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PARAMETRIZATION OF LOCAL CR AUTOMORPHISMS BY FINITE JETS AND APPLICATIONS

BERNHARD LAMEL AND NORDINE MIR

1. Introduction

The goal of this work is to study the structure of the group of all local real-analytic CR automorphisms of a germ of a real-analytic hypersurface (M,p) in $\mathbb{C}^N$ or, equivalently, of all local biholomorphisms $H\colon(\mathbb{C}^N,p)\to(\mathbb{C}^N,p)$ mapping the germ (M,p) into itself; we also deal with the more general situation of a real-analytic CR manifold of arbitrary codimension (see §2). For every fixed point $p\in M$, we refer to this group of local biholomorphisms as the *stability group* of (M,p) and denote it by $\mathrm{Aut}(M,p)$. Endowed with the topology of uniform convergence on compact neighbourhoods of p, it becomes a topological group. This natural (inductive limit) topology can be described in the following way: a sequence of germs of biholomorphisms $(H_j)\subset \mathrm{Aut}(M,p)$ converges to another such germ $H\in\mathrm{Aut}(M,p)$ if there exists a neighbourhood U of p in $\mathbb{C}^N$ to which all mappings H_j extend holomorphically and such that $H_j\to H$ uniformly on U.

This group arises in a variety of different circumstances; one of the most important ones where it naturally appears is the *biholomorphic equivalence problem*, which consists in deciding whether two given germs of real-analytic hypersurfaces are biholomorphically equivalent. The biholomorphic equivalence problem has been studied since the beginning of the last century when Poincaré [37] observed, probably for the first time, that two real hypersurfaces in $\mathbb{C}^N$ for $N\geq 2$ are not necessarily locally equivalent via a holomorphic transformation (though they are via a $\mathcal{C}^\infty$ transformation). Today, the biholomorphic equivalence problem for strongly pseudoconvex (or Levi-nondegenerate) hypersurfaces is pretty well understood, based on the works of E. Cartan [14, 15] in dimension two, and the results of Tanaka [42] and Chern–Moser [17] in arbitrary dimension.

Thanks to the works initiated in the 1970s, a number of remarkable properties of the stability group of strongly pseudoconvex real-analytic hypersurfaces have been discovered (see e.g. the surveys by Vitushkin [45], Huang [30] and Baouendi, Ebenfelt and Rothschild [5] for complete discussions on this matter). These properties depend on whether the germ (M,p) can be biholomorphically mapped onto a piece of the unit sphere; if there exists such a biholomorphism, one says that (M,p) is *spherical*. For instance, if (M,p) is not spherical the group $\mathrm{Aut}(M,p)$ is

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compact (see Vitushkin [44]); every element of such a group is uniquely determined by its derivative at p and also extends holomorphically to a fixed neighbourhood (independently of the automorphism) of p in $\mathbb{C}^N$. On the other hand, if (M,p) is spherical the group $\mathrm{Aut}(M,p)$ is not compact and every local CR automorphism of (M,p) is uniquely determined by its 2-jet (but not 1-jet) at p. Beyond their interest in their own right, these local results also have direct applications to global biholomorphic mappings of strongly pseudoconvex bounded domains with smooth real-analytic boundaries by the classical reflection principle [23, 35, 36].

There are two main approaches to deriving results about the stability group of a strongly pseudoconvex (or Levi-nondegenerate) real-analytic hypersurface in $\mathbb{C}^N$. One consists in using the powerful Chern–Moser theory (see for example the results of Burns and Shnider [12] and the survey by Vitushkin [45]). The other one, initiated by Webster [46], uses the invariant family of Segre varieties attached to any real-analytic hypersurface in a complex manifold.

While it seems difficult to extend the first approach to understand the structure of the stability group of Levi-degenerate hypersurfaces (see e.g. Ebenfelt [21]), the second one can be carried over to that setting, as first observed by Baouendi, Ebenfelt and Rothschild [2]. In fact, it was shown in [2] that many interesting properties of the stability group of a germ of a real-analytic hypersurface (M,p) may be obtained after showing that its local CR automorphisms are parametrized (in a suitable sense) by their jets at p of some finite order. Such a program was successfully carried out in [2, 47, 3] for some classes of real-analytic Levi-degenerate hypersurfaces and even CR manifolds of arbitrary codimension. However, the degeneracies allowed for the hypersurface M in these results are quite restrictive and in particular fail to hold at every point of real-analytic hypersurfaces of $\mathbb{C}^N$ containing no complex-analytic subvariety of positive dimension. Such a class of real hypersurfaces is of particular interest since, by a well-known result of Diederich and Fornæss [19], boundaries of bounded domains with smooth real-analytic boundary are of this type.

In dimension $N=2$, the stability group of such real hypersurfaces has recently been studied by Ebenfelt, Zaitsev and the first author in [22], where it was shown that their local CR automorphisms are analytically parametrized by their 2-jets. In this paper, among other results, we show that such a parametrization property by k-jets holds at *every* point of arbitrary real-analytic hypersurfaces containing no complex-analytic subvariety of positive dimension and for some k depending on the point (see Theorem 1.1).

