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Analyse complexe/*Complex Analysis*

Approximation and convergence properties of formal CR-maps

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Abstract Let $M \subset \mathbb{C}^N$ be a minimal real-analytic CR-submanifold and $M' \subset \mathbb{C}^{N'}$ a real-algebraic subset through points $p \in M$ and $p' \in M'$ respectively. We show that that any formal (holomorphic) mapping $f : (\mathbb{C}^N, p) \to (\mathbb{C}^{N'}, p')$, sending M into M', can be approximated up to any given order at p by a convergent map sending M into M'. If M is furthermore generic, we also show that any such map f, that is not convergent, must send (in an appropriate sense) M into the set $\mathcal{E}' \subset M'$ of points of D'Angelo infinite type. Therefore, if M' does not contain any nontrivial complex-analytic subvariety through p', any formal map f sending M into M' is necessarily convergent. ***To cite this article: F. Meylan et al., C. R. Acad. Sci. Paris, Ser. I 335 (2002) 671–676.***

Propriétés d'approximation et de convergence des applications CR formelles

Résumé Soient $M \subset \mathbb{C}^N$ une sous-variété CR analytique réelle minimale et $M' \subset \mathbb{C}^{N'}$ un sous-ensemble algébrique réel avec $p \in M$ et $p' \in M'$. On montre que pour toute application (holomorphe) formelle $f : (\mathbb{C}^N, p) \to (\mathbb{C}^{N'}, p')$, envoyant M dans M', et pour tout entier positif k donné, il existe un germe d'application holomorphe en p, envoyant M dans M' et dont le jet en p d'ordre k correspond à celui de f. Si M est de plus générique, on montre qu'une telle application f, non convergente, envoie nécessairement M (en un sens approprié) dans le sous-ensemble $\mathcal{E}' \subset M'$ des points de type infini au sens de D'Angelo. Ceci implique en particulier la convergence de toutes les applications formelles envoyant M dans M', si M' ne contient pas de sous-ensemble analytique complexe irréductible de dimension positive passant par p'. ***Pour citer cet article : F. Meylan et al., C. R. Acad. Sci. Paris, Ser. I 335 (2002) 671–676.***

Version française abrégée

Un théorème célèbre d'Artin [1] affirme que pour tout système d'équations analytiques, pour toute solution formelle f d'un tel système et pour tout entier positif k, il existe une solution convergente dont la série de Taylor d'ordre k correspond à celle de f. Dans cette Note, on s'intéresse aux propriétés analogues d'approximation et de convergence pour les applications formelles holomorphes $f : (\mathbb{C}^N, p) \to (\mathbb{C}^{N'}, p')$

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envoyant une sous-variété analytique réelle $M \subset \mathbb{C}^N$ dans une autre telle sous-variété $M' \subset \mathbb{C}^{N'}$. Dans un tel contexte, le théorème d'Artin ne peut s'appliquer directement, et, en fait, il résulte des travaux de Moser–Webster [15] que l'énoncé d'approximation correspondant n'est pas vrai en général. En effet, d'après [15], il existe deux germes de surfaces algébriques réelles dans $\mathbb{C}^2$ qui sont formellement équivalents mais non biholomorphiquement équivalents. Notre premier résultat montre qu'un tel phénomène ne peut se produire si $M \subset \mathbb{C}^N$ est une sous-variété CR analytique réelle minimale (au sens de Tumanov [16]).

THÉORÈME 0.1. – *Soient $M \subset \mathbb{C}^N$ une sous-variété CR analytique réelle minimale et $M' \subset \mathbb{C}^{N'}$ un sous-ensemble algébrique réel avec $p \in M$ et $p' \in M'$. Alors pour toute application formelle $f : (\mathbb{C}^N, p) \to (\mathbb{C}^{N'}, p')$, envoyant M dans M', et pour tout entier positif k, il existe un germe d'application holomorphe $f^k : (\mathbb{C}^N, p) \to (\mathbb{C}^{N'}, p')$, envoyant M dans M', et telle que sa série de Taylor (en p) d'ordre k corresponde à celle de f.*

