows by $\widehat{\mathrm{Y}}$, such that

$$\mathrm{Rk}\left\{\widehat{\mathrm{Y}}_{Z'}\Big(\big(\bar{L}^{\gamma}\overline{\mathbb{V}}\big)_{|\gamma|\leqslant N'}, \big(\bar{L}^{\alpha}\bar{f}\big)_{|\alpha|\leqslant 2N'+1}, \bar{f}, f\Big)\right\} = N'. \tag{3.14}$$

Of course, such a choice depends on each map $f \in \mathcal{S}_q$; however, note again that there are only *finitely many* such possible choices. Furthermore, it follows from (3.13) that the identity

$$\widehat{\mathrm{Y}}\Big(\big(\bar{L}^{\gamma}\overline{\mathbb{V}}\big)_{|\gamma|\leqslant N'}, \big(\bar{L}^{\alpha}\bar{f}\big)_{|\alpha|\leqslant 2N'+1}, \bar{f}, f\Big) = 0 \tag{3.15}$$

holds in $\mathbb{M}_q$.

Complexifying (3.15), we have the following identity:

$$\widehat{\mathrm{Y}}\Big(\big(\mathcal{L}^{\gamma}\overline{\mathbb{V}}(\zeta^1)\big)_{|\gamma|\leqslant N'}, \big(\mathcal{L}^{\alpha}\bar{f}(\zeta^1)\big)_{|\alpha|\leqslant 2N'+1}, \bar{f}(\zeta^1), f(Z)\Big) = 0 \tag{3.16}$$

for $(Z, \zeta^1) \in \mathcal{M}$ near $(q, \bar{q})$ and where the above identity is understood in the field of fractions of germs at $(q, \bar{q})$ of holomorphic functions on $\mathcal{M}$. Complexifying and conjugating (3.12) yields

$$\overline{\mathbb{V}}(\zeta^1) = \overline{\mathcal{R}}\Big(\big(\mathcal{T}^{\alpha}f(Z^1)\big)_{|\alpha|\leqslant N'}, f(Z^1), \bar{f}(\zeta^1)\Big), \tag{3.17}$$

for $(Z^1,\zeta^1)\in\mathcal{M}$ near $(q,\bar q)$. Differentiating (3.17), we see that we may write

$$\begin{cases}\Big(\mathcal{L}^\gamma\overline{\mathbb{V}}(\zeta^1)\Big)_{|\gamma|\leqslant N'}=\mathcal{U}_1\Big(\Theta(Z,\zeta^1,Z^1),\big(\partial^\alpha f(Z^1)\big)_{|\alpha|\leqslant 2N'},\big(\partial^\alpha\bar f(\zeta^1)\big)_{|\alpha|\leqslant N'},f(Z^1),\bar f(\zeta^1)\Big),\\ \big(\mathcal{L}^\alpha\bar f(\zeta^1)\big)_{|\alpha|\leqslant 2N'+1}=\mathcal{U}_2\Big(\Theta(Z,\zeta^1,Z^1),\big(\partial^\alpha\bar f(\zeta^1)\big)_{|\alpha|\leqslant 2N'+1}\Big)\end{cases} \tag{3.18}$$

for some universal admissible map $(\mathcal{U}_1,\mathcal{U}_2)=\mathcal{U}=\mathcal{U}(X,(\Gamma_\alpha)_{|\alpha|\leqslant 2N'},(\Lambda_\alpha)_{|\alpha|\leqslant 2N'+1},Z',\zeta')$, depending, in addition, on the second choice of minor, and whose components belong to the quotient field of $\mathcal{H}^{\mathrm{alg}}(\Omega^{\mathbb{C}})[X,(\Gamma_\alpha)_{|\alpha|\leqslant 2N'},(\Lambda_\alpha)_{|\alpha|\leqslant 2N'+1}]$ and some universal holomorphic map Θ defined on U^3 and depending only on M. Substituting (3.18) into (3.16), clearing denominators and using Lemma 6.1 as well as our specific choice of $\widehat{\Upsilon}$ provides the required conclusion of Theorem 3.3.

3.6 | Proof of Claim 3.6

We now prove Claim 3.6 by contradiction. Hence let us assume that for some $q\in M_0$ and some $f\in\mathcal{S}_q$, we have $\mathrm{Rk}\,\{\bar L^\beta\Upsilon_{Z'}(\overline{\mathbb{V}},(\bar L^\alpha\bar f)_{|\alpha|\leqslant N'+1},\bar f,f)\,:\,|\beta|\leqslant N'\}=N'-\eta$ for some $\eta>0$. We note that from the construction of Υ given in (3.11), it follows that $0<\eta\leqslant\kappa$. Repeating the same arguments borrowed from [20] to derive Lemma 3.4, we get the following:

Lemma 3.7. *With the above notation, for q and f as chosen above, there exist $W^1,\dots,W^\eta\in(\mathbb{M}_q^{CR})^{N'}$, of (generic) rank η such that, for every $j=1,\dots,\eta$,*

$${}^tW^j\cdot\Upsilon_{Z'}\Big(\overline{\mathbb{V}},\big(\bar L^\alpha\bar f\big)_{|\alpha|\leqslant N'+1},\bar f,f\Big)=\sum_{\ell=1}^{N'}W^j_\ell\Upsilon_{Z'_\ell}\Big(\overline{\mathbb{V}},\big(\bar L^\alpha\bar f\big)_{|\alpha|\leqslant N'+1},\bar f,f\Big)=0,\ \text{ in } \mathbb{M}_q.$$

Since $\eta\geqslant 1$, applying the above lemma to one single vector $W:=W^1$ and using how Υ is defined through (3.11), we obtain in $\mathbb{M}_q$, for every $1\leqslant\ell\leqslant\kappa$,

$$\begin{cases}{}^tW\cdot\rho_{Z'}(f,\bar f) & =0,\\ {}^tW\cdot\rho_{Z'\bar Z'}(f,\bar f)\cdot\overline{V^\ell} & =0,\\ {}^tW\cdot V^\ell_{Z'}\cdot\rho_{Z'}(f,\bar f)+{}^tW\cdot\rho_{Z'Z'}(f,\bar f)\cdot V^\ell & =0,\\ {}^tW\cdot(\bar L_\nu V^\ell)_{Z'} & =0,\ \nu=1,\dots,n,\end{cases} \tag{3.19}$$

as identities in $\mathbb{M}_q$. In (3.19), $\rho_{Z'\bar Z'}(f,\bar f)$ (respectively, $\rho_{Z'Z'}(f,\bar f)$) is the $N'\times N'$ hermitian (respectively, symmetric) matrix with entries $(\rho_{Z_i\bar Z'_j}(f,\bar f))_{i,j}$ (respectively $(\rho_{Z_iZ'_j}(f,\bar f))_{i,j}$), $V^\ell_{Z'}$ is the $N'\times N'$ matrix with entries $((V^\ell_j)_{Z'_i})_{i,j}$ and $(\bar L_\nu V^\ell)_{Z'}$ is the $N'\times N'$ matrix with entries $((\bar L_\nu V^\ell_j)_{Z'_i})_{i,j}$ where

$$\Big(V^\ell_j\Big)_{Z'_i}:=\frac{\partial\mathcal{R}_{j\ell}}{\partial Z'_i}\Big(\big(\bar L^\alpha\bar f\big)_{|\alpha|\leqslant N'},\bar f,f\Big),\quad\text{and}\ \Big(\bar L_\nu V^\ell_j\Big)_{Z'_i}:=\bar L_\nu\Big(V^\ell_j\Big)_{Z'_i}.$$