In what follows, for every integer k and every pair of points $p,p'\in\mathbb{C}^N$, we denote by $G^k_{p,p'}(\mathbb{C}^N)$ the group of all k-jets at p of local biholomorphisms $H\colon(\mathbb{C}^N,p)\to(\mathbb{C}^N,p')$. Given coordinates $Z=(Z_1,\ldots,Z_N)$ in $\mathbb{C}^N$ we will usually identify the k-jet j^k_pH of a local biholomorphism H at p with

$$j^k_pH=\left(\frac{\partial^{|\alpha|}H}{\partial Z^\alpha}(p)\right)_{|\alpha|\le k}.$$

If, in addition, M,M' are real-analytic hypersurfaces of $\mathbb{C}^N$ or more generally real-analytic generic submanifolds of the same dimension with $p\in M$ and $p'\in M'$, we denote by $\mathcal{F}(M,p;M',p')$ the set of all germs at p of local biholomorphisms $H\colon(\mathbb{C}^N,p)\to(\mathbb{C}^N,p')$ sending M into M'.

The purpose of this paper is to provide general conditions on a real-analytic generic submanifold $M \subset \mathbb{C}^N$ so that for every $p \in M$ and every M' and p' as above, the set of mappings $\mathcal{F}(M,p;M',p')$ is analytically parametrized by a finite jet at p. Our results are best illustrated in the hypersurface case by the following theorem, which was the main motivation of the present work.

Theorem 1.1. *Let M be a real-analytic hypersurface in $\mathbb{C}^N$ containing no complex-analytic subvariety of positive dimension. Then for every real-analytic hypersurface $M' \subset \mathbb{C}^N$, every $p' \in M'$ and every $p \in M$, there exist an integer ℓ_p (depending only on M and p), an open subset $\Omega \subset \mathbb{C}^N \times G_{p,p'}^{\ell_p}(\mathbb{C}^N)$ and a real-analytic map $\Psi(Z,\Lambda)\colon \Omega \to \mathbb{C}^N$, holomorphic in the first factor, such that for any $H \in \mathcal{F}(M,p;M',p')$, the point $(p, j_p^{\ell_p} H)$ belongs to Ω and the following identity holds:*

$$H(Z) = \Psi(Z, j_p^{\ell_p} H) \text{ for all } Z \in \mathbb{C}^N \text{ near } p.$$

Furthermore, ℓ_p can be chosen so that it depends upper-semicontinuously on p.

Our first application of Theorem 1.1 concerns the group $\mathrm{Aut}(M,p)$. By using standard arguments from [3], we obtain a Lie group structure on $\mathrm{Aut}(M,p)$ compatible with its topology.

Theorem 1.2. *Let M be a real-analytic hypersurface in $\mathbb{C}^N$ containing no complex-analytic subvariety of positive dimension. Then for every $p \in M$, there exists an integer ℓ_p, depending upper-semicontinuously on p, such that the jet mapping*

$$j_p^{\ell_p}\colon \mathrm{Aut}(M,p) \to G_{p,p}^{\ell_p}(\mathbb{C}^N) \tag{1.1}$$

is a continuous group homomorphism that is a homeomorphism onto a real-algebraic Lie subgroup of $G_{p,p}^{\ell_p}(\mathbb{C}^N)$.

We should mention that the real-algebraicity of the Lie group $\mathrm{Aut}(M,p)$ in Theorem 1.2 does not follow directly from Theorem 1.1 but from the more precise version given by Theorem 2.1 in §2 below. Another noteworthy consequence of Theorem 1.2 is the finite jet determination of local CR automorphisms of (M,p) by their ℓ_p-jet at p. As stated, this finite jet determination property would appear as already known since the existence of some integer k for which the finite determination holds follows from the work of Baouendi, Ebenfelt and Rothschild [4]. However the proof given in [4] does not give any information on the dependence of the jet order required with respect to a given point of the hypersurface. The existence of an integer ℓ_p depending *upper-semicontinuously* on p for which the mapping given by (1.1) is injective is completely new and crucial in order to get the following.

Theorem 1.3. *Let M be a compact real-analytic hypersurface in $\mathbb{C}^N$. Then there is an integer k, depending only on M, such that for every $p \in M$ and for every real-analytic hypersurface $M' \subset \mathbb{C}^N$, smooth local CR diffeomorphisms mapping a neighbourhood of p in M into M' are uniquely determined by their k-jets at p.*

Theorem 1.3 follows from the upper-semicontinuity of the map $p \mapsto \ell_p$ in Theorem 1.2, the fact that compact real-analytic hypersurfaces do not contain any holomorphic curves [19] and from the regularity result for smooth CR diffeomorphisms proved in [7]. For proper holomorphic mappings of bounded domains with smooth real-analytic boundaries, we also obtain the following rigidity result as a consequence of Theorem 1.3.

Corollary 1.4. *Let $\Omega \subset \mathbb{C}^N$ be a bounded domain with smooth real-analytic boundary. Then there exists an integer k, depending only on the boundary $\partial\Omega$, such that if $H\colon \Omega \to \Omega$ is a proper holomorphic mapping extending smoothly up to $\partial\Omega$ near some point $p \in \partial\Omega$ which satisfies $H(z) = z + o(|z-p|^k)$ when $z \in \Omega$ and $z \to p$, then H is the identity mapping.*

Corollary 1.4 can be viewed as a boundary version of H. Cartan's uniqueness theorem [16] and seems to be new even in the case of weakly pseudoconvex domains (see e.g. [13, 29, 28]). Note that the smooth extension up to the boundary assumption is known to hold automatically near every point $p \in \partial\Omega$ in many cases, e.g., if Ω is weakly pseudoconvex (see [8] for more on that matter). For bounded strongly pseudoconvex domains with $\mathcal{C}^\infty$ boundary, more precise results are known from the work of Burns and Krantz [13] (see also Huang [29, 28]).