Dans le cas d'applications formelles *inversibles* entre sous-variétés CR analytiques réelles de même dimension, des résultats d'approximation analogues au Théorème 0.1 ont été obtenus dans [5,4]. La nouveauté essentielle réside ici dans l'approximation d'applications formelles quelconques dans des espaces complexes de dimension arbitraire. Signalons aussi que dans la situation du Théorème 0.1, l'application f n'est pas nécessairement convergente en général. Par exemple, si M n'est pas générique, ou si M' contient un sous-ensemble analytique complexe irréductible de dimension positive passant par p', alors des applications formelles non-convergentes envoyant M dans M' existent toujours. Le résultat suivant montre qu'il s'agit essentiellement des deux principales exceptions.

THÉORÈME 0.2. – *Soient $M \subset \mathbb{C}^N$ une sous-variété générique analytique réelle minimale et $M' \subset \mathbb{C}^{N'}$ un sous-ensemble algébrique réel avec $p \in M$ et $p' \in M'$. Désignons par $\mathcal{E}'$ l'ensemble des points de M' par lesquels il passe un sous-ensemble analytique complexe irréductible de dimension positive. Alors toute application formelle $f : (\mathbb{C}^N, p) \to (\mathbb{C}^{N'}, p')$, envoyant M dans M' soit est convergente, soit envoie M dans $\mathcal{E}'$.*

Remarquons que le sous-ensemble $\mathcal{E}' \subset M'$ défini ci-dessus n'est pas en général analytique réel (voir par exemple [11]). Ainsi, dire que f applique M dans $\mathcal{E}'$ dans le Théorème 0.2 signifie que $\varphi(f(x(t))) \equiv 0$ pour tous germes d'applications analytiques réelles $x : (\mathbb{R}_t^{\dim M}, 0) \to (M, p)$ et $\varphi : (M', p') \to (\mathbb{R}, 0)$ telles que φ s'annule sur $\mathcal{E}'$. Du Théorème 0.2, on déduit :

COROLLAIRE 0.3. – *Soient $M \subset \mathbb{C}^N$ une sous-variété générique analytique réelle minimale et $M' \subset \mathbb{C}^{N'}$ un sous-ensemble algébrique réel avec $p \in M$ et $p' \in M'$. Alors toute application formelle $f : (\mathbb{C}^N, p) \to (\mathbb{C}^{N'}, p')$ envoyant M dans M' est convergente si et seulement si M' ne contient pas de sous-ensemble analytique complexe irréductible de dimension positive passant par p'.*

Le Corollaire 0.3 semble nouveau déjà dans le cas où M et M' sont des hypersurfaces et $N = N'$, comme dans le cas où M et M' sont des sphères et $N' \neq N$. Pour des travaux antérieurs traitant de la convergence d'applications formelles entres sous-variétés CR analytiques réelles, nous renvoyons aux articles [6,3,5,13, 14,8,4] et à leur bibliographie.

1. Introduction

A celebrated theorem of Artin [1] states that a formal solution of a system of analytic equations can be replaced by a convergent solution of the same system that approximates the original solution at any prescribed order. In this Note, we are interested in establishing analogous approximation and convergence results for formal (holomorphic) mappings sending (germs of) real-analytic submanifolds $M \subset \mathbb{C}^N$ and $M' \subset \mathbb{C}^{N'}$ into each other. In such a setting, Artin's approximation theorem cannot be used directly, and, in fact, the analogous approximation statement does not hold in general. Indeed, it follows from the work of

Moser and Webster [15] that there exist real-algebraic surfaces $M, M' \subset \mathbb{C}^2$ that are formally equivalent but not biholomorphically equivalent. The first result of this Note shows that this phenomenon cannot happen if M is a (real-analytic) minimal CR-submanifold (in the sense of [16]).