Since W is CR, it follows from the first equation in (3.19) and Remark 3.5 that

$$W = \sum_{\ell=1}^{\kappa} \lambda_\ell V^\ell \tag{3.20}$$

for some $\lambda_\ell \in \mathbb{M}_q$. In fact, as a consequence of Cramer's rule and the fact that each V^ℓ is CR, each λ_ℓ must be CR as well. Hence, using (3.20), (3.19) and the fact that W and the λ_ℓ's are CR, we obtain

$$\begin{cases} {}^tW \cdot \rho_{Z'}(f,\bar{f}) & = 0, \\ {}^tW \cdot \rho_{Z'\bar{Z}'}(f,\bar{f}) \cdot \overline{W} & = 0, \\ {}^tW \cdot \left(\sum_{\ell=1}^{\kappa} \lambda_\ell V^\ell_{Z'}\right) \cdot \rho_{Z'}(f,\bar{f}) + {}^tW \cdot \rho_{Z'Z'}(f,\bar{f}) \cdot W & = 0, \\ \bar{L}_\nu\left({}^tW \cdot \left(\sum_{\ell=1}^{\kappa} \lambda_\ell (V^\ell)_{Z'}\right)\right) & = 0, \ \nu = 1, \dots, n. \end{cases} \tag{3.21}$$

If we set $\Delta := {}^t W \cdot (\sum_{\ell=1}^{\kappa} \lambda_\ell V^\ell_{Z'})$, it follows from the last equation in (3.21) that Δ is a (row) vector in $\mathbb{C}^{N'}$ with components in $\mathbb{M}_q^{CR}$. For $\tau \in \mathbb{C}$, we further set $D(\tau) = f + {}^t W \tau + \frac{1}{2}\tau^2 \Delta$ and note that $D(\tau) \in (\mathbb{M}_q^{CR}[\tau])^{N'}$. Since each component of ρ is real-valued, it follows from the three first equations of (3.21) that, in the ring $\mathbb{M}_q[[\tau,\bar{\tau}]]$, we have

$$\rho(D(\tau), \overline{D(\tau)}) = O(|\tau|^3). \tag{3.22}$$

Pick a point $q_0 \in M$ (arbitrarily close to q) so that W and Δ are real-analytic near q_0 and $W(q_0) \neq 0$. It follows from (3.22) that D is a (germ at $(q_0, 0)$) of a non-trivial 2-approximate CR $\mathcal{S}$-deformation from M into M', a contradiction. The claim is proven.

4 | PROOFS OF THEOREM 1.1, COROLLARIES 1.2, 1.5, 1.6 AND REMARKS

4.1 | Proof of Theorem 1.1

We need the following result which follows from inspecting [24].

Proposition 4.1. *Let $M \subset \mathbb{C}^N$ be a real-analytic CR submanifold, $M' \subset \mathbb{C}^{N'}$ a Nash submanifold, and $\mathcal{S}$ a subsheaf of $\mathcal{A}^\infty_{\mathrm{CR}}(M, M')$. Assume that M is minimal and there is no germ of a non-trivial 2-approximate CR $\mathcal{S}$-deformation from M into M'. Then $\mathcal{S} \subset \mathcal{A}^\omega_{CR}(M, M')$.*

Proof. By contradiction, let us assume that there exists a germ of a C^∞-smooth CR $f : (M, p) \to M'$, with $f \in \mathcal{S}_p$, that is not real-analytic at p. Then according to [24, Theorem 1.8], there exist a point $q \in M$ (arbitrarily close to p) and a germ of a real-analytic CR map $\Phi : (M \times \mathbb{C}^k_\tau, (q, h(q))) \to \mathbb{C}^{N'-k}$, where $f = (g, h) \in \mathbb{C}^{N'-k} \times \mathbb{C}^k$, $1 \leqslant k < N'$, such that $\Phi(\xi, h(\xi)) = g(\xi)$ for ξ near q and such that $(M \times \mathbb{C}^k, (q, h(q))) \ni (\xi, \tau) \mapsto (\Phi(\xi, \tau), \tau) \in M'$. Hence the germ at $(q, 0)$ of the C^∞-smooth CR map $B(\xi, t) = (\Phi(\xi, t + h(\xi)), t + h(\xi))$ defines a 2-approximate CR $\mathcal{S}$-deformation of M into M', a contradiction. $\square$

When M is generic, Theorem 1.1 follows from combining Proposition 4.1, Theorem 3.3 and Theorem 2.2. When M is not generic, then the conclusion follows easily from the generic case since any such M is locally biholomorphically equivalent to a $\widetilde{M} \times \{0\} \subset \mathbb{C}^{N-r} \times \mathbb{C}^r$ for some (real-analytic) generic submanifold $\widetilde{M}$ of $\mathbb{C}^{N-r}$ (see, for example, [3]). The proof is complete.

4.2 | Proof of Corollaries 1.2 and 1.6

Corollaries 1.2 and 1.6 follow from Theorem 1.1 and Lemma 3.2.

4.3 | Proof of Corollary 1.5

Corollary 1.5 follows as an immediate consequence of Theorem 1.1 and the following result, which is a consequence of [21, Proposition 6.4].

Proposition 4.2. *Let $M \subset \mathbb{C}^N$ and $M' \subset \mathbb{C}^{N'}$ be connected real-analytic Levi–non-degenerate hypersurfaces, of signature ℓ and ℓ', respectively, with $N, N' \geqslant 2$. Let $\mathcal{S}$ denote the subsheaf of $\mathcal{A}^{\infty}_{\mathrm{CR}}(M, M')$ consisting of CR transversal maps. If there exists a 2-approximate CR $\mathcal{S}$-deformation from M into M', then M and M' have different signatures and cosignatures.*

4.4 | Remarks

In our statement of Theorem 1.1, we have assumed that the target manifold M' is Nash. As the reader might have observed, our proof also applies to the more general setting of a real-algebraic submanifold M' that can be covered by finitely many open subsets $M'_1, \ldots, M'_m$ satisfying, for each $1 \leqslant j \leqslant m$, $M'_j = \{Z' \in \Omega_j : \rho_j(Z', \bar{Z}') = 0\}$ for some $\mathbb{R}^{d'}$-valued real-algebraic map ρ_j of rank d' over Ω_j.

The methods previously developed to study finite jet determination can also be used to derive another result, of independent interest, regarding the extendability properties of germs of CR maps. More precisely, the following result follows by combining Theorem 3.3 and Proposition 2.7.

Proposition 4.3. *Let $M \subset \mathbb{C}^N$ be a real-analytic generic submanifold, $M' \subset \mathbb{C}^{N'}$ a Nash submanifold, and $\mathcal{S}$ a subsheaf of $\mathcal{A}^{\infty}_{\mathrm{CR}}(M, M')$. Assume that M is minimal and there is no germ of a 2-approximate CR $\mathcal{S}$-deformation from M into M'. Then for every point $p_0 \in M$, there exists a neighbourhood V of p_0 in $\mathbb{C}^N$ such that for every $q \in V$, every $f \in \mathcal{S}_q$ extends as a meromorphic correspondence over V.*

It is unknown, in general, whether every f as in Proposition 4.3 extends as a meromorphic map over V. However, in the case where M' is the sphere, such a conclusion is known to be true (see [27]).