2. Results for CR manifolds of higher codimension and main tools

We state here our results for real-analytic generic submanifolds of higher codimension from which all theorems mentioned in the introduction will be derived. Recall that a CR submanifold M of $\mathbb{C}^N$ is called *generic* if $T_pM + J(T_pM) = T_p\mathbb{C}^N$ where J is the complex structure map of $\mathbb{C}^N$ and T_pM (resp. $T_p\mathbb{C}^N$) denotes the tangent space of M (resp. of $\mathbb{C}^N$) at p (see e.g. [10, 8]).

In order to state our results, we need to impose two nondegeneracy conditions on a given real-analytic generic submanifold $M \subset \mathbb{C}^N$. We shall assume that M is *essentially finite* in the sense of [7] and *minimal* in the sense of [43] at each of its points (see §7.1 for precise definitions). Such conditions are very natural and hold in general situations for compact as well as noncompact real-analytic CR submanifolds in complex space. For instance, if M does not contain any complex-analytic subvariety of positive dimension, then M is necessarily essentially finite and also automatically minimal when M is furthermore of (real) codimension one (see e.g. [8]).

Our first result in this section is the following.

Theorem 2.1. *Let M be a real-analytic generic submanifold of $\mathbb{C}^N$ that is essentially finite and minimal at each of its points. Then for every real-analytic generic submanifold $M' \subset \mathbb{C}^N$, every point $p' \in M'$ and every $p \in M$, there exists an integer ℓ_p (depending only on M and on p), an open subset $\Omega \subset \mathbb{C}^N \times G^{\ell_p}_{p,p'}(\mathbb{C}^N)$ and a real-analytic map $\Psi(Z,\Lambda)\colon \Omega \to \mathbb{C}^N$ holomorphic in the first factor, such that the following hold :*

(i) *for any $H \in \mathcal{F}(M,p;M',p')$ the point $(p, j_p^{\ell_p}H)$ belongs to Ω and the following identity holds:*
$$H(Z) = \Psi(Z, j_p^{\ell_p}H) \text{ for all } Z \in \mathbb{C}^N \text{ near } p;$$
(ii) *the map $M \ni p \mapsto \ell_p \in \mathbb{N}$ is upper-semicontinuous;*
(iii) *the map Ψ has the following formal Taylor expansion:*

$$\Psi(Z,\Lambda) = \sum_{\alpha\in\mathbb{N}^N} \frac{P_\alpha(\Lambda,\overline{\Lambda})}{(D(\Lambda^1))^{s_\alpha}(\overline{D(\Lambda^1)})^{r_\alpha}}\,(Z-p)^\alpha, \tag{2.1}$$

where for every $\alpha \in \mathbb{N}^N$, s_α and r_α are nonnegative integers, P_α and D are polynomials in their arguments, Λ^1 denotes the linear part of the jet Λ and $D(j_0^1H) \neq 0$ for every $H \in \mathcal{F}(M,p;M',p')$.

To our knowledge, Theorem 2.1 is already new even in the case of real hypersurfaces in $\mathbb{C}^N$. (For $N = 2$, Theorem 2.1 (i) with $\ell_p = 2$ is one of the main results of [22].) On the other hand, for CR manifolds of codimension greater than one, only special cases of Theorem 2.1 (including the Levi-nondegenerate case) were known as consequences of the works in [47, 3].

As in §1, we also derive from Theorem 2.1 a Lie group structure on the stability group of any essentially finite minimal real-analytic generic submanifold of $\mathbb{C}^N$.

Theorem 2.2. *Let M be a real-analytic generic submanifold of $\mathbb{C}^N$ that is essentially finite and minimal at each of its points. Then for every $p \in M$ there exists an integer ℓ_p, depending upper-semicontinuously on p, such that the jet mapping*

$$j_p^{\ell_p} \colon \mathrm{Aut}(M,p) \to G_{p,p}^{\ell_p}(\mathbb{C}^N) \tag{2.2}$$

is a continuous group homomorphism that is a homeomorphism onto a real-algebraic Lie subgroup of $G_{p,p}^{\ell_p}(\mathbb{C}^N)$.

Even in the case of real hypersurfaces $M \subset \mathbb{C}^N$ which do not contain any complex-analytic subvariety of positive dimension with $N \geq 3$, the fact that $\mathrm{Aut}(M,p)$ is a Lie group was an open problem. Note also that the fact that the image of $\mathrm{Aut}(M,p)$ under the jet mapping $j_p^{\ell_p}$ is a *real-algebraic* subgroup of $G_{p,p}^{\ell_p}(\mathbb{C}^N)$ follows from the rational dependence of the parametrization Ψ with respect to the jet Λ in Theorem 2.1 (iii) (see §11), which is new even in the case $N = 2$. For submanifolds of higher codimension only special cases of Theorem 2.2 were known from the works [47, 3]. We should point out that even in these special cases the jet order ℓ_p given by our Theorem 2.2 can be computed explicitly (from integer-valued biholomorphic invariants of (M,p)) and is always smaller than the jet order required in the papers [47, 3]. For details on that matter we refer the reader to §§7.2–7.3. We would also like to recall that the statement concerning the injectivity of the mapping given by (2.2) is also new by itself since a choice of an upper-semicontinuous bound on the jet order required to get injectivity does not follow from the work [4]. The following finite jet determination result for compact real-analytic CR manifolds can be viewed as a higher-codimensional version of Theorem 1.3 and is a direct consequence of the possibility of an upper-semicontinuous choice of ℓ_p.

