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NORDINE MIR

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GERMS OF HOLOMORPHIC MAPPINGS BETWEEN REAL ALGEBRAIC HYPERSURFACES

by Nordine MIR

1. Introduction and formulation of main result.

A real algebraic hypersurface in $\mathbb{C}^n$ is the zero set of a real polynomial with non-vanishing gradient. A holomorphic function defined in an open set in $\mathbb{C}^n$ is called algebraic if it is algebraic over the field of rational functions over $\mathbb{C}$, or equivalently if it satisfies a polynomial equation of the form

$$A_k(Z)f^k(Z) + \ldots + A_0(Z) \equiv 0,$$

where the functions $A_j(Z)$ are holomorphic polynomials, with $k \geqslant 1$ and $A_k \not\equiv 0$. In recent years, several papers appeared concerning algebraicity of holomorphic mappings or CR mappings between algebraic Cauchy-Riemann manifolds ([16], [7], [5], [22], [3]). For mappings in the same dimension, Baouendi and Rothschild [7], [5] (for the hypersurface case) and Baouendi, Ebenfelt and Rothschild [3] proved that holomorphic nondegeneracy is a necessary and sufficient condition for algebraicity of germs of biholomorphic maps between minimal generic CR submanifolds of $\mathbb{C}^n$. Here, holomorphic nondegeneracy and minimality must be understood in the sense of Stanton [24] and Tumanov [25]. We note also that in the work [3], the authors consider real algebraic sets (i.e. with singularities). However, when one drops the assumption of nondegeneracy, no information is given about the eventual algebraicity of some components of the map. In this paper, we consider the following setting which will be called the *general situation*:

Key words: Algebraic real hypersurface – Holomorphic mapping – Segre variety – Holomorphic nondegeneracy.
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Let (M,p_0) and (M',p_0') be two germs of real algebraic hypersurfaces in $\mathbb{C}^{N+1}, N \geqslant 1$, with M not Levi-flat, and $H : \mathbb{C}^{N+1} \to \mathbb{C}^{N+1}$ a germ at p_0 of a holomorphic map of generic maximal rank (i.e. $\mathrm{Jac}(H) \not\equiv 0$*) such that $H(p_0) = p_0'$ and $H(M) \subseteq M'$.*

Then we address the following question. What can it be said about the mapping H without assuming any nondegeneracy condition on the manifolds? Theorem 1.1 below gives an answer to this question. We emphasize that the situation here is much more different than in [7], [5] and [3], because we do suppose nothing else than the non-flatness assumption on our hypersurfaces.

We shall now describe our main result. In the general situation described above, after a translation, we may assume that p_0' is sent to 0 and M' is given near 0 by

$$M' = \{Z' \in \mathbb{C}^{N+1} \ / \ \rho'(Z', \bar{Z}') = 0\}, \quad Z' = (z', w') \in \mathbb{C}^N \times \mathbb{C},$$

where ρ' is a real polynomial and $\dfrac{\partial \rho'}{\partial y'_{N+1}}(0) \neq 0$ $(w' = x'_{N+1} + i y'_{N+1})$. By the algebraic (complex) implicit function theorem, M' can also be defined near 0 by the equation

$$(1) \qquad w' = Q'(\bar{z}', \bar{w}', z'),$$

with Q' holomorphic algebraic of its arguments and $Q'(0) = 0$ (see [12]). For any holomorphic function χ defined in a neighborhood of 0 in $\mathbb{C}^k$, $k \geqslant 1$, we put $\bar{\chi}(p) = \overline{\chi(\bar{p})}$ for p close to 0. Following the philosophy of the Schwarz reflection principle ([4], [17]), we define the reflection function $\mathcal{R}$ near $(p_0, 0) \in \mathbb{C}^{N+1} \times \mathbb{C}^N$ to be the map $(Z, \lambda) \to \bar{Q}'(H(Z), \lambda)$. Let $\mathcal{A}_{N+1}$ denote the ring of germs at 0 in $\mathbb{C}^{N+1}$ of holomorphic functions which are algebraic over the field of rational functions over $\mathbb{C}$ and $\mathcal{F}_{N+1}$ denote its quotient field (this is a field of abelian functions). Write $\bar{Q}'$ in the following way: $\bar{Q}'(z', w', \xi) = \sum_{\alpha \in \mathbb{N}^N} \rho'_\alpha(z', w')\xi^\alpha$. Following [19], we denote by $\mathcal{K}(M')$ the smallest field contained in $\mathcal{F}_{N+1}$, containing $\mathbb{C}$ and the family $(\rho'_\beta)_{\beta \in \mathbb{N}^N}$. We are now ready to state our main result.

THEOREM 1.1. — *In the general situation described above and with the previous notations, one has:*

i) *The reflection function $\mathcal{R}$ is holomorphic algebraic near $(p_0, 0)$.*

ii) *For every element $q \in \mathcal{A}_{N+1}$ which belongs to the algebraic closure of $\mathcal{K}(M')$, the function $Z \to q \circ H(Z)$ is holomorphic algebraic.*

Moreover, if the coordinates (z', w') are normal with respect to M' (i.e. $\bar{Q}'(z', w', 0) \equiv w'$), the normal component of H is always algebraic.

We emphasize that here, we have only done a translation from the original defining function of the target hypersurface. As an immediate application of an algebraic criterion of holomorphic non-degeneracy obtained by the author in [19], Theorem 1.1 leads to the following well-known corollary.

COROLLARY 1 ([7], [5]). — *Assume in the general situation that the source hypersurface is holomorphically nondegenerate at p_0. Then H is holomorphic algebraic.*

The paper is organized as follows. In Section 2, we recall briefly some basic facts from algebraic hypersurfaces and the Segre varieties associated to them. Section 3 is devoted to the proof of an algebraic proposition of some interest. Section 4 contains some technical lemmas and in Section 5, we first prove part (ii) of Theorem 1.1, and then part (i). We conclude with the proof of the corollary and some examples illustrating the spirit of our main result.

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2. Algebraic real hypersurfaces, Segre varieties.

Let (M, p_0) be a germ of a real algebraic hypersurface in $\mathbb{C}^{N+1}$. As in the introduction, we may assume that, after a translation, M is given by

$$M = \{Z \in \mathbb{C}^{N+1} \ / \ \rho(Z, \bar{Z}) = 0\}, \quad Z = (z, w) \in \mathbb{C}^N \times \mathbb{C},$$

where ρ is a real polynomial and with the corresponding statements of the introduction without primes. For any point (z, w) near 0, we define a complex algebraic hypersurface $Q_{(z,w)}$, called the *Segre variety* associated to (z, w) by

$$Q_{(z,w)} = \{(\xi, \tau) \in \mathbb{C}^{N+1} \ / \ \rho((\xi, \tau), (\bar{z}, \bar{w})) = 0\},$$

where we have complexified ρ. These manifolds were introduced by Segre [23], and were extensively used by many mathematicians in mapping problems such as Webster [26], Diederich-Webster [12], Diederich-Fornaess