THEOREM 1.1. – *Let $M \subset \mathbb{C}^N$ be a real-analytic minimal CR-submanifold and $M' \subset \mathbb{C}^{N'}$ a real-algebraic subset with $p \in M$ and $p' \in M'$. Then for any formal (holomorphic) mapping $f : (\mathbb{C}^N, p) \to (\mathbb{C}^{N'}, p')$ sending M into M' and any positive integer k, there exists a germ of a holomorphic map $f^k : (\mathbb{C}^N, p) \to (\mathbb{C}^{N'}, p')$ sending M into M', whose Taylor series at p agrees with f up to order k.*

For the case of *invertible* formal maps between real-analytic CR-submanifolds of the same dimension, similar approximation results have been obtained in [5,4]. The main novelty in Theorem 1.1 consists in considering the case of arbitrary formal maps in complex spaces of possibly different dimension. It is worth noticing that in the setting of Theorem 1.1, the map f need not be convergent in general. For instance, if M is not generic, or if M' contains an irreducible complex-analytic subvariety of positive dimension through p', there always exist non-convergent formal maps sending M into M'. Our next result shows that these are essentially the only exceptions.

THEOREM 1.2. – *Let $M \subset \mathbb{C}^N$ be a real-analytic minimal generic submanifold and $M' \subset \mathbb{C}^{N'}$ a real-algebraic subset with $p \in M$ and $p' \in M'$. Denote by $\mathcal{E}'$ the set of all points of M' through which there exist irreducible complex-analytic subvarieties of M' of positive dimension. Then any formal (holomorphic) mapping $f : (\mathbb{C}^N, p) \to (\mathbb{C}^{N'}, p')$ sending M into M' is either convergent or sends M into $\mathcal{E}'$.*

In the context of Theorem 1.2, by saying that f sends M into $\mathcal{E}'$, we mean that $\varphi(f(x(t))) \equiv 0$ holds for all germs of real-analytic maps $x : (\mathbb{R}_t^{\dim M}, 0) \to (M, p)$ and $\varphi : (M', p') \to (\mathbb{R}, 0)$ such that φ vanishes on $\mathcal{E}'$. Such a precision is noteworthy since the subset $\mathcal{E}'$ defined above is not in general a real-analytic subset of M' (see [11] for an example). As an immediate application of Theorem 1.2, we obtain:

COROLLARY 1.3. – *Let $M \subset \mathbb{C}^N$ be a minimal real-analytic generic submanifold and $M' \subset \mathbb{C}^{N'}$ a real-algebraic subset with $p \in M$ and $p' \in M'$. Then all formal maps $f : (\mathbb{C}^N, p) \to (\mathbb{C}^{N'}, p')$ sending M into M' are convergent if and only if M' does not contain any irreducible complex-analytic subvariety of positive dimension through p'.*

Corollary 1.3 seems to be new already in the case where $M, M' \subset \mathbb{C}^N$ are hypersurfaces as well as in the case where M and M' are spheres and $N' \neq N$. For previous work in the direction of Theorem 1.2 and Corollary 1.3, the reader is referred to the papers [6,3,5,13,14,8,4] and also references there.

2. Meromorphic extension of ratios of formal power series

We describe in this section one of the main tools (Theorem 2.4 below and its application Theorem 2.5) needed for the proofs of the above mentioned results (see [12] for details). One of the main novelties of this part compared to previous related work lies in the study of convergence properties of ratios of formal power series rather than of the series themselves. It is natural to call such a ratio convergent if it is equivalent to a ratio of convergent power series. However, for our purposes, we need a refined version of convergence along a given submanifold which may be described as follows.

For any formal power series $F = F(t) \in \mathbb{C}[\![t]\!]$ in $t = (t_1, \ldots, t_n)$ and any nonnegative integer k, we denote by $j^k F$ its k-jet, i.e., *the formal power series mapping corresponding to the collection of all partial derivatives of F up to order k*. For $F(t), G(t) \in \mathbb{C}[\![t]\!]$, we write $(F : G)$ for a pair of two formal power series thinking of it as a ratio, where we allow both series to be zero.