Theorem 2.3. *Let M be a compact real-analytic CR submanifold of $\mathbb{C}^N$ minimal at each of its points. Then there is an integer k, depending only on M, such that for every $p \in M$ and for every real-analytic CR submanifold $M' \subset \mathbb{C}^N$ with the same CR dimension as that of M, smooth local CR diffeomorphisms mapping a neighbourhood of p in M into M' are uniquely determined by their k-jets at p.*

Let us explain now in some greater detail the main new difficulties faced in this paper compared to previous related work. The first main ingredient of the proof of Theorem 2.1 relies on the full machinery of the Segre sets technique introduced by Baouendi, Ebenfelt and Rothschild [8, 3]. In order to use such a tool one needs to show that every local biholomorphism fixing a germ of a submanifold (M,p) (satisfying the conditions of the theorem) as well as all its jets, when restricted to any Segre set attached to p, is analytically parametrized (in a suitable manner) by its jet at p of some finite precise order. However, in order to realize such a program, one immediately runs into the problem of inverting certain families of parametrized maps with singularities in multidimensional complex space, a phenomenon that does not appear in all previous related work. Here the existence of such singularities comes directly from the high Levi-degeneracy allowed for the submanifold M in Theorem 2.1. To overcome this difficulty, we will be led to consider precise types of linear and nonlinear singular analytic systems of equations near the origin in $\mathbb{C}^n$. The most difficult types of singular systems we end up with are of the form $A(w(z)) = b(z)$, where $A, b\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ are germs of holomorphic maps with $A = A(w)$ of generic rank n and one wants to solve such a system for a biholomorphism $w\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$; the solution(s) should, for our application, preserve the dependence of b on arbitrary deformations, that is, any continuous (resp. smooth, resp. holomorphic) perturbation of the term b should lead to a solution $w = w(z)$ that behaves in the same manner. Unfortunately, there is, to our knowledge, no general theory to deal with this situation, so we have to develop our own machinery to parametrize the invertible solutions of such singular systems.

Since our solution to the above problem may be of independent interest, we introduce some notation which allows us to present it in a particularly simple manner. It is useful here to work in the context of holomorphic maps of infinite-dimensional complex vector spaces. Let $\mathcal{H}(\mathbb{C}^n, 0)$ be the space of germs of holomorphic maps $H : (\mathbb{C}^n, 0) \to \mathbb{C}^n$, $n \geq 1$, endowed with the topology of uniform convergence on compact neighbourhoods of the origin; here again, a sequence of germs of holomorphic maps (H_j) converges to another such germ H if there exists a neighbourhood W of the origin in $\mathbb{C}^N$ to which all H_j extend holomorphically and such that $H_j \to H$ uniformly on W. With such a topology, the space $\mathcal{H}(\mathbb{C}^n, 0)$ carries the structure of an infinite-dimensional topological complex vector space being an inductive limit of Fréchet spaces (see §5.1 for more details). Let $\mathcal{H}(\mathbb{C}^n, 0)^0$ be the subspace of elements of $\mathcal{H}(\mathbb{C}^n, 0)$ preserving the origin and $\mathcal{B}^n$ be the open subset of $\mathcal{H}(\mathbb{C}^n, 0)^0$ consisting of the germs of biholomorphic maps (preserving the origin).

Given any $A \in \mathcal{H}(\mathbb{C}^n, 0)$, composition with A naturally yields a (holomorphic) map $A_* : \mathcal{B}^n \to \mathcal{H}(\mathbb{C}^n, 0)$ given by $A_*(u) := A \circ u$ for $u \in \mathcal{B}^n$. Our above-mentioned problem calls for the construction of a holomorphic "left inverse" to A_*, that is, a holomorphic map $\Psi\colon \mathcal{H}(\mathbb{C}^n, 0) \to \mathcal{B}^n$ satisfying $\Psi(A_* u) = u$. Here holomorphy of a map $f\colon X \supset \Omega \to Y$ between locally convex topological complex vector spaces X, Y with Ω open in X is understood in the sense of [20, Definition 3.16]: f is holomorphic if it is continuous and its composition with any continuous linear functional on Y is holomorphic along finite-dimensional affine subspaces of X intersected with Ω.

Since the mapping A_* is obviously not injective in general, there is of course no hope to find such a left inverse. However, it was recently shown by the first author in [34] that the holomorphic map $A_* \times j_0^1\colon \mathcal{B}^n \ni u \mapsto (A_*(u), u'(0)) \in \mathcal{H}(\mathbb{C}^n, 0) \times \mathsf{GL_n}(\mathbb{C})$ is injective *when A is of generic rank n.* (As usual, $\mathsf{GL_n}(\mathbb{C})$ denotes the group of invertible $n \times n$ matrices and $j_0^1 u = u'(0)$ the differential or 1-jet of u at the origin.) It thus makes sense to try to find a left inverse to that latter map. Our solution provides such an inverse that we also call parametrization.