5 | HYPERSURFACES FOLIATED BY COMPLEX SUBMANIFOLDS AND PROOF OF COROLLARY 1.7

In this section, we want to discuss in further details applications of Theorem 1.1 to the case where the target manifold M' is a weakly pseudoconvex real hypersurface, that is, in addition, everywhere Levi-degenerate. The considerations below are valid with M' being real-analytic (instead of Nash).

It is a classical fact that near uniformly pseudoconvex points (that is, points where the rank of the Levi form is locally constant) such hypersurfaces carry a foliation η (the so-called *Levi foliation*) by complex manifolds η_p induced by the distribution of the Levi kernels $\mathcal{N}_q \subset T^{(1,0)}_q M'$, that is, defined through $T_q \eta_p = \mathcal{N}_q$, for $q \in \eta_p$ (see [13]).

Given *any* foliation η of M' by complex manifolds, Greilhuber and the first author introduced in [16] an invariant ν on M' measuring the (CR) dimension of images of possible maps which would allow for a deformation in the direction of the leaves, given by

$$\nu_{p'} := \max_{0 \neq V_{p'} \in T_{p'}\eta} \dim_{\mathbb{C}} \ker \left(\bar{L}_{p'} \to \mathbb{P}_{T\mathbb{C}^{N'}/T\eta}(\bar{L}_{p'} V) \right) - \dim_{\mathbb{C}} \eta. \tag{5.1}$$

This definition is independent of any particular choices of extensions (in the appropriate bundles), that is, the map

$$R : T^{(0,1)}_{p'} M' \times T_{p'}\eta \to {}^{T_{p'}\mathbb{C}^{N'}}\!\big/_{T_{p'}\eta} \quad R(\bar{L}_{p'}, V_{p'}) = \mathbb{P}_{T_{p'}\mathbb{C}^{N'}/T_{p'}\eta}(\bar{L}_{p'} V)$$

is a well-defined linear map (see [16] for details). The relevance of the invariant ν to study approximate CR deformations is as follows. Given, in addition, a real-analytic CR submanifold $M \subset \mathbb{C}^N$, the existence of a 2-approximate CR deformation $B = B(\xi, t)$ from M into M' (deforming $B_0(\xi) := B(\xi, 0)$ as in Definition 3.1), near a point $p \in M$, would necessarily lead to a 'large' invariant $\nu_{B_0(p)}$, if one can show that the (1,0)-vector fields $B_{t_j}(\xi, 0) \in T_{B_0(\xi)}\eta$, $1 \leqslant j \leqslant k$, since these are CR in the directions along $B_0(M)$. For the convenience of the reader, we translate some of the results in [16] into the language we are using here, and give short proofs. As soon as we talk about a uniformly pseudoconvex hypersurface, we will consider it as endowed with the Levi foliation η.

Given a CR map $H : M \to M'$, its CR rank at the point $p \in M$ is defined as the rank of the (complex-linear) map $H'|_{T^{(1,0)}_p M} : T^{(1,0)}_p M \to T^{(1,0)}_{H(p)} M'$ and is denoted $\mathrm{CR{-}rk}_p\, H$. We also define the CR rank of the germ of H at p to be the maximum rank achieved over sufficiently small neighbourhoods of p in M.

Lemma 5.1. *Assume that M and M' are as above, and that, in addition, M' is a uniformly pseudoconvex hypersurface. Let $B : (M \times \mathbb{C}, (p, 0)) \to \mathbb{C}^{N'}$ be a (germ of a) 2-approximate CR deformation from M into M' and write $B(\xi, t) = B_0(\xi) + B_1(\xi)t + B_2(\xi)t^2 + O(t^2)$. Then for every $\xi \in M$ near p, $B_1(\xi) \in T_{B_0(\xi)}\eta$, and any CR vector of the form $\bar{L}'_{B_0(\xi)} = B_0'(\xi)\bar{L}_\xi$ satisfies $\bar{L}'_{B_0(\xi)} B_1(\xi) = 0$. In particular we have that for every ξ*

$$\nu_{B_0(\xi)} \geqslant \mathrm{CR{-}rk}_\xi\, B_0 - \dim_{\mathbb{C}} B_0'(\xi)\Big(T^{(1,0)}_\xi M\Big) \cap T_{B_0(\xi)}\eta = \dim_{\mathbb{C}} {}^{B_0'(\xi)\left(T^{(1,0)}_\xi M\right)}\!\big/_{T_{B_0(\xi)}\eta}.$$

Proof. We have already observed that $B_1(\xi)$ is Levi-null for every ξ, and since M' is pseudoconvex, this implies that $B_1(\xi)$ is in the Levi-kernel, that is, $B_1(\xi) \in T_{B_0(\xi)}\eta$.

As for the second part of the Lemma, let $\bar{L}'_{B_0(\xi)} = B_0'(\xi)\bar{L}_\xi$ and assume without loss of generality that $\bar{L}'_{B_0(\xi)} \neq 0$. We choose a non-singular parametrized real surface $\gamma : \mathbb{C}_\zeta \supset \{|\zeta| < \varepsilon\} \to M$, $\zeta \mapsto \gamma(\zeta, \bar{\zeta})$ such that $\gamma(0) = \xi$ and $\bar{L}_\xi = \gamma_* \frac{\partial}{\partial \bar{\zeta}}\Big|_{\zeta=0}$. Note that the image of $\hat{\gamma} = B_0 \circ \gamma$ is also a non-singular real surface, satisfying $\hat{\gamma}(0) = B_0(\xi)$ and $\hat{\gamma}_* \frac{\partial}{\partial \bar{\zeta}}\Big|_{\zeta=0} = \bar{L}'_{B_0(\xi)}$. By assumption, the map $\hat{B}_1 = B_1 \circ \gamma$ is smooth and takes values in $T\eta$; we can therefore write $\hat{B}_1 = \tilde{B}_1 \circ B_0 \circ \gamma$ for a smooth section $\tilde{B}_1$ of $T\eta$.

It follows that

$$\bar{L}'_{B_0(\xi)} B_1(\xi) = \frac{\partial}{\partial \bar{\zeta}}\Big|_{\zeta=0} \tilde{B}_1(B_0(\gamma(\zeta, \bar{\zeta}))) = \frac{\partial}{\partial \bar{\zeta}}\Big|_{\zeta=0} B_1(\gamma(\zeta, \bar{\zeta})) = 0,$$

because B_1 is CR on M. This proves the first part of the lemma. The second part is a direct application of the first part. □

We therefore obtain the following result about maps into uniformly pseudoconvex hypersurfaces whose invariant ν is everywhere zero:

Proposition 5.2. *Let M, M' be as in Lemma 5.1, and assume that M is minimal and that $\nu \equiv 0$ all over M'. Let $\mathcal{S}$ be a subsheaf of $\mathcal{A}^\infty_{CR}(M, M')$. If, for every $p \in M$, every $h \in \mathcal{S}_p$ does not map (M, p) completely into the leaf $\eta_{h(p)}$ of the Levi foliation of M', then there is no germ of 2-approximate $\mathcal{S}$-deformation from M into M'. In particular, this assumption is satisfied if the CR rank of every $h \in \mathcal{S}_p$ exceeds the (complex) fiber dimension of η.*

