The Wiegerinck Problem in The Class of Hartogs Domains

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Some Notation

- Let $L_h^2(\Omega, \varphi)$ denote the Bergman space of a domain $\Omega \subseteq \mathbb{C}^n$ with weight $e^{-\varphi}$. More precisely,

$$L_h^2(\Omega, \varphi) = \left\{ f \text{ is holomorphic on } \Omega \, : \, \int_\Omega |f|^2 e^{-\varphi} dV < \infty \right\}.$$

- Write $\|f\|_{\Omega,\varphi}$ for the L^2-norm of f with respect to $e^{-\varphi}$.
- We use the convention $\|f\|_\Omega = \|f\|_{\Omega,0}$ and $L_h^2(\Omega) = L_h^2(\Omega, 0)$.
- Denote the Lelong number of a plurisubharmonic function φ at $z = a$ by

$$\nu(\varphi, a) = \lim_{r \to 0} \frac{(2\pi)^{-1} \Delta\varphi(B(a,r))}{dV_{2n-2}(B(a,r) \cap \mathbb{C}^{n-1})}$$

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- The Bergman space of a domain in the complex plane must be either trivial or have infinite dimension.
- For each natural number k, there exists a Reinhardt domain $\Omega_k \subseteq \mathbb{C}^2$ whose Bergman space has dimension k.

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"Does there exist a pseudoconvex domain whose Bergman space is nontrivial and finite-dimensional?"

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- (Carleson 1983) $L_h^2(\Omega)$ is nontrivial for $\Omega \subseteq \mathbb{C}$ if and only if Ω^c has positive logarithmic capacity.

What's Next After The Complex Plane?

- For a domain $G \subseteq \mathbb{C}^M$ and function φ, set

$$D_\varphi(G) = \left\{(z, w) \in G \times \mathbb{C}^N \,:\, \|w\| < e^{-\varphi(z)}\right\}.$$

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- The f_α are holomorphic functions on G with

$$\|f_\alpha w^\alpha\|^2_{D_\varphi(G)} = C_{\alpha,N} \|f_\alpha\|^2_{G,2(N+|\alpha|)\varphi}.$$

The Wiegerinck Problem on Hartogs Domains

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Problem

Show that $L_h^2(D_\varphi(G))$ is trivial or infinite-dimensional whenever $G \subset \mathbb{C}$ has polar complement and φ is harmonic on G.

Other Reasons We Like Hartogs Domains

- A balanced domain is a domain of the form $\{z \in \mathbb{C}^n : h(z) < 1\}$, where $h(\lambda z) = |\lambda| h(z)$, for $\lambda \in \mathbb{C}$.

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- Pflug and Zwonek (2017) used this isomorphism to show that a balanced pseudoconvex domain in $\mathbb{C}^2$ has trivial Bergman space if it is either $\mathbb{C}^2$ or h is of the form $h(z) = |Az|^t |Bz|^{1-t}$, where $A, B : \mathbb{C}^2 \to \mathbb{C}$ are nontrivial linear mappings and $t \in [0, 1]$.

Hartogs Domains With Base in $\mathbb{C}^M$, $M > 1$

Theorem

Let $G \subseteq \mathbb{C}^M$ be pseudoconvex and $\varphi \in PSH(G)$. Assume that $U \subseteq G$ is an open set such that $\varphi - c|\cdot|^2$ is plurisubharmonic on U for some $c > 0$, and $\nu(\varphi, \cdot) = 0$ on U. Then $L^2_h(D_\varphi(G))$ has infinite dimension.

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Theorem (Gallagher, Harz, Herbort (2017))

Let $G \subseteq \mathbb{C}^M$ be a pseudoconvex domain and let $\Phi \in PSH(G)$. Assume that

- *$U \subseteq G$ is open such that $\Phi - c|\cdot|^2$ is plurisubharmonic on U for some $c > 0$, and*
- *$v \in L^2_{(0,1)}(G, \Phi)$ is a smooth form such that $\bar{\partial} v = 0$ and supp $v \subseteq U$.*

Then there exists a smooth form $u : G \to \mathbb{C}$ such that $\bar{\partial} u = v$ and

$$\|u\|^2_{G,\Phi} \leq \frac{1}{c}\|v\|^2_{G,\Phi}.$$

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Theorem (Shortened Statement)

$\exists c > 0 \ni \varphi - c|\cdot|^2 \in PSH(U)$ *implies* $\dim L^2_H(D_\varphi(G)) = \infty$.

Sketch of Proof.

- It suffices to find for infinitely many $\alpha \in \mathbb{Z}_+^N$ a nontrivial $f_\alpha \in \mathcal{O}(G)$ with $\|f_\alpha\|_{G,2(N+|\alpha|)\varphi} < \infty$.

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- Set $v(z) := \bar{\partial}\chi(z-p)$, and

$$\Phi_\alpha := 2(N+|\alpha|)\varphi + M\cdot\chi(z-p)\log|z-p|.$$

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- Hörmander's estimate above yields a smooth function u_α such that $\bar{\partial}u_\alpha = v$ and $\|u_\alpha\|^2_{G,\Phi_\alpha} \leq \|v\|^2_{G,\Phi_\alpha}$.

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- Setting $f_\alpha := \chi(z-p) - u_\alpha(z)$ yields a nontrivial member of $L^2_h(G, 2(N+|\alpha|)\varphi)$. □

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By taking advantage of various generalizations of the Ohsawa-Takegoshi extension theorem, we have

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Suppose $G \subset \Omega \times \mathbb{C}^{M-1}$ is a pseudoconvex domain, where $\Omega \subset \mathbb{C}$ is bounded. Then for any $\varphi \in PSH(G)$, $\dim L^2_h(D_\varphi(G)) = \infty$ whenever $L^2_h(D_\varphi(\{z_1 = 0\} \cap G))$ is infinite dimensional.

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More generally, Ω may have nonpolar complement in $\mathbb{C}$.

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A similar theorem, with the restriction on G replaced by a restriction on φ, is

Theorem

Suppose that $G \subseteq \mathbb{C}^M$ is pseudoconvex and $\varphi \in PSH(G) \cap \mathcal{C}^2(G)$. Further suppose that there exists a complex hyperplane $A \subset \mathbb{C}^M$ such that

$$\inf_{p \in A} H_p(\varphi, N_p) > 0,$$

where N_p is the unit complex normal vector to A at $p \in A$. Then $\dim L^2_h(D_\varphi(G)) = \infty$ whenever $L^2_h(D_\varphi|_{A \cap G}(A \cap G))$ has infinite dimension.

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Problem

Is it possible to replace the hyperplane in the above theorem with a hypersurface?

Other Questions

A previous theorem implies that if $\dim D_\varphi(\mathbb{C}^M) < \infty$, then the Monge-Ampère operator of φ is a sum of point-masses.

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Are there necessary and sufficient conditions on the weights of the point-masses which yields the nontrivial or infinite-dimensionality of $D_\varphi(\mathbb{C}^M)$?

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This is still not known in the case where φ is smooth:

Problem

Give necessary and sufficient conditions for a domain $D_\varphi(\mathbb{C}^M)$ to have trivial or infinite-dimensional Bergman space when φ is smooth.

Thank you!