Theorem 2.4. *Let $A : (\mathbb{C}^n, 0) \to \mathbb{C}^n$ be a germ of a holomorphic map of generic rank n. Then there exists a holomorphic map $\Psi\colon \mathcal{H}(\mathbb{C}^n, 0) \times \mathsf{GL_n}(\mathbb{C}) \to \mathcal{B}^n$ such that Ψ satisfies $\Psi(A_* u, u'(0)) = u$ for all $u \in \mathcal{B}^n$; furthermore, Ψ can be chosen so that it satisfies $j_0^1 (\Psi(b, \lambda)) = \lambda$ for every $b \in \mathcal{H}(\mathbb{C}^n, 0)$ and $\lambda \in \mathsf{GL_n}(\mathbb{C})$.*

This result provides us the necessary information about the structure of the invertible solutions to the equation $A(w) = b(z)$ and allows us to proceed with the construction of the parametrization of the automorphism group of CR manifolds. The fact that we are even able to parametrize the solutions of the above type of equations precisely by their 1-jets is one of the main reasons why we succeed in getting an upper-semicontinuous dependence of the integer ℓ_p on p in Theorem 2.1. Another result in the spirit of Theorem 2.4, for a certain kind of linear singular system that we also need is Proposition 6.3, which provides the parametrization of this type of equations. In both cases, we need a very careful construction of such a parametrization (which in the case of Theorem 2.4 is provided by the more precise Theorem 3.1) in order to prove statement (iii) of Theorem 2.1.

The paper is organized as follows. We start by providing the necessary background on parametrization of nonlinear singular systems in §3; the proof of the precise results in that section and the proof of Theorem 2.4 are given in §4–§5 (following the development of some necessary material on homogeneous polynomials in §4.1); we also include an additional application of Theorem 2.4 in §5.2. In §6, we treat the necessary case of a different (and simpler) type of singular systems, namely linear ones, following the approach in the nonlinear case.

After that, we proceed with the construction of the parametrization of the automorphism group of CR manifolds. In §7 we introduce a class of real-analytic generic submanifolds which is more general than the class of essentially finite ones. After discussing a few properties of this class of submanifolds and giving several examples, we state the most general parametrization theorem of this paper (Theorem 7.3) which holds for the stability group of any submanifold of the above-mentioned class. The proof of Theorem 7.3 is furnished in §8–§10 and is completed in §11, where we also prove all the remaining theorems stated in §1–§2.

Remark 2.5 (Jet space notation). Jet spaces of various kinds will be encountered frequently in this paper. For a complete discussion, we refer the interested reader to e.g. [25] or [11]. Here we will use the following notation: for positive integers n, k, l, we denote by $J_0^l(\mathbb{C}^n, \mathbb{C}^k)$ the jet space at the origin of order l of holomorphic mappings from $\mathbb{C}^n$ to $\mathbb{C}^k$. For a germ of a holomorphic map $h\colon (\mathbb{C}^n, 0) \to \mathbb{C}^k$, we denote by $j_0^l h$ the l-jet of h at 0. If $k = n$, we simply write $J_0^l(\mathbb{C}^n)$ for $J_0^l(\mathbb{C}^n, \mathbb{C}^n)$. The jet space of holomorphic mappings $(\mathbb{C}^n, 0) \to (\mathbb{C}^k, 0)$ will be denoted by $J_{0,0}^l(\mathbb{C}^n, \mathbb{C}^k)$, or simply $J_{0,0}^l(\mathbb{C}^n)$ if $n = k$. We have already introduced $G_{0,0}^l(\mathbb{C}^n)$ above as our notation for the jet group of order l of biholomorphic mappings $(\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$.

3. Parametrizing invertible solutions of nonlinear singular analytic equations

We will derive Theorem 2.4 in §5 from the more precise result given in Theorem 3.1 below. For every $\mathbb{C}^k$-valued (homogeneous polynomial) map $f\colon \mathbb{C}^n \to \mathbb{C}^k$, and for any bounded set $F \subset \mathbb{C}^n$, we define

$$\|f\|_F := \max_{1 \le j \le k} \sup_{z \in F} |f^j(z)|, \quad f = (f^1, \ldots, f^k).$$

Theorem 3.1. *Let $A : (\mathbb{C}^n, 0) \to \mathbb{C}^n$ be a germ of a holomorphic map of generic rank n. Then there exists an integer ℓ_0 and for every integer $\ell > 0$ polynomial mappings $p_\ell : \mathbb{C}^n \times \mathsf{GL_n}(\mathbb{C}) \times J_0^{\ell_0+\ell}(\mathbb{C}^n) \to \mathbb{C}^n$, homogeneous of degree ℓ in their first variable, and an integer $\kappa(\ell)$ such that for every germ of a holomorphic map $b\colon (\mathbb{C}^n, 0) \to \mathbb{C}^n$, if $u : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ is a germ of a biholomorphism satisfying*

$A(u(z)) = b(z)$, then necessarily

$$u_\ell(z) = \frac{1}{(\det u'(0))^{\kappa(\ell)}}\, p_\ell(z, u'(0), j_0^{\ell_0+\ell} b), \tag{3.1}$$

where $u(z) = \sum_{\ell \geq 1} u_\ell(z)$ is the decomposition of u into homogeneous $\mathbb{C}^n$-valued polynomial terms. Furthermore, the polynomial maps p_ℓ and the integers $\kappa(\ell)$ can be chosen with the following two properties:

(i) *for every germ of a holomorphic map $b\colon (\mathbb{C}^n, 0) \to \mathbb{C}^n$ and for every $\lambda \in \mathsf{GL_n}(\mathbb{C})$,*

$$p_1(z, \lambda, j_0^{\ell_0+1} b) = \lambda z, \ \ \textit{and}\ \kappa(1) = 0; \tag{3.2}$$