[10], [9], Diederich-Pinchuk [11], to name a few. Recall also, as in the introduction, that by the algebraic implicit theorem, the Segre variety $Q_{(z,w)}$ can be given near 0 by $Q_{(z,w)} = \{(\xi,\tau) \in U^0 \ / \ \tau = \bar{Q}(\bar{z},\bar{w},\xi)\}$, where Q is holomorphic algebraic of its arguments and U^0 is a sufficiently small neighborhood of 0 in $\mathbb{C}^{N+1}$. Recall also that the polar $\mathcal{M}$ of M (i.e. its complexification) is the complex algebraic hypersurface in $\mathbb{C}^{2N+2}$ given by $\{(z,w,\xi,\tau) \in U^1 \times U^0 \ / \ \tau = \bar{Q}(z,w,\xi)\}$ and U^1 is a small neighborhood(1) of 0 in $\mathbb{C}^{N+1}$ with $U^0 \subseteq U^1$. We remind the reader that from the reality of M, one has the following identity:

$$Q(\xi, \bar{Q}(z,w,\xi), z) \equiv w. \tag{2}$$

We will make use of the following basis of holomorphic vector fields tangent to the Segre variety $Q_{(z,w)}$:

$$X^j_{(\bar{z},\bar{w})} = \frac{\partial \rho}{\partial \tau}(\bar{z},\bar{w},\xi,\tau)\frac{\partial}{\partial \xi_j} - \frac{\partial \rho}{\partial \xi_j}(\bar{z},\bar{w},\xi,\tau)\frac{\partial}{\partial \tau}, \quad j = 1,\dots N. \tag{3}$$

3. An algebraic proposition.

In this section, we consider the target hypersurface given as in the introduction. First, define for $\alpha \in \mathbb{N}^N$ $\Xi_\alpha(\xi,\tau,z) = \dfrac{\partial^{|\alpha|} Q'}{\partial z^\alpha}(\xi,\tau,z)$. (We have dropped the ' for the variables.) This defines Ξ_α as an element of $\mathcal{A}_{2N+1}$. We also define $q_\beta = \bar{\rho}'_\beta$, for $\beta \in \mathbb{N}^N$. We denote by K the smallest field contained in $\mathcal{F}_{2N+1}$ and containing $\mathbb{C}$, the families $z = (z_1,\dots,z_N)$ and (Ξ_α). To finish with these notations, let $\bar{\mathcal{K}}(M')$ be the smallest field contained in $\mathcal{F}_{2N+1}$ and containing $\mathbb{C}$ and the family (q_β). Then one has the following

PROPOSITION 1. — *The field $\bar{\mathcal{K}}(M')$ is contained in the algebraic closure of K.*

Proof. — We must show that for each multi-index $\beta \in \mathbb{N}^N$, q_β is algebraic over K. First, recall that if k_1 and k_2 are two fields with $k_1 \subseteq k_2$, a finite subset $\{s_1,\dots,s_p\}$ of k_2 is called algebraically independent over k_1 if the following proposition:

$$(P \in k_1[T_1,\dots,T_p] \text{ and } P(s_1,\dots,s_p) \equiv 0 \text{ implies that } P \equiv 0)$$

(1) Throughout the paper, all neighborhoods will be assumed to be connected.

holds. (For more details about the standard concepts of field theory we shall use, we refer the reader to [15], [27] or [20].) To begin with, choose in the families z and $(\Xi)_\alpha$ a maximal set of algebraically independent elements over $\mathbb{C}$. Such a set is always finite and does not exceed $2N+1$, the transcendence degree of $\mathcal{F}_{2N+1}$ over $\mathbb{C}$. Note that since the z_i, for $i=1,\dots,N$ are algebraically independent over $\mathbb{C}$, we can assume that the set chosen is of the form $(z_1,\dots,z_N,\Xi_{\alpha_1},\dots,\Xi_{\alpha_k})$. This also means that the algebraic closure of K is the algebraic closure of $\mathbb{C}\left(z,\Xi_{\alpha_1},\dots,\Xi_{\alpha_k}\right)$ i.e. the smallest field containing $\mathbb{C}$ and the family $(z,\Xi_{\alpha_1},\dots,\Xi_{\alpha_k})$.(Moreover, one can see that the fact that $\dfrac{\partial Q'}{\partial\tau}$ does not vanish at 0 implies that $k\geqslant 1$; but it has no importance in the sequel of our proof.) Recall that we want to show that for any β, q_β is algebraically dependent over K, i.e. over the family $\left(z,\Xi_{\alpha_1},\dots,\Xi_{\alpha_k}\right)$. To show this, it suffices to see, according to [15] (Theorem III, p. 135, volume 1), that the generic rank of the following Jacobian matrix $\nu=\nu(\xi,\tau,z)$:

$$\begin{pmatrix} & \frac{\partial\Xi_{\alpha_1}}{\partial z_1} & \cdots & \frac{\partial\Xi_{\alpha_k}}{\partial z_1} & 0 \\ I_N & \vdots & \cdots & \vdots & 0 \\ & \frac{\partial\Xi_{\alpha_1}}{\partial z_N} & \cdots & \frac{\partial\Xi_{\alpha_k}}{\partial z_N} & 0 \\ & \frac{\partial\Xi_{\alpha_1}}{\partial \xi_1} & \cdots & \frac{\partial\Xi_{\alpha_k}}{\partial \xi_1} & \frac{\partial q_\beta}{\partial \xi_1} \\ 0 & \vdots & \cdots & \vdots & \vdots \\ & \frac{\partial\Xi_{\alpha_1}}{\partial \tau} & \cdots & \frac{\partial\Xi_{\alpha_k}}{\partial \tau} & \frac{\partial q_\beta}{\partial \tau} \end{pmatrix}$$

is less or equal to $N+k$. We may assume that $k<N+1$. Indeed, in case $k=N+1$, similarly to what has been done in [19], the family $(z,\Xi_{\alpha_1},\dots,\Xi_{\alpha_{N+1}})$ would be a transcendence basis of $\mathcal{F}_{2N+1}$ over $\mathbb{C}$. As a consequence, we then would have that the algebraic closure of K is all $\mathcal{F}_{2N+1}$, and then the proposition follows. We will thus suppose that $k<N+1$. We shall first show that each square submatrix x of order $N+k+1$ extracted from ν has a determinant which vanishes identically. Consider such a submatrix x. For z close to 0, consider the Taylor expansion of $\Xi_\beta=\Xi_\beta(\xi,\tau,\lambda)$ with respect to λ at z, i.e. $\Xi_\beta(\xi,\tau,\lambda)=\sum_{\alpha\in\mathbb{N}^N}\Xi_{\alpha+\beta}(\xi,\tau,z)\dfrac{(\lambda-z)^\alpha}{\alpha!}$. This implies for z close to 0 that $q_\beta(\xi,\tau)=\sum_{\alpha\in\mathbb{N}^N}\Xi_{\alpha+\beta}(\xi,\tau,z)\dfrac{(-z)^\alpha}{\alpha!}$ and hence

$$\frac{\partial q_\beta}{\partial\xi_j}(\xi,\tau)=\sum_{|\alpha|=0}^{\infty}\frac{\partial\Xi_{\alpha+\beta}}{\partial\xi_j}(\xi,\tau,z)\frac{(-z)^\alpha}{\alpha!},\quad\text{for } i=1,\dots,N+1,\tag{4}$$