DEFINITION 2.1. – Let $(F_1 : G_1)$, $(F_2 : G_2)$ be ratios of formal power series in $t = (t_1, \ldots, t_n)$, and $S \subset \mathbb{C}^n$ be a (germ of a) complex submanifold through $0 \in \mathbb{C}^n$. Given a positive integer k, we say that the ratios $(F_1 : G_1)$ and $(F_2 : G_2)$ are *k-similar along S* if $(j^k(F_1 G_2 - F_2 G_1))|_S \equiv 0$.

The precise notion of convergence along a given submanifold is then given by the following.

DEFINITION 2.2. – Let $S \subset \mathbb{C}^n$ be a complex submanifold through the origin and $F(t), G(t) \in \mathbb{C}[\![t]\!]$, $t = (t_1, \ldots, t_n)$. The ratio $(F : G)$ is said to be *convergent along* S if there exist a nonnegative integer l and, for any nonnegative integer k, convergent power series $F_k(t), G_k(t) \in \mathbb{C}\{t\}$, such that the ratio $(F_k : G_k)$ is k-similar to $(F : G)$ along S and $(j^l(F_k, G_k))|_S \not\equiv 0$.

The notion of convergence of a ratio of formal power series along a submanifold introduced in Definition 2.2 extends in an obvious way to formal power series defined on a (germ of a) complex manifold $\mathcal{X}$. With this refined notion of convergence, we are able to conclude in the following proposition the convergence of a given ratio along a submanifold provided its convergence is known to hold along a smaller submanifold and under suitable conditions on the ratio.

PROPOSITION 2.3. – *Let $\mathcal{X}$ and $\mathcal{Y}$ be complex manifolds and $v : \mathcal{Y} \to \mathcal{X}$ a holomorphic submersion with $y_0 \in \mathcal{Y}$. Let $S \subset \mathcal{Y}$ be a complex submanifold through y_0 and $(F : G)$ a ratio of formal power series on $\mathcal{X}$, centered at $x_0 := v(y_0)$, whose pullback under v is convergent along S. Let $\eta : \mathcal{X} \to \mathcal{C}$ be a holomorphic submersion onto a complex manifold $\mathcal{C}$. Define*

$$\widetilde{\mathcal{Y}} := \{(y, x) \in \mathcal{Y} \times \mathcal{X} : \eta(v(y)) = \eta(x)\}, \quad \widetilde{S} := \{(y, x) \in \widetilde{\mathcal{Y}} : y \in S\}, \quad \widetilde{v} : \widetilde{\mathcal{Y}} \ni (y, x) \mapsto x \in \mathcal{X}.$$

Assume that one of the following conditions hold:

(i) *the ratio $(F : G)$ is equivalent to a nontrivial ratio $(\alpha \circ \eta : \beta \circ \eta)$, where α and β are formal power series on $\mathcal{C}$ centered at $\eta(x_0)$;*

(ii) *the ratio $(F : G)$ is equivalent to a nontrivial ratio of the form $(\Phi(Y(\eta(x)), x) : \Psi(Y(\eta(x)), x))$, where Y is a $\mathbb{C}^r$-valued formal power series on $\mathcal{C}$ centered at $\eta(x_0)$ and Φ, Ψ are convergent power series centered at $(Y(x_0), x_0) \in \mathbb{C}^r \times \mathcal{X}$.*

Then the pullback of $(F : G)$ under $\widetilde{v}$ is convergent along $\widetilde{S}$.

Another novelty of our techniques consists of applying the convergence results given by the previous proposition to ratios defined on iterated complexifications of real-analytic submanifolds (in the sense of [17,18]) rather than on single Segre sets (in the sense of [2]) associated to given fixed points. The choice of iterated complexifications is needed to guarantee the nonvanishing of the relevant ratios that may not hold when restricted to the Segre sets. These tools are then used to obtain the convergence of a certain type of ratios of formal power series that appear naturally in the proofs of Theorems 1.1 and 1.2. This is done in Theorem 2.5 that is, in turn, derived from Theorem 2.4 below which is established in the following more general context of a pair of submersions of a complex manifold.