(ii) *there exists a polydisc $\Delta \subset \mathbb{C}^n$ centered at the origin such that for every $C > 0$ and every compact subset $L \subset \mathsf{GL_n}(\mathbb{C})$ there exists $K > 0$ such that for every $\lambda \in L$ and every germ of a holomorphic map $b\colon (\mathbb{C}^n, 0) \to \mathbb{C}^n$ written into homogeneous terms $b(z) = \sum_{k \geq 0} b_k(z)$ satisfying $\|b_k\|_\Delta \leq C^k$ for all $k \in \mathbb{N}$, we have for all $\ell \geq 1$,*

$$\left\| \frac{p_\ell(\cdot, \lambda, j_0^{\ell_0+\ell} b)}{(\det \lambda)^{\kappa(\ell)}} \right\|_\Delta \leq K^\ell. \tag{3.3}$$

As explained in §5, to every choice of polynomial mappings (p_ℓ) and integers $(\kappa(\ell))$ satisfying the conclusion of Theorem 3.1, the map

$$\Psi(b, \lambda) := \sum_{\ell \geq 1} \frac{p_\ell(\cdot, \lambda, j_0^{\ell_0+\ell} b)}{(\det \lambda)^{\kappa(\ell)}}, \quad b \in \mathcal{H}(\mathbb{C}^n, 0), \quad \lambda \in \mathsf{GL_n}(\mathbb{C}), \tag{3.4}$$

will be a holomorphic parametrization satisfying the conclusions of Theorem 2.4.

Let us add that it follows from the proof of Theorem 3.1 that we can choose $\kappa(\ell) = \ell_0 \ell$. However, the exact value of κ is not important for our purposes. If in Theorem 3.1 we just consider formal maps A instead of holomorphic ones the above theorem remains true in the formal category without the estimates in (ii). In fact, we have a formal counterpart of the above theorem as follows. Consider a formal power series mapping A as in Theorem 3.1 (not necessarily convergent) and the sets $\mathcal{H}(\mathbb{C}^n, 0)_f$ and $\mathcal{B}_f^n$ of $\mathbb{C}^n$-valued formal holomorphic maps and formal biholomorphic maps preserving the origin respectively. For A as above, let (p_ℓ) and $(\kappa(\ell))$ satisfy the conclusion of Theorem 3.1 (i). Then for every $\lambda \in \mathsf{GL_n}(\mathbb{C})$ and every $b \in \mathcal{H}(\mathbb{C}^n, 0)_f$, formula (3.4) defines a formal power series mapping that belongs to $\mathcal{B}_f^n$ and satisfies $\Psi(A_* u, u'(0)) = u$ for $u \in \mathcal{B}_f^n$. If we consider b's which are elements of $\mathcal{R}[[z]]$ for some algebra $\mathcal{R}$ over $\mathbb{C}$, then we get that $\Psi(b, \lambda) \in \mathcal{R}\,(\lambda)\,[[z]]$. This means that the obtained parametrization Ψ also preserves the dependence on parameters in a formal way.

As mentioned before, the obtained parametrization in Theorem 2.4 or Theorem 3.1 allows us to consider arbitrary perturbations on the right-hand side of the singular system. For the purposes of this paper, the dependence on holomorphic parameters given in Corollary 3.2 below is crucial: the precise result can be derived from Theorem 3.1 and Lemma 5.1, but we note that Corollary 3.2 (i) also follows directly from Theorem 2.4.

Corollary 3.2. *Let $A : (\mathbb{C}^n, 0) \to \mathbb{C}^n$ be a germ of a holomorphic map of generic rank n, X a complex manifold, and $b = b(z, \omega)$ be a $\mathbb{C}^n$-valued holomorphic map defined on an open neighbourhood of $\{0\} \times X \subset \mathbb{C}^n \times X$. Then there exists a* *holomorphic map* $\Gamma = \Gamma(z,\lambda,\omega)\colon \mathbb{C}^n \times \mathsf{GL_n}(\mathbb{C}) \times X \to \mathbb{C}^n$*, defined on an open neighbourhood* Ω *of* $\{0\} \times \mathsf{GL_n}(\mathbb{C}) \times X$*, such that if* $u : (\mathbb{C}^n,0) \to (\mathbb{C}^n,0)$ *is a germ of a biholomorphism satisfying* $A(u(z)) = b(z,\omega_0)$ *for some* $\omega_0 \in X$*, then necessarily* $u(z) = \Gamma(z,u'(0),\omega_0)$*. Furthermore, the map* Γ *can be chosen with the following properties :*

(i) *for every* $(\lambda,\omega) \in \mathsf{GL_n}(\mathbb{C}) \times X$*,* $\Gamma(0,\lambda,\omega) = 0$ *and* $\Gamma_z(0,\lambda,\omega) = \lambda$*;*
(ii) *if we write the Taylor expansions*

$$\Gamma(z,\lambda,\omega) = \sum_{\alpha\in\mathbb{N}^n} \Gamma_\alpha(\lambda,\omega) z^\alpha, \quad b(z,\omega) = \sum_{\alpha\in\mathbb{N}^n} b_\alpha(\omega) z^\alpha,$$

then there exists an integer ℓ_0 *and for every* $\alpha \in \mathbb{N}^n$ *nonnegative integers* k_α *and polynomial mappings* $p_\alpha\colon \mathsf{GL_n}(\mathbb{C}) \times J_0^{\ell_0+|\alpha|}(\mathbb{C}^n) \to \mathbb{C}^n$ *such that*

$$\Gamma_\alpha(\lambda,\omega) = \frac{p_\alpha(\lambda,(b_\beta(\omega))_{|\beta|\le \ell_0+|\alpha|})}{(\det\lambda)^{k_\alpha}}. \tag{3.5}$$

An outline of the proof of Theorem 3.1 is as follows. First (in Lemma 4.5), we show that we may assume that the lowest-order homogeneous terms of A are generically independent. Such a reduction then allows us to linearize our initial system by considering homogeneous expansions of the power series involved in the system. By using elementary linear algebra on vector spaces of homogeneous polynomials, one may then conclude the formal part of the proof. However, for the last and more tedious part of the proof, namely the convergence part (given by Theorem 3.1 (ii)), a more careful analysis of the formal part is required in order to get the needed estimates, which are obtained by using a majorant method.