where we note $\xi_{N+1} = \tau$. Now, consider for $n \in \mathbb{N}$, the following element u_n^β of $\mathcal{A}_{2N+1}$ defined by $u_n^\beta(\xi, \tau, z) = \sum_{|\alpha| \leqslant n} \Xi_{\alpha+\beta}(\xi, \tau, z)\frac{(-z)^\alpha}{\alpha!}$. By the choice of the family $(z, \Xi_{\alpha_1}, \ldots, \Xi_{\alpha_k})$, one sees that each term of the sum is algebraically dependent over this family, and hence u_n^β too. This implies (again according to [15] Theorem III, p.135, volume 1, or [20]) that the generic rank of the following matrix:

$$\begin{pmatrix} & \frac{\partial \Xi_{\alpha_1}}{\partial z_1} & \cdots & \frac{\partial \Xi_{\alpha_k}}{\partial z_1} & \frac{\partial u_n^\beta}{\partial z_1} \\ I_N & \vdots & \cdots & \vdots & \vdots \\ & \frac{\partial \Xi_{\alpha_1}}{\partial z_N} & \cdots & \frac{\partial \Xi_{\alpha_k}}{\partial z_N} & \frac{\partial u_n^\beta}{\partial z_N} \\ & \frac{\partial \Xi_{\alpha_1}}{\partial \xi_1} & \cdots & \frac{\partial \Xi_{\alpha_k}}{\partial \xi_1} & \frac{\partial u_n^\beta}{\partial \xi_1} \\ 0 & \vdots & \cdots & \vdots & \vdots \\ & \frac{\partial \Xi_{\alpha_1}}{\partial \tau} & \cdots & \frac{\partial \Xi_{\alpha_k}}{\partial \tau} & \frac{\partial u_n^\beta}{\partial \tau} \end{pmatrix}$$

which is the same as the following one:

$$\begin{pmatrix} & \frac{\partial \Xi_{\alpha_1}}{\partial z_1} & \cdots & \frac{\partial \Xi_{\alpha_k}}{\partial z_1} & 0 \\ I_N & \vdots & \cdots & \vdots & \vdots \\ & \frac{\partial \Xi_{\alpha_1}}{\partial z_N} & \cdots & \frac{\partial \Xi_{\alpha_k}}{\partial z_N} & 0 \\ & \frac{\partial \Xi_{\alpha_1}}{\partial \xi_1} & \cdots & \frac{\partial \Xi_{\alpha_k}}{\partial \xi_1} & \frac{\partial u_n^\beta}{\partial \xi_1} \\ 0 & \vdots & \cdots & \vdots & \vdots \\ & \frac{\partial \Xi_{\alpha_1}}{\partial \tau} & \cdots & \frac{\partial \Xi_{\alpha_k}}{\partial \tau} & \frac{\partial u_n^\beta}{\partial \tau} \end{pmatrix}$$

is less or equal to $N + k$. Now, one sees that identity 4 together with the above statement implie that the determinant of x is the limit of a sequence of determinants which all vanish identically. This shows that the generic rank of ν is less than $N + k$ and hence that the family $(z, \Xi_{\alpha_1}, \ldots, \Xi_{\alpha_k}, q_\beta)$ is algebraically dependent over $\mathbb{C}$. By the choice of the family $(z, \Xi_{\alpha_1}, \ldots, \Xi_{\alpha_k})$, this proves that q_β is algebraic over K. This completes the proof of Proposition 1.

4. Algebraicity along Segre varieties.

4.1. Some preliminaries.

We consider now the general situation described in the introduction of the paper. We will assume that the source manifold is given as in Section 2 and the target manifold is given as in the introduction. To begin with, recall the following arguments due to Webster [26]. In the general situation, one has $H(Q_p) \subset Q'_{H(p)}$, where the ' means that we consider the Segre variety of the target manifold. If $H = (f,g) = (f_1, \ldots, f_N, g)$, then for any point $(z,w) \in U^1$ and any point $(\xi,\tau) \in Q_{(\bar z,\bar w)} \cap U^0$, we have

$$g(\xi,\tau) = Q'\left(\overline{f(\bar z,\bar w)}, \overline{g(\bar z,\bar w)}, f(\xi,\tau)\right). \tag{5}$$

Recall that $(\xi,\tau) \in Q_{(\bar z,\bar w)}$, is equivalent to saying that (z,w,ξ,τ) belongs to the polar $\mathcal{M}$. Define for $(z,w,\xi,\tau) \in \mathcal{M} \cap U^1 \times U^0$,

$$D(z,w,\xi,\tau) = D = \det\left(X^j_{(z,w)} f_i(\xi,\tau)\right)_{i,j=1,\ldots,N},$$

and let $X^\alpha_{(z,w)}$ denote $\left(X^1_{(z,w)}\right)^{\alpha_1} \cdots \left(X^N_{(z,w)}\right)^{\alpha_N}$ for each multi-index $\alpha = (\alpha_1, \ldots, \alpha_N)$. Differentiating the identity (5) along the Segre variety $Q_{(\bar z,\bar w)}$ yields the following lemma, whose proof can be found in substance in [6].

LEMMA 1. — *For any multi-index $\beta \in \mathbb{N}^N$ with $|\beta| \geqslant 1$, there exists a universal polynomial $P_\beta \in \mathbb{C}[T_{a(N+1)}]$ with $a = \mathrm{card}\{\alpha \ / \ 1 \leqslant |\alpha| \leqslant |\beta|\}$ such that for any point $(z,w,\xi,\tau) \in \mathcal{M} \cap U^1 \times U^0$, one has*

$$D^{2|\beta|-1}\ \Xi_\beta(\bar f(z,w), \bar g(z,w), f(\xi,\tau)) = P_\beta\left((X^\alpha_{(z,w)} H(\xi,\tau))_{1 \leqslant |\alpha| \leqslant |\beta|}\right).$$

Before following our plan, we would like to point out a simple but crucial fact . In the above lemma, the identity holds on $\mathcal{M} \cap U^1 \times U^0$, but D and the right hand-side of the last equation are defined (and holomorphic) in the whole neighborhood $U^1 \times U^0$ of $\mathbb{C}^{2N+2}$. Furthermore, by the choice of the vector fields tangent to $Q_{(\bar z,\bar w)}$ (recall that ρ is a polynomial), one sees that D and the right hand-side of this last equation are actually elements of $\mathcal{O}(U^0)[Z,W]$ i.e. polynomials in (z,w) with holomorphic coefficients in $(\xi,\tau) \in U^0$.

Since in our defining functions we allow pure terms and terms of order one to exist, we must be a bit careful in our computations. Hence, we have to show the following lemma.