Let $\mathcal{X}$, $\mathcal{Z}$ and $\mathcal{W}$ be complex manifolds and $\lambda : \mathcal{X} \to \mathcal{Z}$, $\mu : \mathcal{X} \to \mathcal{W}$ be holomorphic submersions. Set $\mathcal{X}^{(0)} := \mathcal{X}$ and for any integer $l \geqslant 1$, define the (odd) fiber product

$$\mathcal{X}^{(l)} := \{(z_1, \ldots, z_{2l+1}) \in \mathcal{X}^{2l+1} : \mu(z_{2s-1}) = \mu(z_{2s}),\ \lambda(z_{2s}) = \lambda(z_{2s+1}),\ 1 \leqslant s \leqslant l\}. \tag{2.1}$$

It is easy to see that $\mathcal{X}^{(l)} \subset \mathcal{X}^{2l+1}$ is a complex submanifold. Let $\pi_j^{(l)}$ be the restriction to $\mathcal{X}^{(l)}$ of the natural projection to the j-th component, $1 \leqslant j \leqslant 2l+1$, and denote by $\widetilde{\lambda} : \mathcal{X}^{(l)} \to \mathcal{Z}$ and $\widetilde{\mu} : \mathcal{X}^{(l)} \to \mathcal{W}$ the maps defined by $\widetilde{\lambda} := \lambda \circ \pi_1^{(l)}$, $\widetilde{\mu} := \mu \circ \pi_{2l+1}^{(l)}$. Then, for every $x \in \mathcal{X}$, we set $x^{(l)} := (x, \ldots, x) \in \mathcal{X}^{(l)}$ and $D_l(x) := \widetilde{\lambda}^{-1}(\widetilde{\lambda}(x^{(l)}))$ and $E_l(x) := \widetilde{\mu}^{-1}(\widetilde{\mu}(x^{(l)}))$ are complex submanifolds of $\mathcal{X}^{(l)}$. We say in this setting that the pair (λ, μ) of submersions is *of finite type* at a point $x_0 \in \mathcal{X}$ if there exists $l_0 \geqslant 1$ such that the map $\widetilde{\mu}_{l_0}|_{D_{l_0}(x_0)}$ has rank equal to $\dim \mathcal{W}$ at some points of the intersection $D_{l_0}(x_0) \cap E_{l_0}(x_0)$ that are arbitrarily close to $x_0^{(l_0)}$. We may now formulate the main tool of this section which is the following meromorphic extension property for ratios of formal power series. It was inspired by an analogous result from [10] in a different context but its proof is however completely different and consists of repeatedly applying Proposition 2.3.

THEOREM 2.4. – *Let $\mathcal{X}$, $\mathcal{Z}$, $\mathcal{W}$ be complex manifolds and $\lambda : \mathcal{X} \to \mathcal{Z}$, $\mu : \mathcal{X} \to \mathcal{W}$ be a pair of holomorphic submersions of finite type at a point $x_0 \in \mathcal{X}$. Consider formal power series $F(x), G(x)$ on $\mathcal{X}$ centered at x_0 of the form $F(x) = \Phi(Y(\lambda(x)), x)$, $G(x) = \Psi(Y(\lambda(x)), x)$, where Y is a $\mathbb{C}^r$-valued formal power series on $\mathcal{W}$ centered at $\lambda(x_0)$ and Φ, Ψ are convergent power series on $\mathbb{C}^r \times \mathcal{X}$ centered at $(Y(\lambda(x_0)), x_0)$. Suppose that $G \not\equiv 0$ and that $\mathcal{L}(F/G) \equiv 0$ holds for any holomorphic vector field $\mathcal{L}$ on $\mathcal{X}$ annihilating μ. Then $(F : G)$ is equivalent to a nontrivial ratio of convergent power series on $\mathcal{X}$ (centered at x_0).*