Let us again summarize the organization of the material in the following sections. In §4.1, we introduce the notation and collect all the necessary facts about homogeneous polynomials needed for the proof of Theorem 3.1; a particular normalization of the equations is discussed in §4.2 and the proof of Theorem 3.1 is completed in §4.3. In §5, we show how to derive Theorem 2.4 from Theorem 3.1. §5.2 offers another application of the above results, which can be seen as the continuous version of Corollary 3.2. In §6, we illustrate how the method of proof of Theorem 3.1 can be used to give a precise result, which we also need, for the parametrization of solutions of other types of singular systems (which are simpler, because they are linear). This section actually serves as a guide to our proof of Theorem 3.1, the steps being followed in precisely the same manner, but the technicalities involved are much easier to deal with. As a consequence, the reader may prefer to start with that part before going through the details of the proof of Theorem 3.1.

4. Proof of Theorem 3.1

4.1. **Preliminary results about homogeneous polynomials.** Let $\mathcal{P}_{n,d}$ be the vector space of homogeneous polynomials of degree $d \ge 0$ in $z = (z_1,\ldots,z_n) \in \mathbb{C}^n$, $n \ge 1$. If n is fixed, we are dropping the n and simply write $\mathcal{P}_d$. Given any positive integer m, note that the space $\mathcal{P}_d^m$ can be identified with the space of all homogeneous $\mathbb{C}^m$-valued polynomial maps of degree d. Given any bounded open set $\Omega \subset \mathbb{C}^n$ containing the origin, we are going to consider the following norms on the vector space $\mathcal{P}_d$:

$$\|f\|_1 := \sum_{|\alpha|=k} |f_\alpha|\,,\ \text{where } f(z) = \sum_{|\alpha|=d} f_\alpha z^\alpha,$$
$$\|f\|_\Omega := \sup_{z\in\Omega} |f(z)|\,.$$

Given $(k_1, \ldots, k_m) \in \mathbb{N}^m$, we consider the following norms on the cartesian product $\mathcal{P}_{k_1} \times \ldots \times \mathcal{P}_{k_m}$:

$$\|f\|_1 := \max_{j=1,\ldots,m} \left\|f^j\right\|_1, \quad f = (f^1, \ldots, f^m), f^j \in \mathcal{P}_{k_j},$$
$$\|f\|_\Omega := \max_{j=1,\ldots,m} \left\|f^j\right\|_\Omega, \quad f = (f^1, \ldots, f^m), f^j \in \mathcal{P}_{k_j}.$$

Let us first note the following fact:

Lemma 4.1. *Let $M = M(z)$ be an $m\times m$ matrix with holomorphic coefficients near the origin in $\mathbb{C}^n$, $z = (z^1, \ldots, z^n)$. Assume that $\det M \not\equiv 0$ (in any neighbourhood of 0) and that for every $j = 1, \ldots, m$, the j-th row of M is formed by homogeneous polynomials of degree $k_j \geq 0$. Then there exists a bounded open set $\Omega \subset \mathbb{C}^n$ containing the origin and a constant $C > 0$, only depending on M, such that for every $d \in \mathbb{N}$ and for all $k_1, \ldots, k_m$ there exists a linear operator $B : \mathcal{P}_{k_1+d} \times \ldots \times \mathcal{P}_{k_m+d} \to \mathcal{P}_d^m$ which is inverse to matrix multiplication by M of norm at most C; that is, B satisfies*

(4.1)
$\|Bs\|_\Omega \leq C\,\|s\|_\Omega\,,$ *for all* $s \in \mathcal{P}_{k_1+d}\times\ldots\times\mathcal{P}_{k_m+d}$, *and* $B(M\cdot p) = p$ *for all* $p \in \mathcal{P}_d^m$.

After a linear change of coordinates (normalizing $\det M$ in the z^n-direction), we may actually assume that Ω is a polydisc.

Proof. Let $\widehat{M}(z)$ denote the transpose of the comatrix of $M(z)$, i.e., the matrix satisfying

$$M(z) \cdot \widehat{M}(z) = \widehat{M}(z) \cdot M(z) = (\det M(z))\, I_m,$$

where I_m is the identity matrix. In particular, for any $d \in \mathbb{N}$ and for every $p \in \mathcal{P}_d^m$ we have

$$(\det M(z))p(z) = \widehat{M}(z) \cdot M(z) \cdot p(z).$$

Since $\det M \not\equiv 0$, the Weierstrass division theorem (see e.g. [27]) implies that we can find a linear change of coordinates (making the function $\det M$ z^n-regular of order $r \geq 1$) and, in the new coordinates, a polydisc $\Delta \subset \mathbb{C}^n$ centered at the origin and a constant $\tilde{C}$ (depending only on M) such that for any holomorphic function f holomorphic in a neighbourhood of $\overline{\Delta}$,