Proof. By Lemma 5.1, in order to exclude 2-approximate $\mathcal{S}$-deformations, we only have to check that, for every $p \in M$ and every $h \in \mathcal{S}_p$, there exists $\xi \in M$ close to p such that $h'(\xi)(T^{(1,0)}_\xi M) \not\subset T_{h(\xi)}\eta$. So assume that this is not the case, that is, for a full neighbourhood U of p in M, we have that $h'(\xi)(T^{(1,0)}_\xi M) \subset T_{h(\xi)}\eta$. Identifying $T^{(1,0)}M$ with T^cM, we hence have $h'(\xi)(T^c_\xi M) \subset T_{h(\xi)}\eta$ for each $\xi \in U$. Hence, every polygonal path $\gamma \subset U$ starting at p tangent to T^cM must be mapped under h to the leaf $\eta_{h(p)}$. By minimality (see [3]), every point in a neighbourhood $\widetilde{U} \subset U$ of p in M can be obtained as the endpoint of such a path, and therefore, we have $h(\widetilde{U}) \subset \eta_{h(p)}$. The proof of the proposition is complete. □

We are now going to assume that the Levi form of M' has n'_+ positive eigenvalues (since we are assuming M' to be uniformly pseudoconvex, everywhere), that is, $N' = n'_+ + n_0 + 1$ with n_0 denoting the dimension of each fiber of the null bundle $\mathcal{N}$. Similarly, we assume from now on that M is a real pseudoconvex hypersurface and write $n_+(\xi)$ for the number of positive Levi eigenvalues of M at any point $\xi \in M$.

Lemma 5.3. *Let M, M' be real-analytic pseudoconvex hypersurfaces in $\mathbb{C}^N$ and $\mathbb{C}^{N'}$, respectively, with $M' \subset \mathbb{C}^{N'}$ uniformly pseudoconvex. If $B : (\mathbb{C}^N \times \mathbb{C}, (p, 0)) \to \mathbb{C}^{N'}$ is a 2-approximate CR deformation from M into M' such that B_0 is CR transversal at $\xi \in M$ near p, then* $\max\{\nu_{B_0(\xi)}, n'_+\} \geqslant n_+(\xi)$.

Proof. By the transversality assumption, the Levi form $\mathcal{L}$ of M can be computed from the Levi form $\mathcal{L}'$ of M' by $\mathcal{L}_\xi(X,Y) = c\mathcal{L}'_{\xi'}(B_0'(\xi)X, B_0'(\xi)Y)$ for some $c > 0$ and all vectors $X, Y \in T_\xi^{1,0}M$ (see, for example, [3]). Since $\mathcal{L}_\xi$ has $n_+(\xi)$ eigenvalues, the range of $B_0'(\xi)$ modulo $T_{B_0(\xi)}\eta$ is of dimension at least $n_+(\xi)$, and hence by Lemma 5.1, we must have $\nu_{B_0(\xi)} \geqslant n_+(\xi)$. The other inequality $n_+' \geqslant n_+$ easily follows from the same identity. $\square$

A particularly important class of target manifolds M' we are now considering are the boundaries of the classical domains. In that situation, the invariant ν has been computed in [16], and we recall its values. To start with, we recall that the classical domains of type I-IV are the sets of matrices given by

$$
\begin{aligned}
D_I^{m,n} &= \{Z \in \mathbb{C}^{m\times n} : \mathbb{I}_m - ZZ^* > 0\},\\
D_{II}^m &= \left\{Z \in \mathbb{C}^{m\times m} : Z^T = -Z, \mathbb{I}_m - Z^*Z > 0\right\},\\
D_{III}^m &= \left\{Z \in \mathbb{C}^{m\times m} : Z^T = Z, \mathbb{I}_m - Z^*Z > 0\right\},\\
D_{IV}^m &= \left\{z \in \mathbb{C}^m : z^*z < 1, 1 + |z^Tz|^2 - 2z^*z > 0\right\}.
\end{aligned}
$$

We will denote the regular part of their boundaries by $M_I^{m,n}$, M_{II}^m, M_{III}^m and M_{IV}^m. We recall that the domains of types I–III contain matrices whose singular values are strictly less than 1, and the regular part of their boundaries contain those which have exactly one (or two) of their singular values equal to one. The type IV domain is biholomorphic to the tube over the light cone. The regular parts of the boundaries are homogeneous real hypersurfaces, and in particular, ν and n_+' remain constant throughout; the following table gives the values of these invariants (the computation can be found in [16]). We also include, for easy reference, the corresponding (complex) dimensions of the Levi foliation.

	$\boldsymbol{\nu}$	$\boldsymbol{n_+'}$	$\mathbf{dim}\, \boldsymbol{T\eta}$
$M_I^{m,n}$	$m+n-4$	$m+n-2$	$(m-1)(n-1)$
M_{II}^m	$2m-8$	$2m-4$	$(m-2)(m-3)/2$
M_{III}^m	$m-2$	$m-1$	$m(m-1)/2$
M_{IV}^m	0	$m-2$	1

A direct consequence of Lemma 5.3 and the above table is the following proposition; note in particular that we get only information for one particular source signature in the case of the type III domains.

Proposition 5.4. *Let $M \subset \mathbb{C}^N$ be a connected, real-analytic, pseudoconvex hypersurface and set $n_+ := \max\{n_+(\xi) : \xi \in M\}$. Let $M' \subset \mathbb{C}^{N'}$ be a real-analytic hypersurface and denote by $\mathcal{S}$ the subsheaf of $\mathcal{A}^\infty_{\mathrm{CR}}(M,M')$ of generically CR transversal maps. Then there is no 2-approximate CR $\mathcal{S}$-deformation if:*

I) $M' = M_I^{m,n}$, *$m, n \geqslant 2$, and $m+n-4 < n_+ \leqslant m+n-2$;*
II) $M' = M_{II}^m$, *$m \geqslant 4$, and $2m-8 < n_+ \leqslant 2m-4$;*
III) $M' = M_{III}^m$, *$m \geqslant 2$, and $n_+ = m-1$;*
IV) $M' = M_{IV}^m$, *$m \geqslant 2$, $0 < n_+ \leqslant m-2$.*

We can now give the proof of Corollary 1.7.

Proof of Corollary 1.7. We first note $n_+(\xi) = n_+$ for ξ on a dense open subset of M. We also note that all four types of boundaries of classical domains considered here are Nash real hypersurfaces (as can be seen from [16, 29]). Parts (i), (ii) and (iii) of Corollary 1.7 then follow from applying Theorem 1.1 and Proposition 5.4. Regarding part (iv), the reader may check that the minimality of M together with $m \geqslant N$ implies that $0 < n_+ \leqslant m-2$ which allows to conclude the same way as above. □

Let us also mention an immediate application of Theorem 1.1, Proposition 5.2 and the above table, for targets of type IV and in which the source need not be of hypersurface type.

Corollary 5.5. *Let $M \subset \mathbb{C}^N$ be a real-analytic minimal real-analytic CR submanifold and $\mathcal{S}$ be the subsheaf of $\mathcal{A}^\infty_{\mathrm{CR}}(M, M^m_{IV})$ whose stalk $\mathcal{S}_p$ at any point $p \in M$ consists of those germs whose generic rank is $\geqslant 3$. Then $(M, M^m_{IV}, \mathcal{S})$ has the strong finite jet determination property.*

Note that in the above corollary, the notion of generic rank for a germ $f \in \mathcal{S}_p$ is well defined as a consequence of the minimality assumption on M.