As mentioned above, the main example of this setting is given by the case when $\mathcal{X}$ is the complexification $\mathcal{M} \subset \mathbb{C}^N \times \mathbb{C}^N$ of a real-analytic generic submanifold $M \subset \mathbb{C}^N$ through the origin, where a pair of submersions (λ, μ) on $\mathcal{M}$ is given by the projections on the last and the first component $\mathbb{C}^N$ respectively. In this case, we call any holomorphic vector field in $\mathbb{C}^N \times \mathbb{C}^N$ annihilating the submersion μ a $(0, 1)$ vector field and the construction of $\mathcal{X}^{(l)}$ yields the iterated complexification $\mathcal{M}^{2l}$ as defined in [17]. The images $\widetilde{\mu}(D_l(x))$ turn out to be the Segre sets in the sense of Baouendi, Ebenfelt and Rothschild [2] and their minimality criterion says that M is minimal if and only if the Segre sets of sufficiently high order have nonempty interior. The last condition can also be expressed in terms of ranks and turns out to be equivalent to saying that the corresponding pair of submersions (λ, μ) defined above is of finite type at $0 \in \mathcal{M}$ (see [2,3]). Therefore, an immediate application of Theorem 2.4 to that setting yields the following.

THEOREM 2.5. – *Let $M \subset \mathbb{C}^N$ be a real-analytic generic submanifold through 0 and $\mathcal{M} \subset \mathbb{C}^N_Z \times \mathbb{C}^N_\zeta$ its complexification. Consider formal power series $F(Z,\zeta), G(Z,\zeta) \in \mathbb{C}[\![Z,\zeta]\!]$ of the form $F(Z,\zeta) = \Phi(Y(\zeta), Z)$, $G(Z,\zeta) = \Psi(Y(\zeta), Z)$, where $Y(\zeta)$ is a $\mathbb{C}^r$-valued formal power series and Φ, Ψ are convergent power series centered at $(Y(0), 0) \in \mathbb{C}^r \times \mathbb{C}^N$ with $G(Z,\zeta) \not\equiv 0$ for $(Z,\zeta) \in \mathcal{M}$. Suppose that M is minimal at 0 and that $\mathcal{L}(F/G) \equiv 0$ on $\mathcal{M}$ (i.e., $F\mathcal{L}G - G\mathcal{L}F \equiv 0$ on $\mathcal{M}$) for any $(0, 1)$ holomorphic vector field tangent to $\mathcal{M}$. Then there exist convergent power series $\widetilde{F}(Z), \widetilde{G}(Z) \in \mathbb{C}\{Z\}$, with $\widetilde{G}(Z) \not\equiv 0$, such that the ratios $(F : G)$ and $(\widetilde{F} : \widetilde{G})$ are equivalent as formal power series on $\mathcal{M}$ (i.e., $F\widetilde{G} - \widetilde{F}G \equiv 0$ on $\mathcal{M}$).*

3. Description of the proof of Theorems 1.1 and 1.2

Let $f : (\mathbb{C}^N_Z, p) \to (\mathbb{C}^{N'}_{Z'}, p')$ be a formal (holomorphic) map, i.e., the data of N' formal power series $f(Z) = (f_1(Z), \ldots, f_{N'}(Z))$ centered at p, with $f(p) = p'$. In what follows, we will assume that p and p' are the origin in $\mathbb{C}^N$ and $\mathbb{C}^{N'}$ respectively and denote by $\mathbb{C}\{t\}$ (resp. $\mathbb{C}[\![t]\!]$) the ring of convergent power series (resp. formal power series) in n variables, $t \in \mathbb{C}^n$. For f as above, we say that a germ at $(0, 0) \in \mathbb{C}^N \times \mathbb{C}^{N'}$ of a holomorphic function h vanishes on the graph of f if the identity $h(Z, f(Z)) = 0$ holds in the ring $\mathbb{C}[\![Z]\!]$. We define the *Zariski closure* associated to f (with respect to the ring $\mathbb{C}\{Z\}[Z']$) as the germ $\mathcal{Z}_f \subset \mathbb{C}^N \times \mathbb{C}^{N'}$ at $(0, 0)$ of a complex-analytic set defined by the zero-set of all elements in $\mathbb{C}\{Z\}[Z']$ vanishing on the graph of f. We shall denote by $\mu(f)$ the dimension of the Zariski closure $\mathcal{Z}_f$. The integer $\mu(f)$ is relevant for the study of the convergence of f, since it follows easily from Artin's approximation theorem [1] that $\mu(f) = N$ if and only if f is convergent (see, e.g., [3,13]). Moreover, we may assume that the Zariski closure $\mathcal{Z}_f$ can be represented by an irreducible closed analytic subset (over the ring $\mathbb{C}\{Z\}[Z']$ and also denoted by $\mathcal{Z}_f$) of $\Delta^N_0 \times \mathbb{C}^{N'}$, where Δ^N_0 is a sufficiently small polydisc centered at $0 \in \mathbb{C}^N$.