LEMMA 2. — *D does not vanish identically in $\mathcal{M} \cap U^1 \times U^0$.*

Proof. — First, we choose a point $q \in M$ (arbitrarily close to 0) such that the Jacobian determinant of H does not vanish at q (this is possible since M is a set of uniqueness for holomorphic functions defined near M). As one can easily check, the rank of the Jacobian matrix of H at q is the same as the rank of the following $N+1 \times N+1$ matrix:

$$A(\bar{q}, q) = \begin{pmatrix} \left(X_{\bar{q}}^j f_i(q)\right)_{i,j \leqslant N} & \left(\frac{\partial \rho}{\partial \tau}(\bar{q}, q) \frac{\partial f_k}{\partial \tau}(q)\right)_{k \leqslant N} \\ (X_{\bar{q}}^j g(q))_{j \leqslant N} & \frac{\partial \rho}{\partial \tau}(\bar{q}, q) \frac{\partial g}{\partial \tau}(q) \end{pmatrix}.$$

Differentiating (5) along the Segre Varieties and evaluating at the point $(\bar{q}, q) \in \mathcal{M} \cap U^1 \times U^0$, we get for $j = 1, \ldots N$,

$$X_{\bar{q}}^j g(q) = \sum_{i=1}^{N} Q'_{z'_i}(\bar{H}(\bar{q}), f(q)) X_{\bar{q}}^j f_i(q). \tag{6}$$

If $D(\bar{q}, q) = 0$, then from (6), one sees that the rank of the following $N+1 \times N$ matrix:

$$\begin{pmatrix} \left(X_{\bar{q}}^j f_i(q)\right)_{i,j \leqslant N} \\ (X_{\bar{q}}^j g(q))_{j \leqslant N} \end{pmatrix}$$

is less or equal to $N-1$. This implies that the rank of $A(\bar{q}, q)$ is less or equal to N, a contradiction. Hence, D can not vanish identically on the polar.

We follow our plan by applying Proposition 1. This latter means that for any $\beta \in \mathbb{N}^N$, there exists a positive integer $k(\beta)$ and holomorphic polynomials of their arguments R_β^j (with $0 \leqslant j \leqslant k(\beta)$) such that near 0, one has

$$\sum_{j=0}^{k(\beta)} R_\beta^j \Big((\Xi_{\alpha_p}(\xi', \tau', z')_{p=1,\ldots,r}, z' \Big) q_\beta^j(\xi', \tau') \equiv 0, \tag{7}$$

with $R_\beta^{k(\beta)} \Big((\Xi_{\alpha_p}(\xi', \tau', z'))_{p=1,\ldots,r}, z' \Big) \not\equiv 0$, and $\Big((\Xi_{\alpha_p}(\xi', \tau', z'))_{p=1,\ldots,r}, z' \Big)$ is a maximal set of algebraically independent elements as in the proof of Proposition 1. For $(z,w,\xi,\tau) \in \mathcal{M} \cap U^1 \times U^0$, putting $z' = f(\xi,\tau)$, $\xi' = \bar{f}(z,w)$, and $\tau' = \bar{g}(z,w)$ in the previous equation yields

(8)
$$\sum_{j=0}^{k(\beta)} R_\beta^j\Big((\Xi_{\alpha_p}(\bar{f}(z,w),\bar{g}(z,w),f(\xi,\tau))_{p=1,\dots,r}, f(\xi,\tau)\Big) q_\beta^j(\bar{H}(z,w)) \equiv 0,$$

which can be rewritten in the following way:

$$\sum_{j=0}^{k(\beta)} \delta_\beta^j(z,w,\xi,\tau) q_\beta^j(\bar{H}(z,w)) \equiv 0 \quad \text{in } \mathcal{M} \cap U^1 \times U^0.$$

To continue, we will need the following.

LEMMA 3. — *The holomorphic map $\delta_\beta^{k(\beta)}$ does not vanish identically on $\mathcal{M} \cap U^1 \times U^0$.*

Proof. — Recall that

$$\delta_\beta^{k(\beta)}(z,w,\xi,\tau) = R_\beta^{k(\beta)}\Big((\Xi_{\alpha_p}(\bar{f}(z,w),\bar{g}(z,w),f(\xi,\tau))_{p=1,\dots,r}, f(\xi,\tau)\Big).$$

Moreover, we know that $R_\beta^{k(\beta)}\Big((\Xi_{\alpha_p}(\xi',\tau',z')_{p=1,\dots,r}, z'\Big) \not\equiv 0$. Hence, to show that $\delta_\beta^{k(\beta)}$ does not vanish identically it suffices to see that the holomorphic map u, defined by $(z,w,\xi,\tau) \in \mathcal{M} \cap U^1 \times U^0 \to \Big(\bar{f}(z,w),\bar{g}(z,w),f(\xi,\tau)\Big) \in \mathbb{C}^{2N+1}$ is of generic complex rank 2N+1 near 0. But if we take a point $q \in M$ (arbitrarily close to 0) satisfying the conditions of the proof of the preceding lemma, we easily get that our map u is precisely of complex maximal rank at the point $(\bar{q},q) \in \mathcal{M}$. This achieves the proof of the lemma.

We shall now use lemma 1. For $(z,w,\xi,\tau) \in \Omega = \{\, v \in \mathcal{M} \cap U^1 \times U^0 \;/\; D(v) \neq 0\}$, we have the following identity for δ_β^j:

$$(9)\quad \delta_\beta^j(z,w,\xi,\tau) = R_\beta^j\left(\left(\frac{P_{\alpha_p}\Big((X^\alpha_{(z,w)}H(\xi,\tau))_{1\leqslant|\alpha|\leqslant|\alpha_p|}\Big)}{D^{2|\alpha_p|-1}}\right), f(\xi,\tau)\right).$$

Note also in the case where one of the α_i equals 0, we just replace the corresponding term by $g(\xi,\tau)$ according to Equation 5. Now, multiplying the last equation by enough powers of D and reminding the reader the remark after Lemma 1, one sees that for each β, we have

$$\sum_{j=0}^{k(\beta)} u_\beta^j(z,w,\xi,\tau)q_\beta^j(\bar{H}(z,w)) \equiv 0, \tag{10}$$

for $(z,w,\xi,\tau) \in \Omega$. Moreover, $k(\beta) \geqslant 1$ and each u_β^j is holomorphic in the whole neighborhood $U^1 \times U^0$, and more precisely belongs to $\mathcal{O}(U^0)[Z,W]$. This implies (together with Lemma 2) that the identity (10) holds on $\mathcal{M} \cap U^1 \times U^0$. To finish, note also that from Lemma 3 and Lemma 2, we see that for each multi-index β, $u_\beta^{k(\beta)}$ does not vanish identically on $\mathcal{M} \cap U^1 \times U^0$.

Equation 10 means, in a certain sense, that each function $Z \to q_\beta(\bar{H}(Z))$ is algebraic along the Segre varieties of M. Our aim, now, is to show that this implies the algebraicity of the latter function. But, here, the identity 10 is a very weak statement concerning algebraicity compared to the ones than one can find in the literature ([7], [5], [3]). Nevertheless, we will show, by using the fact that the u_β^j are actually polynomials in (z,w), that 10 will be sufficient for us to prove algebraicity of the desired map.

From now, we only have to consider the following situation. Let M be a real algebraic hypersurface given as in Section 2, with M not Levi-flat (near $p_0 = 0$), and h a holomorphic function defined in neighborhood of 0 (in $\mathbb{C}^{N+1}$), which is algebraic along the Segre Varieties in the sense that h satisfies an identity of the form

$$\sum_{j=0}^{k} v_j(z,w;\xi,\tau)h^j(z,w) \equiv 0, \tag{11}$$

for $(z,w,\xi,\tau) \in \mathcal{M} \cap U^1 \times U^0$, with $k \geqslant 1$, $v_j \in \mathcal{O}(U^0)[Z,W]$ for $j = 1,\ldots,k$, and v_k does not vanish identically on the polar.