We now discuss examples to show that the range of values for n_+ in Corollary 1.7 is in some sense optimal; these are adapted to the present finite jet determination setting from the examples given, for example, in [16], which we refer to for the detailed arguments.

Example 5.6. Consider the set Σ_I of rank 1 matrices of norm 1 in $\mathbb{C}^{(m-1)\times(n-1)}$, that is, the intersection of the (euclidean) unit sphere with the complex submanifold $\mathcal{X}_1$ of rank 1 matrices, which is of dimension $m+n-3$, and therefore can locally be identified with $\mathbb{C}^{m+n-3}$. Furthermore, Σ_I is a strictly pseudoconvex hypersurface of $\mathcal{X}_1$. Pick any matrix $A \in \Sigma_I$. For any holomorphic function φ on $\mathcal{X}_1$ near A, vanishing at A, the (local) holomorphic map

$$\mathcal{X}_1 \ni Z \mapsto \begin{pmatrix} Z & 0 \\ 0 & \varphi(Z) \end{pmatrix}$$

maps (Σ_I, A) (transversally) into $M^{m,n}_I$. This shows that finite jet determination does not hold for CR transversal maps between Σ_I into $M^{m,n}_I$. Note that in this example, $n_+ = m+n-4$.

Example 5.7. A similar construction can be made for the set Σ_{II} of anti-symmetric rank two matrices of norm 1 and of size $(m-2)\times(m-2)$. The set $\mathcal{X}_2$ of anti-symmetric rank two matrices and of size $(m-2)\times(m-2)$ is a $(2m-7)$-dimensional complex submanifold, and therefore can be locally identified with $\mathbb{C}^{2m-7}$. Furthermore, Σ_{II} is a strictly pseudoconvex hypersurface of $\mathcal{X}_2$. Pick any matrix $A \in \Sigma_{II}$. For any holomorphic function φ on $\mathcal{X}_2$ near A, vanishing at A, the (local) holomorphic map

$$\mathcal{X}_2 \ni Z \mapsto \begin{pmatrix} Z & 0 & 0 \\ 0 & 0 & \varphi(Z) \\ 0 & -\varphi(Z) & 0 \end{pmatrix}$$

maps (Σ_{II}, A) (transversally) into M^m_{II} and therefore, finite jet determination does not hold for CR transversal maps between Σ_{II} and M^m_{II}. Note that in this example, $n_+ = 2m-8$.

Example 5.8. A similar construction can be made for the set Σ_{III} of symmetric rank one matrices of size $(m-1)\times(m-1)$. The set $\mathcal{X}_3$ of symmetric rank one matrices of size $(m-1)\times(m-1)$ is an $(m-1)$-dimensional complex submanifold, locally equivalent to $\mathbb{C}^{m-1}$. Σ_{III} is a strictly pseudoconvex hypersurface of $\mathcal{X}_3$. As above, pick any $A \in \Sigma_{III}$ and any holomorphic function φ on $\mathcal{X}_3$ near A, vanishing at A, then the (local) holomorphic map

$$\mathcal{X}_3 \ni Z \mapsto \begin{pmatrix} Z & 0 \\ 0 & \varphi(Z) \end{pmatrix}$$

maps (Σ_{III}, A) (transversally) into M_{III}^m and thus finite jet determination does not hold for CR transversal maps between Σ_{III} and M_{III}^m. Note that in this case $n_+ = m-2$.

Of course, Lemma 5.3 gives additional information even if the source has a non-trivial Levi foliation, as well. In the following table, we summarize the range of dimensions for which finite jet determination holds for CR transversal maps from the boundary of a classical domain to another one. Targets are denoted with primes. We do not discuss the actual *existence* of such maps here, hence, part of the entries might be void. Because the type IV target is covered by Proposition 5.2, we omit it from the list of targets.

	$\boldsymbol{M_I^{m',n'}}$	$\boldsymbol{M_{II}^{m'}}$	$\boldsymbol{M_{III}^{m'}}$
$M_I^{m,n}$	$m+n \leqslant m'+n' \leqslant m+n+1$	$m+n+2 \leqslant 2m' \leqslant m+n+5$	$m' = m+n-1$
M_{II}^m	$m'+n'+1 \leqslant 2m \leqslant m'+n'+2$	$m \leqslant m' \leqslant m+1$	$m' = 2m-3$
M_{III}^m	$m+1 \leqslant m'+n' \leqslant m+2$	$m+3 \leqslant 2m' \leqslant m+6$	$m' = m$
M_{IV}^m	$m \leqslant m'+n' \leqslant m+1$	$m+2 \leqslant 2m' \leqslant m+5$	$m' = m-1$

We note that the entries in the diagonal are again optimal: indeed, the preceding examples can be adapted to consider sources which are (regular parts of) boundaries of bounded symmetric domains. To be more specific, the maps

$$M_I^{m,n} \ni Z \mapsto \begin{pmatrix} Z & 0 \\ 0 & \varphi(Z) \end{pmatrix} \in M_I^{m+1,n+1}$$

$$M_{II}^m \ni Z \mapsto \begin{pmatrix} Z & 0 & 0 \\ 0 & 0 & \varphi(Z) \\ 0 & -\varphi(Z) & 0 \end{pmatrix} \in M_{II}^{m+2}$$

$$M_{III}^m \ni Z \mapsto \begin{pmatrix} Z & 0 \\ 0 & \varphi(Z) \end{pmatrix} \in M_{III}^{m+1}$$

show that finite jet determination is no longer possible for higher rank targets.

6 | UNIVERSALITY PROPERTIES OF POLYNOMIAL EQUATIONS SATISFIED BY POWER SERIES

This section is devoted to some universality properties of polynomial equations satisfied by convergent power series. Even though the tools we are using are standard in commutative algebra, we have not been able to find the precise results needed for this paper in the literature.

In what follows, we denote by $\mathbb{C}((x))$ the field of fractions of $\mathbb{C}\{x\}$, where $x = (x_1, \dots, x_n)$.

Lemma 6.1. *Let $P(X,T) = (P_1(X,T), \dots, P_\ell(X,T))$ be a complex-algebraic map defined on some connected open set $\Omega \subset \mathbb{C}^m_X \times \mathbb{C}^\ell_T$. Then there exists a finite collection of $\mathbb{C}^\ell$-valued polynomial maps $\Delta^{(1)}, \dots, \Delta^{(e)} \in \mathbb{C}[X,S]$, with $S \in \mathbb{C}$, depending only on P with the following property: For every $g(x) \in (\mathbb{C}\{x\})^m$ and $h(x) \in (\mathbb{C}\{x\})^\ell$, $x \in \mathbb{C}^n$, satisfying $(g(0), h(0)) \in \Omega$ and*

$$P(g(x), h(x)) = 0, \quad \det\left(\frac{\partial P}{\partial T}(g(x), h(x))\right) \not\equiv 0, \tag{6.1}$$

there exists $b \in \{1, \dots, e\}$ such that

$$\Delta_j^{(b)}\big(g(x), h_j(x)\big) = 0, \quad \frac{\partial \Delta_j^{(b)}}{\partial S}\big(g(x), h_j(x)\big) \not\equiv 0, \ j = 1, \dots, \ell, \tag{6.2}$$

where $\Delta^{(b)} = (\Delta_1^{(b)}, \dots, \Delta_\ell^{(b)})$.