By the construction of the Zariski closure $\mathcal{Z}_f$, any (germ at $(0, 0)$ of a) holomorphic function vanishing on $\mathcal{Z}_f$ vanishes on the graph of f and therefore, an application of Artin's approximation theorem [1] yields for any positive integer κ, a convergent (holomorphic) power series f^κ, whose graph is contained in $\mathcal{Z}_f$. We may assume that each series f^κ is convergent in a polydisc $\Delta^N_{0,\kappa} \subset \Delta^N_0$ containing the origin and we denote the graph of f^κ by $\Gamma_{f^\kappa} := \{(Z, f^\kappa(Z)) : Z \in \Delta^N_{0,\kappa}\}$. The main point of the proof consists of showing that if f sends a generic real-analytic minimal submanifold $M \subset \mathbb{C}^N$ into a real-algebraic subset $M' \subset \mathbb{C}^{N'}$ (both through the origin), then necessarily there exists a suitable union of local real-analytic irreducible components of the real-analytic subset $\mathcal{Z}_f \cap (M \times \mathbb{C}^{N'})$ that is contained in $M \times M'$ and that contains $\Gamma_{f^\kappa} \cap (M \times \mathbb{C}^{N'})$, for all κ sufficiently large. The precise statement is given by the following result.

THEOREM 3.1. – *Let $f : (\mathbb{C}^N_Z, 0) \to (\mathbb{C}^{N'}_{Z'}, 0)$ be a formal map, $\mathcal{Z}_f$ the Zariski closure associated to f and $(f^\kappa)_{\kappa \geqslant 0}$ the convergent maps given above. Let $M \subset \mathbb{C}^N$ be a minimal real-analytic generic submanifold through the origin. Assume that f sends M into M' where $M' \subset \mathbb{C}^{N'}$ is a real-algebraic subset through the origin. Then, after shrinking M around the origin if necessary, there exist a positive integer κ_0 and an appropriate union Z_f of local real-analytic irreducible components of $\mathcal{Z}_f \cap (M \times \mathbb{C}^{N'})$ such that for any $\kappa \geqslant \kappa_0$, $\Gamma_{f^\kappa} \cap (M \times \mathbb{C}^{N'}) \subset Z_f \subset M \times M'$ and such that Z_f satisfies the following straightening property: for any $\kappa \geqslant \kappa_0$, there exists a neighborhood M^κ of 0 in M such that for any point Z_0 in a dense open subset of M^κ, there exists a neighborhood $U^\kappa_{Z_0}$ of $(Z_0, f^\kappa(Z_0))$ in $\mathbb{C}^N \times \mathbb{C}^{N'}$ and a holomorphic change of coordinates in $U^\kappa_{Z_0}$ of the form $(\widetilde{Z}, \widetilde{Z}') = \Phi^\kappa(Z, Z') = (Z, \varphi^\kappa(Z, Z')) \in \mathbb{C}^N \times \mathbb{C}^{N'}$ such that $Z_f \cap U^\kappa_{Z_0} = \{(Z, Z') \in U^\kappa_{Z_0} : Z \in M,\ \widetilde{Z}'_{m+1} = \cdots = \widetilde{Z}'_{N'} = 0\}$, where $m = \mu(f) - N$.*

One of the ingredients of the proof of Theorem 2.5 strongly relies on the meromorphic extension property proved in Theorem 2.5. With Theorem 3.1 at our disposal, it is then not difficult to derive Theorem 1.2 from the closedness of the set $\mathcal{E}'$ (see, e.g., [7,9]) as well as Theorem 1.1 (see [12] for details).

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