First, choose a point $p_0 \in U^0 \cap M$ such that M is of finite type at this point. Indeed, this is possible since M is assumed not to be Levi-flat (near 0), and hence M contains minimal points arbitrarily close to 0. (See for example [13].) In a second time, note that since the set $T = \{(z,w,\bar{z},\bar{w}) \in U^1 \times U^0 \ / \ (z,w) \in M\}$ is a maximally real (algebraic) CR submanifold of $\mathcal{M} \cap U^1 \times U^0$, it is a set of uniqueness for holomorphic functions defined on the polar [21], and hence v_k can not vanish on any open subset of T; and we may thus choose a point $p_1 = (z_1, w_1) \in M$ (arbitrarily close to p_0) such that v_k does not vanish in an open set $V^1 \times V^0$ of $\mathbb{C}^{2N+2}$, with $(p_1, \bar{p}_1) \in V^1 \times V^0 \subseteq U^1 \times U^0$. Moreover, since M is minimal at p_0, we can assume that M is also minimal at p_1 (recall that minimality is an open property since it is equivalent to the finite type condition of Kohn [18] and Bloom-Graham [8]). Now, we restrict the identity (11) to the complex submanifold near $(p_1, \bar{p}_1)$ given by $(\mathcal{M} \cap V^1 \times V^0) \cap \{(z, w, \xi, \tau) \in V^1 \times V^0 \ / \ (\xi, \tau) \in Q_{p_1}\} = (\mathcal{M} \cap V^1 \times V^0) \cap \{(z, w, \xi, \tau) \in V^1 \times V^0 \ / \ \tau = \bar{Q}(z_1, w_1, \xi)\}$. This gives that for $(z, \xi) \in \mathbb{C}^N \times \mathbb{C}^N$ near $(z_1, \bar{z}_1)$, one has

$$(12) \qquad \sum_{j=0}^{k} v_j(z, Q(\xi, \bar{Q}(z_1, w_1, \xi), z); \xi, \bar{Q}(z_1, w_1, \xi))\psi^j(z, \xi) \equiv 0,$$

where $\psi(z, \xi) = h(z, Q(\xi, \bar{Q}(z_1, w_1, \xi), z))$. Our next goal is to show the following proposition, which will be the main purpose of Subsection 4.2.

PROPOSITION 2. — *ψ is holomorphic algebraic near $(z_1, \bar{z}_1)$.*

4.2. Proof of Proposition 2.

Recall first that for $j = 1, \dots k$, $v_j \in \mathcal{O}(V_1)[Z, W]$. This implies together with Equation 12 that we have for $(z, \xi) \in \mathbb{C}^N \times \mathbb{C}^N$ near $(z_1, \bar{z}_1)$ the following identity:

$$(13) \qquad \sum_{j=0}^{k} \sum_{\substack{\alpha \in \mathbb{N}^N, |\alpha| \leqslant b \\ \nu = 0, \dots, c}} z^\alpha \Big(Q(\xi, \bar{Q}(z_1, w_1, \xi), z)\Big)^\nu u_{\alpha,\nu,j}(\xi)\psi^j(z, \xi) \equiv 0,$$

where $b, c \in \mathbb{N}^*$, the $u_{\alpha,\nu,j}$ are holomorphic near $\bar{z_1}$, and

$$\sum_{\substack{\alpha \in \mathbb{N}^N, |\alpha| \leqslant b \\ \nu = 0, \dots, c}} (z_1)^\alpha (w_1)^\nu u_{\alpha,\nu,k}(\bar{z_1}) = v_k(z_1, w_1; \bar{z_1}, \bar{w_1}) \neq 0.$$

(We have used the identity (2).) We define for $(\sigma, \epsilon) \in \mathbb{C}^N \times \mathbb{C}^N$ close to 0 the following holomorphic function $\varphi(\sigma, \epsilon) = \psi(\sigma + z_1, \epsilon + \bar{z}_1)$. After this translation was made, it is enough to prove that φ is algebraic near 0 to prove Proposition 2. Now, from (13), one sees that φ satisfies near 0 a relation of the form (we omit the parameter (z_1, w_1))

$$(14) \qquad \sum_{j=0}^{k} \sum_{\substack{\alpha \in \mathbb{N}^N, |\alpha| \leqslant b' \\ \nu = 0, \dots, c'}} \sigma^\alpha (\theta(\sigma, \epsilon))^\nu W_{\alpha,\nu,j}(\epsilon)\varphi^j(\sigma, \epsilon) \equiv 0,$$

with θ algebraic holomorphic near 0 (recall that Q is algebraic) and $\sum_{\nu=0,\dots,c'} \theta(0)^\nu W_{0,\nu,k}(0) = \sum_{\alpha \in \mathbb{N}^N, |\alpha| \leqslant b} \sum_{\nu=0,\dots,c} (z_1)^\alpha (w_1)^\nu u_{\alpha,\nu,k}(\bar{z_1}) \neq 0$. The crucial point of our proof is now the following lemma.

Lemma 4. — *For any given $n \in \mathbb{N}$, there exists a family $(x^n_{\alpha,\nu,j})$ $(|\alpha| \leqslant b', \nu = 0,\dots,c', j = 0,\dots,k)$, such that*

i) *this family agrees with $(W_{\alpha,\nu,j})$ up to order n (at 0);*

ii) *each $x^n_{\alpha,\nu,j}$ is algebraic holomorphic;*

iii) $\sum_{j=0}^{k} \sum_{\substack{\alpha\in\mathbb{N}^N,|\alpha|\leqslant b' \\ \nu=0,\dots,c'}} \sigma^\alpha(\theta(\sigma,\epsilon))^\nu x^n_{\alpha,\nu,j}(\epsilon)\varphi^j(\sigma,\epsilon) \equiv 0$, *near 0.*

Proof of Lemma 4. — Before beginning the proof, let us deal first with (14). Expanding it in a Taylor series with respect to σ at 0 yields

$$\sum_{j=0}^{k} \sum_{\substack{\alpha\in\mathbb{N}^N,|\alpha|\leqslant b' \\ \nu=0,\dots,c'}} \sigma^\alpha \sum_{|\gamma|=0}^{\infty} \frac{\sigma^\gamma}{\gamma!}\theta_{\gamma,\nu}(\epsilon)W_{\alpha,\nu,j}(\epsilon) \sum_{|\mu|=0}^{\infty} \varphi_{\mu,j}(\epsilon)\frac{\sigma^\mu}{\mu!} \equiv 0,$$

for (σ,ϵ) belonging to a small neigborhood $W^1 \times W^0$ of 0 in $\mathbb{C}^N \times \mathbb{C}^N$, and $\theta_{\gamma,\nu}$ is holomorphic algebraic. We may thus rewrite the previous equation in the following form:

(15)
$$\sum_{|u|=0}^{\infty} \sigma^u \sum_{\substack{\alpha\in\mathbb{N}^N,\mu\in\mathbb{N}^N \\ |\alpha|\leqslant b',\alpha+\mu\leqslant u}} \sum_{\substack{\nu=0,\dots,c' \\ j=0,\dots,k}} \frac{1}{\mu!(u-\alpha-\mu)!}\theta_{u-\alpha-\mu,\nu}(\epsilon)W_{\alpha,\nu,j}(\epsilon)\varphi_{\mu,j}(\epsilon) \equiv 0,$$

where $\alpha + \mu \leqslant u$ means that for each $i = 1,\dots,N$, $\alpha_i + \mu_i \leqslant u_i$. Now, for each multi-index $u \in \mathbb{N}^N$, we may define the following holomorphic function J_u in $W^0 \times \mathbb{C}^s$ ($s = \mathrm{card}\{\ (\alpha,\nu,j)\ /\ |\alpha| \leqslant b', \nu = 0,\dots,c', j = 0,\dots,k\}$),

$$\begin{aligned} &J_u(\epsilon,(T_{\alpha,\nu,j})) \\ &= \sum_{\substack{\alpha\in\mathbb{N}^N,\mu\in\mathbb{N}^N \\ |\alpha|\leqslant b',\alpha+\mu\leqslant u}} \sum_{\substack{\nu=0,\dots,c' \\ j=0,\dots,k}} \frac{1}{\mu!(u-\alpha-\mu)!}\theta_{u-\alpha-\mu,\nu}(\epsilon)(T_{\alpha,\nu,j}+W_{\alpha,\nu,j}(0))\varphi_{\mu,j}(\epsilon). \end{aligned}$$

Note that it is possible to define all these holomorphic functions in a common neighborhood of 0 in $\mathbb{C}^N \times \mathbb{C}^s$ since the $\theta_{\gamma,\nu}$ and the $\varphi_{\mu,j}$ correspond to the derivatives associated to θ and φ. By the Noetherian property [14], there exists $l_0 \in \mathbb{N}^*$ such that the ideal generated by the $(J_u)_{u\in\mathbb{N}^N}$ is the same as the ideal generated by the $(J_u)_{|u|\leqslant l_0}$. To continue, we have to show the following lemma.

Lemma 5. — *For any μ and for any j, $\varphi_{\mu,j}$ is algebraic holomorphic.*

Proof of Lemma 5. — Recall that by construction, $\varphi_{\mu,j}(\epsilon) = \dfrac{\partial^{|\mu|}\varphi^j}{\partial\sigma^\mu}(0,\epsilon)$ and that $\varphi(\sigma,\epsilon) = \psi(\sigma + z_1, \epsilon + \bar{z}_1)$. We treat the case $j = 1$, since as we shall see, the general case follows from the same lines. We thus have that the derivatives of φ evaluated at $(0,\epsilon)$ are the same as the derivatives of ψ evaluated at $(z_1,\epsilon+\bar{z}_1)$. We claim that these derivatives are algebraic functions of ϵ. Indeed, recall that $\psi(z,\xi)=h(z,Q(\xi,\bar{Q}(z_1,w_1,\xi),z))$. Hence, all the derivatives of ψ evaluated at $(z_1,\epsilon+\bar{z}_1)$ involve the derivatives of Q (which is known to be algebraic) and all the derivatives of h evaluated at the point $(z_1,Q(\epsilon+\bar{z}_1,\bar{Q}(z_1,w_1,\epsilon+\bar{z}_1),z_1))=(z_1,w_1)$ (recall (2)). Hence, these latter derivatives are independent of ϵ, and we are done.

We come back to the proof of Lemma 4. Since for $|\gamma|\leqslant l_0$, $|\mu|\leqslant l_0$, $j=0,\dots,k$ and $\nu=0,\dots,c'$, the $\varphi_{\mu,j}-\varphi_{\mu,j}(0)$ and the $\theta_{\gamma,\nu}-\theta_{\gamma,\nu}(0)$ are algebraic, they satisfy the following non-trivial polynomial system:

$$\Psi_{\mu,j}(\epsilon;\varphi_{\mu,j}-\varphi_{\mu,j}(0))\equiv 0,\quad \Phi_{\gamma,\nu}(\epsilon;\theta_{\gamma,\nu}-\theta_{\gamma,\nu}(0))\equiv 0. \tag{16}$$

The $\Psi_{\mu,j}$ and the $\Phi_{\gamma,\nu}$ are polynomials of their arguments. We shall now use a procedure which has already been used (in another context) by Baouendi and Rothschild ([7] Lemma 1.11). We first choose $n_0\in\mathbb{N}$ large enough so that two families of (germs at 0) holomorphic functions $(\chi_{\mu,j},s_{\gamma,\nu})$ and $(\eta_{\mu,j},t_{\gamma,\nu})$ which satisfy (16) and which agree up to order n_0 must be equal. Let us consider now $n\geqslant n_0$, and the following system of equations in ϵ, in the unknowns $R=(R_{\gamma,\nu})$, $\varpi=(\varpi_{\mu,j}),T=(T_{\alpha,\nu,j})$, with $|\gamma|\leqslant l_0$, $|\mu|\leqslant l_0$, $\nu=0,\dots,c'$, $j=0,\dots,k$, $|\alpha|\leqslant b'$:

$$\begin{cases}\displaystyle\sum_{\substack{\alpha\in\mathbb{N}^N,\mu\in\mathbb{N}^N\\|\alpha|\leqslant b',\alpha+\mu\leqslant u\\ \nu=0,\dots,c'\\ j=0,\dots,k}}\kappa_{u,\alpha,\mu}(R_{u-\alpha-\mu,\nu}+\theta_{u-\alpha-\mu,\nu}(0))(T_{\alpha,\nu,j}+W_{\alpha,\nu,j}(0))\\ \qquad\qquad\qquad\qquad\times(\varpi_{\mu,j}+\varphi_{\mu,j}(0))\equiv 0,\\ \Psi_{\mu,j}(\epsilon;\varpi_{\mu,j})\equiv 0,\quad \Phi_{\gamma,\nu}(\epsilon;R_{\gamma,\nu})\equiv 0,\end{cases}$$

for all $u\in\mathbb{N}^N$ such that $|u|\leqslant l_0$. ($\kappa_{u,\alpha,\mu}=\dfrac{1}{\mu!(u-\alpha-\mu)!}$.) According to (15) and (16), $Y=\Big((\theta_{\gamma,\nu}-\theta_{\gamma,\nu}(0)),(\varphi_{\mu,j}-\varphi_{\mu,j}(0)),(W_{\alpha,\nu,j}-W_{\alpha,\nu,j}(0))\Big)$ is a convergent power series solution of this system. Since we deal with a polynomial system, according to a result of Artin [2], there exists an algebraic formal power series $(R^{0,n},\varpi^{0,n},T^{0,n})$, which satisfies the above system and which agrees with the original solution Y up to order n at 0. The algebraicity of $T^{0,n}$ gives a family of non-trivial polynomials (i.e all at least of degree one) $E^n_{\alpha,\nu,j}=E^n_{\alpha,\nu,j}(\epsilon;X)$ such that $E^n_{\alpha,\nu,j}(\epsilon;T^{0,n}_{\alpha,\nu,j}(\epsilon))\equiv 0$ (in the sense of formal power series). We consider now the following new system in the same unknowns as the first one given by