Proof. Since P must satisfy $\det(\frac{\partial P}{\partial T}(X,T)) \not\equiv 0$, it follows from standard commutative algebra (see [17] or [25, §3]) that for every $j = 1, \dots, \ell$, there exist a (non-zero) polynomial $R_j(X,Y,S)$, $Y = (Y_1, \dots, Y_\ell)$, such that for $(X,T) \in \Omega$, we have

$$R_j(X, P(X,T), T_j) = 0. \tag{6.3}$$

For every fixed j, we write

$$R_j(X,Y,S) = \sum_\nu R_{j,\nu}(X,Y)S^\nu.$$

Hence for every $g(x), h(x)$ as in the lemma, we have

$$R_j(g(x), P(g(x), h(x)), h_j(x)) = R_j(g(x), 0, h_j(x)) = 0.$$

We claim that the finite family of polynomials $(\partial_X^\alpha \partial_Y^\beta \partial_S^\delta R_j)(X,0,S)$ with $|\alpha| + |\beta| \leqslant \deg R_j$ and $\delta < \deg_S R_j$ satisfies the conclusion of the lemma.

If there exists $\nu \geqslant 1$ such that $R_{j,\nu}(g(x), 0) \not\equiv 0$, the desired polynomial $\Delta_j^{(b)}$ may be chosen to be among the polynomials $(\partial_S^\delta R_j)(X,0,S)$ for some $0 \leqslant \delta < \deg_S R_j$.

On the hand, assume that for all ν we have $R_{j,\nu}(g(x), 0) = 0$, that is, that $R_j(g(x), 0, S) = 0$. Differentiating (6.3) with respect to T, evaluating at $(X,T) = (g(x), h(x))$ and using (6.1) and that $R_j(g(x), 0, S) = 0$, we get that

$$\frac{\partial R_j}{\partial Y}\big(g(x), 0, h_j(x)\big) = 0. \tag{6.4}$$

Again, if there exists $\nu \geqslant 1$ such that $\frac{\partial R_{j,\nu}}{\partial Y}(g(x), 0) \not\equiv 0$, then we are done as before. If this is not the case, that is, if $\frac{\partial R_j}{\partial Y}(g(x), 0, S) = 0$, differentiating (6.3) with respect to X and evaluating as before, we get

$$\frac{\partial R_j}{\partial X}\big(g(x),0,h_j(x)\big)=0.$$

Using the same dichotomy as before, either we reach the desired conclusion or we get

$$\frac{\partial R_j}{\partial X}(g(x),0,S)=\frac{\partial R_j}{\partial Y}(g(x),0,S)=0.$$

Repeating the previous procedure, and because there exists $\alpha\in\mathbb{N}^m$ and $\beta\in\mathbb{N}^\ell$ with $|\alpha|+|\beta|\leqslant\deg R_j$ such that $\frac{\partial^{|\alpha|+|\beta|}R_j}{\partial X^\alpha\partial Y^\beta}(g(x),0,S)\not\equiv 0$ (as R_j is a non-zero polynomial), we get that there exists α and β as above such that

$$\frac{\partial^{|\alpha|+|\beta|}R_j}{\partial X^\alpha\partial Y^\beta}(g(x),0,h_j(x))=0$$

with

$$\frac{\partial^{|\alpha|+|\beta|}R_j}{\partial X^\alpha\partial Y^\beta}(g(x),0,S)\not\equiv 0.$$

The proof of the lemma is complete. □

For the next results, it will be useful to have the following preliminary fact.

Lemma 6.2. *Let $P(X,T)\in\mathbb{C}[X,T]$, where $(X,T)=(X_1,\dots,X_m,T)$, and write $P'(X,T)=\frac{\partial P}{\partial T}(X,T)$. Then there exist finitely many polynomials $(R_\ell(X,T))_{1\leqslant\ell\leqslant a}$ and $(U_j(X,T))_{1\leqslant j\leqslant b}$ (depending only on P) such that for every $g(x)\in(\mathbb{C}((x)))^m$, $x\in\mathbb{C}^n$, satisfying $P'(g(x),T)\not\equiv 0$, the following holds:*

(i) *The greatest common divisor of $P(g(x),T)$ and $P'(g(x),T)$ in $\mathbb{C}((x))[T]$ is equal to $R_\ell(g(x),T)$ for some integer $\ell\in\{1,\dots,a\}$.*
(ii) *There exists $j\in\{1,\dots,b\}$ such that $P(g(x),T)=c(x)\,R_\ell(g(x),T)\,U_j(g(x),T)$ for some non-zero $c\in\mathbb{C}((x))$. Furthermore, let $D_j(X)$ be the discriminant of $U_j(X,T)$. Then the discriminant in $\mathbb{C}((x))$ of $U_j(g(x),T)$ is $D_j(g(x))$ and is non-zero.*

Proof. Lemma 6.2 follows from Euclid's algorithm. The details are as follows.

For the proof of (i), we pick $g(x)\in(\mathbb{C}\{x\})^m$ satisfying $P'(g(x),T)\not\equiv 0$ and we may assume that $\deg_T P(X,T)=\deg_T P(g(x),T)$. Indeed, if it is not the case, we may replace P by a truncation of it of some order to reach the desired assumption. Furthermore, truncation of P to some order involves substituting P with only finitely many polynomials depending only on P and therefore does not affect the conclusion of the lemma.

From the Euclidean division algorithm in the ring $\mathbb{C}(X)[T]$, we have unique polynomials $Q,S_0\in\mathbb{C}(X)[T]$ such that

$$P(X,T)=Q(X,T)P'(X,T)+S_0(X,T), \tag{6.5}$$

where $\deg_T S_0 < \deg_T P'$. Clearing denominators, we may find a unique non-zero polynomial $\Delta_0(X) \in \mathbb{C}[X]$ of minimal degree, monic in its highest degree monomial (using lexicographic ordering), depending only on P, such that

$$\Delta_0(X)P(X,T) = \widetilde{Q}(X,T)P'(X,T) + \widetilde{S}_0(X,T), \tag{6.6}$$

with $\widetilde{Q}, \widetilde{S}_0 \in \mathbb{C}[X,T]$.

If $\Delta_0(g(x)) \not\equiv 0$, then the greatest common divisor of $P(g(x),T)$ and $P'(g(x),T)$ in $\mathbb{C}((x))[T]$ is the same as that of $P'(g(x),T)$ and $\widetilde{S}_0(g(x),T)$, where we note that $\widetilde{S}_0$ depends uniquely on P.

If $\Delta_0(g(x)) \equiv 0$, because Δ_0 is monic in its highest order monomial, we can choose $\alpha \in \mathbb{N}^m$ of minimal length, $|\alpha| \leqslant \deg \Delta_0$, such that $(\partial_X^\alpha \Delta_0)(g(x)) \not\equiv 0$. Because $P'(g(x),T) \not\equiv 0$ and $\deg_T P'(X,T) = \deg_T P'(g(x),T)$, it follows from (6.6) that $(\partial_X^\beta \widetilde{Q})(g(x),T) = 0$ and $(\partial_X^\beta \widetilde{S}_0)(g(x),T) = 0$ for $|\beta| < |\alpha|$. Hence it follows from (6.6) that

$$(\partial_X^\alpha \Delta_0)(g(x))P(g(x),T) = (\partial_X^\alpha \widetilde{Q})(g(x),T)P'(g(x),T) + (\partial_X^\alpha \widetilde{S}_0)(g(x),T), \tag{6.7}$$

and therefore the greatest common divisor of $P(g(x),T)$ and $P'(g(x),T)$ in $\mathbb{C}((x))[T]$ is the same as that of $P'(g(x),T)$ and $(\partial^\alpha \widetilde{S}_0)(g(x),T)$, for some $|\alpha| \leqslant \deg \Delta_0$. The polynomials $(\partial^\alpha \widetilde{S}_0)(g(x),T)$, $|\alpha| \leqslant \deg \Delta$, form a finite family of polynomials depending only on P, each of them of degree in T strictly less than P', and proceeding inductively with Euclid's algorithm, we clearly reach the conclusion of the Lemma. The proof of (i) is complete.

For the proof of (ii), we will proceed in a similar manner. We first note that, enlarging the family of polynomials (R_ℓ) if necessary, we may assume that for every $g(x)$ as in the lemma, there exists $\ell \in \{1,\dots,a\}$ such that (i) holds and $\deg_T R_\ell(g(x),T) = \deg_T R_\ell(X,T)$. Fix $g(x)$ as above and a corresponding polynomial $R_\ell(X,T)$. Euclidean division in the ring $\mathbb{C}(X)[T]$ yields unique polynomials A and B depending only on P such that

$$P(X,T) = R_\ell(X,T)\,A(X,T) + B(X,T), \tag{6.8}$$

with $\deg_T B < \deg_T R_\ell$. As before, clearing denominators in (6.8), we may write

$$\Delta(X)\,P(X,T) = R_\ell(X,T)\,\widetilde{A}(X,T) + \widetilde{B}(X,T), \tag{6.9}$$

for some non-zero polynomial of minimum degree $\Delta(X) \in \mathbb{C}[X]$ (and unique as a monic polynomial in its highest degree) and where $\widetilde{A}(X,T), \widetilde{B}(X,T) \in \mathbb{C}[X,T]$. Now, a procedure as the one used in (i) shows that for some suitable choice of multi-index $\alpha \in \mathbb{N}^m$ with $|\alpha| \leqslant \deg \Delta$, we have

$$(\partial_X^\alpha \Delta)(g(x))\,P(g(x),T) = R_\ell(g(x),T)\,(\partial_X^\alpha \widetilde{A})(g(x),T) + (\partial_X^\alpha \widetilde{B})(g(x),T), \tag{6.10}$$

with $(\partial_X^\alpha \Delta)(g(x)) \not\equiv 0$. Since $R_\ell(g(x),T)$ divides $P(g(x),T)$ in $\mathbb{C}((x))[T]$ and $\deg_T R_\ell(g(x),T) = \deg_T R_\ell(X,T)$, it must hold that $(\partial_X^\alpha \widetilde{B})(g(x),T) = 0$. To finish the proof of (ii), note that we may assume, enlarging the family (U_j) if necessary, that $\deg_T U_j(g(x),T) = \deg_T U_j(X,T)$. Now the second statement in (ii) follows from standard properties about discriminants of polynomials. The proof of the lemma is complete. $\square$

The next lemma states that a polynomial relation between power series gives rise to universal polynomial relations between their derivatives.

Lemma 6.3. *Let $P(X,T) \in \mathbb{C}[X,T]$, where $(X,T) = (X_1, \dots, X_m, T)$ and $n \in \mathbb{Z}_+$. Then for every $\gamma \in \mathbb{N}^n$, there exist finitely many polynomials $\Upsilon_1, \dots, \Upsilon_{e_\gamma}$, depending only on P, such that for every $g(x) \in (\mathbb{C}\{x\})^m$ and $h(x) \in \mathbb{C}\{x\}$, $x \in \mathbb{C}^n$, satisfying*

$$P(g(x), h(x)) = 0, \qquad \frac{\partial P}{\partial T}(g(x), h(x)) \not\equiv 0, \tag{6.11}$$

there exists $d \in \{1, \dots, e_\gamma\}$ such that

$$\Upsilon_d((\partial^\alpha g(x))_{|\alpha|\leqslant|\gamma|}, \partial^\gamma h(x)) = 0, \qquad \frac{\partial \Upsilon_d}{\partial T}((\partial^\alpha g(x))_{|\alpha|\leqslant|\gamma|}, \partial^\gamma h(x)) \not\equiv 0. \tag{6.12}$$

Proof. It is enough to prove the lemma for $|\gamma| = 1$, since the general case follows inductively.

Let (R_ℓ) and (U_j) be the finite collection of polynomials obtained by applying Lemma 6.2 to P. For $g(x)$ satisfying (6.11) we choose ℓ and j such that the conclusions of Lemma 6.2 hold, and write U for U_j. Using (6.11), it can be easily checked that

$$U(g(x), h(x)) = 0, \qquad \frac{\partial U}{\partial T}(g(x), h(x)) \not\equiv 0. \tag{6.13}$$

Note that U may have been chosen so that $s := \deg_T\, U(X,T) = \deg_T\, U(g(x),T)$ (see the proof of Lemma 6.2). Write

$$U(X,T) = U_s(X)T^s + \cdots + U_0(X)$$

with $U_s(g(x)) \not\equiv 0$. Differentiating the equality in (6.13), we obtain, from the chain rule,

$$\partial^\gamma h(x)\, \frac{\partial U}{\partial T}(g(x), h(x)) + \widetilde{U}((\partial^\alpha g(x))_{|\alpha|\leqslant 1}, h(x)) = 0, \tag{6.14}$$

for some universal polynomial $\widetilde{U} = \widetilde{U}(X,Y,T)$ depending only on U, and hence on P. For $X \in \mathbb{C}^m$ outside the zero set of U_s, let $\sigma_1(X), \dots, \sigma_s(X)$ denote the s roots of $U(X,T)$ (counted with multiplicity). Set

$$V(X,Y,T) := \prod_{i=1}^{s} \left(T \frac{\partial U}{\partial T}(X, \sigma_i(X)) + \widetilde{U}(X, Y, \sigma_i(X)) \right).$$

By Newton's theorem on symmetric polynomials, we may write

$$V(X,Y,T) = W\left(X, Y, \left(\frac{U_k(X)}{U_s(X)} \right)_{0\leqslant k\leqslant s-1}, T \right)$$

for some universal polynomial W depending only on U. Hence for a sufficiently high power $\delta \in \mathbb{Z}_+$, $\widetilde{V}(X,Y,T) = U_s(X)^\delta V(X,Y,T)$ is a polynomial depending only on U (and hence on P)

which, in view of (6.14) and (6.13), satisfies

$$\widetilde{V}((\partial^\alpha g(x))_{|\alpha|\leqslant 1}, \partial^\gamma h(x)) = 0.$$

Furthermore, by construction, the coefficient of highest degree in T (that is, in front of T^s) of $\widetilde{V}$ is equal to $(U_s(X))^\nu D(X)$ for some power ν where $D(X)$ is the discriminant of $U(X,T)$. Since $U_s(g(x)) \not\equiv 0$ and $D(g(x)) \not\equiv 0$ (as a consequence of Lemma 6.2 (ii)), it follows that $\widetilde{V}((\partial^\alpha g(x))_{|\alpha|\leqslant 1}, T) \not\equiv 0$. Hence, we may choose the desired polynomial Υ_d among the polynomials $\frac{\partial^c \widetilde{V}}{\partial T^c}$, $c \in \{0, \dots, s-1\}$. The proof of the lemma is complete. □

The next lemma allows us to universally choose good polynomial relations from a given one if we have some additional information about the solution.