$$\begin{cases} \displaystyle\sum_{\substack{\alpha\in\mathbb{N}^N,\mu\in\mathbb{N}^N\\ |\alpha|\leqslant b',\alpha+\mu\leqslant u\\ \nu=0,\dots,c'\\ j=0,\dots,k}} \kappa_{u,\alpha,\mu}(R_{u-\alpha-\mu,\nu}+\theta_{u-\alpha-\mu,\nu}(0))(T_{\alpha,\nu,j}+W_{\alpha,\nu,j}(0)) \\ \qquad\qquad\qquad\qquad\qquad\qquad \times(\varpi_{\mu,j}+\varphi_{\mu,j}(0))\equiv 0,\\ \Psi_{\mu,j}(\epsilon;\varpi_{\mu,j})\equiv 0, \quad \Phi_{\gamma,\nu}(\epsilon;R_{\gamma,\nu})\equiv 0, \quad E^n_{\alpha,\nu,j}(\epsilon;T_{\alpha,\nu,j})\equiv 0. \end{cases}$$

($|u|\leqslant l_0$, $|\alpha|\leqslant b'$, $|\gamma|\leqslant l_0$, $|\mu|\leqslant l_0$, $\nu=0,\dots,c'$, $j=0,\dots,k$). $(R^{0,n},\varpi^{0,n},T^{0,n})$ is a formal power series solution of this new system, and this formal power series vanishes at 0 (recall that it agrees with Y up to order n). According to another theorem due to Artin [1], there is a convergent power series $(R^{1,n},\varpi^{1,n},T^{1,n})$ solution of this new system which agrees up to order n with $(R^{0,n},\varpi^{0,n},T^{0,n})$ and hence with Y. By the choice of n_0, we get that $R^{1,n}=(\theta_{\gamma,\nu}-\theta_{\gamma,\nu}(0))$ and $\varpi^{1,n}=(\varphi_{\mu,j}-\varphi_{\mu,j}(0))$. As a consequence, for any multi-index u such that $|u|\leqslant l_0$, $T^{1,n}$ satisfies the following identity in a neighborhood W_n^1 of 0 in $\mathbb{C}^N$:

$$\sum_{\substack{\alpha\in\mathbb{N}^N,\mu\in\mathbb{N}^N\\ |\alpha|\leqslant b',\alpha+\mu\leqslant u\\ \nu=0,\dots,c'\\ j=0,\dots,k}} \kappa_{u,\alpha,\mu}\theta_{u-\alpha-\mu,\nu}(\epsilon)(T^{1,n}_{\alpha,\nu,j}(\epsilon)+W_{\alpha,\nu,j}(0))\varphi_{\mu,j}(\epsilon)\equiv 0.$$

This precisely means that we have $J_u(\epsilon,T^{1,n}(\epsilon))\equiv 0$, for $\epsilon\in W_n^1$ and $|u|\leqslant l_0$. By the choice of l_0, we get that $J_u(\epsilon,T^{1,n}(\epsilon))\equiv 0$ for $\epsilon\in W_n^1$ and $u\in\mathbb{N}^N$. But $J_u(\epsilon,T^{1,n}(\epsilon))$ is precisely the term of order u of the Taylor expansion with respect to σ of

$$\sum_{j=0}^{k}\sum_{\substack{\alpha\in\mathbb{N}^N,|\alpha|\leqslant b'\\ \nu=0,\dots,c'}} \sigma^\alpha(\theta(\sigma,\epsilon))^\nu(T^{1,n}_{\alpha,\nu,j}(\epsilon)+W_{\alpha,\nu,j}(0))\varphi^j(\sigma,\epsilon).$$

Hence we get part (iii) of the lemma by putting $(x^n_{\alpha,\nu,j})$ to be equal to $(T^{1,n}_{\alpha,\nu,j}+W_{\alpha,\nu,j}(0))$. Since $T^{1,n}$ agrees up to order n with $(W_{\alpha,\nu,j}-W_{\alpha,\nu,j}(0))$, we then get part (i). To finish, recall that $T^{1,n}$ satisfies the following non-trivial polynomial system $E^n(\epsilon;T^{1,n}(\epsilon))\equiv 0$ in W_n^1, and hence, we get that $T^{1,n}$ is algebraic holomorphic. The proof of Lemma 4 is thus complete.

Completion of the proof of Proposition 2. — Recall that to show this proposition, it suffices to see that φ is algebraic near 0. Apply Lemma 4 by taking for example $n=n_0\geqslant 1$ as in the proof of this lemma. To obtain the algebraicity of φ, it suffices to see (according to the transitivity of the property of being algebraic [7]), that the identity (iii) is not trivial. But this is clear because $k \geqslant 1$ and

$$\sum_{\nu=0,\dots,c'} \theta(0)^\nu x^{n_0}_{\alpha,\nu,k}(0) = \sum_{\nu=0,\dots,c'} \theta(0)^\nu W_{0,\nu,k}(0) \neq 0.$$

Hence, we are done. We have thus obtained the algebraicity of ψ which was defined by $\psi(z,\xi) = h(z, Q(\xi, \bar{Q}(z_1, w_1, \xi), z)))$ near $(z_1, \bar{z}_1)$. To conclude the algebraicity of h near p_1, we shall use a procedure similar to the one used in [5]. First, note that if $y(z,\xi) = Q(\xi, \bar{Q}(z_1, w_1, \xi), z)$, the gradient of y with respect to ξ does not vanish identically near $(z_1, \bar{z}_1)$. Indeed, if it was not the case we would have $Q(\xi, \bar{Q}(z_1, w_1, \xi), z)) = \chi(z)$, near $(z_1, \bar{z}_1)$, for some holomorphic function χ. The latter identity with $\xi = \bar{z}_1$ together with the fact that $p_1 \in M$ gives that $\chi(z) = Q(\bar{z}_1, \bar{w}_1, z)$. Hence, one has $Q(\bar{z}, \overline{\chi(z)}, z) = \chi(z)$, which means that M contains the complex hypersurface $\{ w = \chi(z) \}$ through $p_1 = (z_1, w_1) \in M$. This contradicts the minimality assumption of M at p_1. Choose (z^0, ξ^0) near $(z_1, \bar{z}_1)$ such that for example $\frac{\partial y}{\partial \xi_1}(z^0, \xi^0) \neq 0$. By the algebraic implicit function theorem, there exists an algebraic holomorphic function j defined in a neighborhood of the point $(z^0, y(z^0, \xi^0)) = (z^0, w^0)$ such that the following identity holds near this point:

$$Q\Big(j(z,w), `\xi^0, \bar{Q}(z_1, w_1, j(z,w), `\xi^0, z)\Big) \equiv w,$$

where $\xi^0 = (\xi^0_1, `\xi^0) \in \mathbb{C} \times \mathbb{C}^{N-1}$. We may now consider the following map near $(z^0, w^0)$, $(z,w) \to (z, j(z,w), `\xi^0) \to h(z, y(z, j(z,w), `\xi^0)) = h(z,w)$. This is a composition of algebraic maps. This implies the algebraicity of h near (z^0, w^0), and hence everywhere where h is defined near 0.