Lemma 6.4. *Given $P(X,T), Q(X,T) \in \mathbb{C}[X,T]$, where $(X,T) = (X_1, \dots, X_m, T)$ and $n \in \mathbb{Z}_+$, there exists finitely many polynomials $R^1(X,T), \dots, R^J(X,T)$, depending only on P and Q, such that for every $g(x) \in (\mathbb{C}\{x\})^m$ and $h(x) \in \mathbb{C}\{x\}$, $x \in \mathbb{C}^n$, satisfying*

$$Q(g(x), h(x)) = 0, \quad Q(g(x), T) \not\equiv 0, \quad P(g(x), h(x)) \not\equiv 0, \tag{6.15}$$

there exists $j \in \{1, \dots, J\}$ such that $R^j(g(x), h(x)) = 0$, $R^j(g(x), T) \not\equiv 0$, and the resultant of $P(g(x), T)$ and $R^j(g(x), T)$ is non-zero (in $\mathbb{C}\{x\}$).

Proof. If the resultant of $Q(g(x),T)$ and $P(g(x),T)$ is non-zero in $\mathbb{C}((x))$, there is nothing to prove. Let us assume therefore that this is not the case. This means that the greatest common divisor of the (non-zero) polynomials $Q(g(x),T)$ and $P(g(x),T)$ in $\mathbb{C}((x))[T]$ is of degree $\geqslant 1$. Following the proof of Lemma 6.2(i), that is, a suitable use of Euclid's algorithm, one may find finitely many polynomials $A_1, \dots, A_a$ and $B_1, \dots, B_b$, depending only on P and Q, such that for some $\alpha \in \{1, \dots, a\}$, $A_\alpha(g(x),T) = \mathrm{GCD}(Q(g(x),T), P(g(x),T))$ and such that $Q(g(x),T) = \delta(x) A_\alpha(g(x),T) B_\beta(g(x),T)$ for some β and some non-zero $\delta(x) \in \mathbb{C}((x))$. Note that $\deg_T B_\beta(g(x),T) < \deg_T Q(g(x),T)$. Because of (6.15), it must hold that $B_\beta(g(x), f(x)) = 0$. If the resultant of $P(g(x),T)$ and $B_\beta(g(x),T)$ in $\mathbb{C}((x))$ is non-zero, we are done. If not, we repeat the previous procedure with $Q(g(x),T)$ replaced by $B_\beta(g(x),T)$, which is a non-zero polynomial of degree strictly smaller than that of $Q(g(x),T)$. Hence, such a substitution procedure must terminate and it provides the desired result. The proof of the lemma is complete. □

Lemma 6.4 is used for the following result.

Lemma 6.5. *Let $P(X,T,S) \in \mathbb{C}[X,T,S]$, $\Theta_1(Y,T_1) \in \mathbb{C}[Y,T_1], \dots, \Theta_r(Y,T_r) \in \mathbb{C}[Y,T_r]$ where $(Y,S) = (Y_1, \dots, Y_\ell, S)$, $(X,T,S) = (X_1, \dots, X_m, T_1, \dots, T_r, S)$, and let $n \in \mathbb{Z}_+$. Then there exist finitely many polynomials $\Psi_1, \dots, \Psi_K \in \mathbb{C}[X,Y,S]$, depending only on P and the Θ_j's, such that for every $u(x) \in (\mathbb{C}\{x\})^m$, $w(x) \in (\mathbb{C}\{x\})^\ell$, $v(x) \in (\mathbb{C}\{x\})^r$, $h(x) \in \mathbb{C}\{x\}$, $x \in \mathbb{C}^n$, satisfying*

$$P(u(x), v(x), h(x)) = 0, \quad \frac{\partial P}{\partial S}(u(x), v(x), h(x)) \not\equiv 0, \tag{6.16}$$

$$\Theta_j(w(x), v_j(x)) = 0, \quad \frac{\partial \Theta_j}{\partial T_j}(w(x), v_j(x)) \not\equiv 0, \ j = 1, \dots, r, \tag{6.17}$$

one has for some $k \in \{1, \ldots, K\}$

$$\Psi_k(u(x), w(x), h(x)) = 0, \quad \frac{\partial \Psi_k}{\partial S}(u(x), w(x), h(x)) \not\equiv 0. \tag{6.18}$$

Proof. It is enough to prove the lemma in the case $r = 1$, as the general case follows from a repeated use of the case $r = 1$. Hence, in what follows, $r = 1$, and therefore we write Θ for Θ_1.

We write $P(X, T, S) = P_\mu(X, T)S^\mu + \cdots, P_0(X, T)$. Pick u, v, w, h as in the lemma. We may assume without loss of generality that $P_\mu(u(x), v(x)) \not\equiv 0$. Otherwise, we repeat the proof given below for P_μ for some other polynomial P_i as there must exist $i \in \{1, \ldots, \mu\}$ such that $P_i(u(x), v(x)) \not\equiv 0$.

We apply Lemma 6.4 to $P_\mu = P_\mu(X, T) = P(X, Y, T)$ and $\Theta = \Theta(Y, T) = \Theta(X, Y, T)$ (since $\Theta(w(x), T) \not\equiv 0$). There exist finitely many polynomials $R^1, \ldots, R^J \in \mathbb{C}[X, Y, T]$, depending only on P and Θ, such that $R^j(u(x), w(x), v(x)) = 0$ for some j and the resultant (in T) of $P_\mu(u(x), T)$ and $R^j(u(x), w(x), T)$ is non-zero. Furthermore, inspecting the proof of Lemma 6.4, R^j may be chosen so that $\deg_T R^j(X, Y, T) = \deg_T R^j(u(x), w(x), T)$.

We now write $R^j(X, Y, T) = R^j_\nu(X, Y)T^\nu + \cdots + R^j_0(X, Y)$, where $R^j_\nu(X, Y) \not\equiv 0$. For (X, Y) outside the zero locus of R^j_ν, denote by $\sigma_1(X, Y), \ldots, \sigma_\nu(X, Y)$ the ν roots of R^j (counted with multiplicity) and consider

$$W(X, Y, S) := \prod_{i=1}^{\nu} P(X, \sigma_i(X, Y), S) = \prod_{i=1}^{\nu} \left(P_\mu(X, \sigma_i(X, Y))S^\mu + \cdots + P_0(X, \sigma_i(X, Y))\right).$$

For some appropriate power γ, $(R^j_\nu(X, Y))^\gamma \cdot W(X, Y, S) = \mathrm{Res}_T(P(X, T, S), R^j(X, Y, T))$, where the latter is the polynomial resultant, with respect to T, of P and R^j. The polynomial coefficient $C(X, Y)$ in front of $S^{\mu\nu}$ in the polynomial $\mathrm{Res}_T(P, R^j)$ is equal, by construction, to $(R^j_\nu(X, Y))^{\tilde{\gamma}} \cdot \mathrm{Res}_T(P_\mu, R^j)$ for some power $\tilde{\gamma}$. It then follows that $C(u(x), w(x)) \not\equiv 0$ and, therefore, necessarily for one of the polynomials $\partial^c_S(\mathrm{Res}_T(P, R^j))$, $0 \leqslant c < \deg_S(\mathrm{Res}_T(P, R^j))$ the conclusion must hold. The proof of the lemma is complete. $\square$

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