5. Proof of Theorem 1.1 and its corollary - Examples.

We first deal with part (ii) of Theorem 1.1. Recall that the above result implies that for each multi-index $\beta \in \mathbb{N}^N$, the holomorphic function $Z \to q_\beta(\bar{H}(Z))$ is algebraic, which implies the same statement for $Z \to \rho'_\beta(H(Z))$. Pick now q in the algebraic closure of $\mathcal{K}(M')$. This means that one has a relation of the form

$$\sum_{j=0,\dots,r} \omega_j\Big((\rho'_{\alpha_i}(z', w'))_{i=1,\dots,p}\Big) q^j(z', w') \equiv 0,$$

where $p, r \geqslant 1$ and the ω_j are holomorphic polynomials of their arguments and $\omega_r((\rho_{\alpha_i})_i)$ is not identically zero. After substituing (z', w') by $H(z,w)$, we see that, by the transitivity of algebraicity, it suffices to show that $\omega_r((\rho_{\alpha_i}(H))_{i=1,\dots,p})$ is not identically zero. But this is clear since H is of generic maximal rank. The second part of (ii) of Theorem 1.1 follows easily.

We deal now with part (i). Using the fact that $\bar{Q}'$ is algebraic, we have a relation of the form

$$\sum_{j=0,\dots,r} \Omega_j(Z', W', \lambda)\bar{Q}'^j(Z', W', \lambda) \equiv 0,$$

where as usual the Ω_j are holomorphic polynomials with $r \geqslant 1$ and Ω_r is not identically zero. Hence, we get

$$\sum_{j=0,\dots,r} \sum_{|\alpha| \leqslant l} \Omega_j^\alpha(H(Z))\lambda^\alpha \mathcal{R}^j(Z, \lambda) \equiv 0,$$

for Z close to 0 in $\mathbb{C}^{N+1}$. Now, since the derivatives with respect to λ at 0 of $\mathcal{R}$ are precisely the $\rho'_\beta(H(Z))$, and since we know that these latter terms are algebraic (according to part (ii) proved before), one sees that a similar procedure as the one used in Lemma 4 gives the following[(2)]: For any given positive integer n, there exists a family $\left(x_{j,\alpha}^n(Z)\right)$ (with $j = 0, \dots, r$ and $|\alpha| \leqslant l$) of algebraic elements, which agree up to order n with $\Omega_j^\alpha(H(Z))$ and such that the following identity holds in a neighborhood of 0 (which depends on n):

$$\sum_{j=0,\dots,r} \sum_{|\alpha| \leqslant l} x_{j,\alpha}^n(Z)\lambda^\alpha \mathcal{R}^j(Z, \lambda) \equiv 0. \tag{17}$$

Suppose that for all n, $\sum_{|\alpha| \leqslant l} x_{r,\alpha}^n(Z)\lambda^\alpha \equiv 0$. Then, each $x_{r,\alpha}^n$ vanishes identically near 0. Hence, each $\Omega_r^\alpha(H(Z))$ vanishes at infinite order at 0, and hence is identically zero. This implies that $\Omega_r(H(Z), \lambda)$ vanishes identically. But here again, since $\mathrm{Jac}(H) \not\equiv 0$, by the choice of Ω_r, we reach a contradiction. Hence, for some n_0, we have a non-trivial algebraic relation for the reflection function. Thus, we are done.

For the proof of the corollary, according to [19], the assumption of holomorphic non-degeneracy on M (and hence on M' too) implies that the algebraic closure of $\mathcal{K}(M')$ is all $\mathcal{F}_{N+1}$. (This has been proved for not

(2) In fact, this is a bit more simpler in this setting.

necessarily the so-called normal coordinates.) This achieves the proof of the corollary. We conclude with some examples.

Example 1. — Consider the algebraic hypersurface in $\mathbb{C}^3$ given by $\Im m\, w = (\Re e\, w)|z_1 z_2|^6$. Theorem 1.1 means that for any (germ at 0 of a) holomorphic map $H = (f_1, f_2, g)$ of generic maximal rank fixing the latter hypersurface and the point 0, one has that g and $f_1 f_2$ are algebraic. This result is the optimal one that one can get. Indeed, for this holomorphically degenerate hypersurface, according to [7], there exists a biholomorphic map $H^0 = (f_1^0, f_2^0, g^0)$ near 0 fixing M which is not algebraic. One then gets that necessarily f_1^0 and f_2^0 are not algebraic (but $f_1^0 f_2^0$ is!).

Example 2. — Consider the algebraic hypersurface in $\mathbb{C}^3$ given by $\Im m\, w = z_1^2 + \bar{z}_1{}^2 + |z_2|^2$. Theorem 1.1 gives that any holomorphic mapping $H = (f_1, f_2, g)$ with $\mathrm{Jac}(H) \not\equiv 0$ fixing the latter manifold satisfies the following property: f_2 and $g - 2if_1^2$ are algebraic. Applying again the result of Baouendi and Rothschild cited above, we get a biholomorphism $H^0 = (f_1^0, f_2^0, g^0)$ fixing our hypersurface with the following property: f_1^0 and g^0 are not algebraic but f_2^0 and $g^0 - 2i(f_1^0)^2$ are!

Example 3. — Let M be the hypersurface in $\mathbb{C}^5$ given by $\Im m\, w = \sum_{j=1}^{3} |\prod_{i=1}^{j} z_i|^{2p_j}$, $p_j \in \mathbb{N}^*$. Note that the holomorphic vector field $\dfrac{\partial}{\partial z_4}$ is tangent to the latter hypersurface. Theorem 1.1 implies that any holomorphic map $H = (f_1, f_2, f_3, f_4, g)$ which sends M into itself with $\mathrm{Jac}(H) \not\equiv 0$ must have its components f_1, f_2, f_3, g algebraic. Here, again this will be the optimal information that one can get since, according to [7], there exists such a biholomorphism H^0 which is not algebraic. As a consequence, one knows that necessarily the component f_4^0 is the only component of H^0 which is not algebraic.

Remark. — Theorem 1.1 is obviously false in the Levi-flat case as it can be seen with the following simple example. Considerer the following local biholomorphism H near 0 in $\mathbb{C}^{N+1}$ given by $(z, w) \to (z, \exp(w) - 1)$. Note that H maps the real hyperplane $\Im m\, w = 0$ into itself. However, the transversal component is not algebraic.

Note added in proof. — A long time after this paper was completed, we knew about the paper of J. Merker, "On the Schwarz symmetry principle in three dimensional complex euclidean space", where similar considerations to part (i) of our Theorem 1.1 can be found in the real analytic category. Also, after this paper was written, F. Meylan and A.B. Sukhov informed us about their joint work with B. Coupet "Holomorphic maps of algebraic CR manifolds". We also received the preprint "Algebraicity of local holomorphisms between real algebraic submanifolds of complex spaces" from D. Zaitsev. Both papers deal with holomorphic maps of algebraic manifolds in complex spaces of (possibly) different dimension.

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Nordine MIR,
Université de Rouen
Département de Mathématiques
UPRES-A CNRS 6085
Site Colbert
76821 Mont Saint-Aignan Cedex (France).
mir@ams.univ-rouen.fr.