$$f = q \det M + h,$$

where h is a unique polynomial in z^n of degree $< k$, and q (the quotient) is a unique holomorphic function in Δ that satisfies $\|q\|_\Delta \leq \tilde{C}\|f\|_\Delta$. We define the map B by associating to $s \in \mathcal{P}_{k_1+d} \times \ldots \times \mathcal{P}_{k_m+d}$ the quotient given by Weierstrass division of $\widehat{M}(z) \cdot s(z)$ by $\det M$. It is easily checked that this is a homogeneous polynomial map of degree d, and it satisfies $\|Bs\|_\Delta \leq \tilde{C}\left\|\widehat{M}s\right\|_\Delta \leq C\|s\|_\Delta$, for some constant $C > 0$. This proves (4.1) on the open set Ω which corresponds to the polydisc in the original coordinates. $\square$

In what follows we will make extensive use of Faa di Bruno's formula, which will only be needed for formal power series (so that we actually only use the multinomial formula; we will still refer to it as Faa di Bruno's formula). We give a statement in this case (see e.g. [33]):

Proposition 4.2. *Assume that $f(t) = \sum_r a_r t^r$, $g(x) = \sum_q b_q x^q$, and $h(t) = \sum_s c_s t^s$ are three complex-valued formal power series in one variable with $c_0 = 0$ and $f(t) = g(h(t))$. Then for every positive integer r, one has*

$$a_r = \sum \frac{q!\, b_q}{k_1! \dots k_r!} c_1^{k_1} \dots c_r^{k_r}, \tag{4.2}$$

where $q = \sum_{1 \le j \le r} k_j$ and the sum goes over all nonnegative integers $k_1, \dots, k_r$ such that

$$k_1 + 2k_2 + \dots + rk_r = r.$$

We now turn to some estimates for homogeneous parts of compositions of certain types of formal power series mappings. We use the following notation for decomposing a formal power series mapping $u = (u^1, \dots, u^n) : (\mathbb{C}^N, 0) \to (\mathbb{C}^n, 0)$, $N \ge 1$, in Taylor series or into polynomial homogeneous terms:

$$u(\zeta) = \sum_{\alpha \in \mathbb{N}^N} u_\alpha \zeta^\alpha = \sum_{j=1}^{\infty} u_j(\zeta), \quad u_j(t\zeta) = t^j u_j(\zeta) \text{ for all } (t, \zeta) \in \mathbb{C} \times \mathbb{C}^N. \tag{4.3}$$

In other words, for every $j \in \mathbb{N}^*$, $u_j \in (\mathcal{P}_{N,j})^n$. From time to time, we also use the notation $u|_j$ to denote u_j (especially when the expression of the power series mapping u is long). For each component u^k of u, we use an analogous notation for the decomposition of u^k into homogeneous terms. Note that for every polydisc $\Delta \subset \mathbb{C}^n$ centered at the origin, we have $\left\|u_j^k\right\|_\Delta \le \|u_j\|_\Delta$ for every $k = 1, \dots, n$.

Given a homogeneous polynomial $P \in \mathcal{P}_{n,d}$ (of degree $d \ge 1$) and a formal power series mapping $u \colon (\mathbb{C}^N, 0) \to (\mathbb{C}^n, 0)$, we want to estimate, for any $j \in \mathbb{N}^*$, the homogeneous part of order $d+j-1$ of the composition $P \circ u$ which, in agreement with the former notation, is denoted by $(P \circ u)_{d+j-1}$.

Lemma 4.3. *Let $P \in \mathcal{P}_{n,d}$ and assume that $u : (\mathbb{C}^N, 0) \to (\mathbb{C}^n, 0)$ is a formal power series mapping whose decomposition into homogeneous terms is given by (4.3), $d \ge 1$. Then for every polydisc Δ centered at the origin in $\mathbb{C}^N$ and every $j \ge 1$, we have*

$$\|(P \circ u)_{d+j-1}\|_\Delta \le \|P\|_1 \sum \frac{d!}{k_1! \dots k_j!} \|u_1\|_\Delta^{k_1} \dots \|u_j\|_\Delta^{k_j}, \tag{4.4}$$

where the sum goes over all $k_1, \dots, k_j$ such that $k_1 + 2k_2 + \dots + jk_j = d + j - 1$ and $k_1 + \dots + k_j = d$, and if $j \ge 2$, we also have

$$\|(P \circ u)_{d+j-1} - (P' \circ u_1) \cdot u_j\|_\Delta \le \|P\|_1 \sum \frac{d!}{k_1! \dots k_{j-1}!} \|u_1\|_\Delta^{k_1} \dots \|u_{j-1}\|_\Delta^{k_{j-1}}, \tag{4.5}$$

where the sum goes over all $k_1, \dots, k_{j-1}$, such that $k_1 + 2k_2 + \dots + (j-1)k_{j-1} = d+j-1$ and $k_1 + \dots + k_{j-1} = d$. Here $((P' \circ u_1) \cdot u_j)$ is the homogeneous polynomial of degree $d + j - 1$ given by $\sum_{\nu=1}^n \left(\frac{\partial P}{\partial w_\nu} \circ u_1\right) u_j^\nu$.

Proof. We set $v(t) := \sum_{j\in\mathbb{N}^*} \|u_j\|_\Delta t^j \in \mathbb{C}[[t]]$. Writing $P(w) := \sum_{|\alpha|=d} P_\alpha w^\alpha$, we have

$$\begin{aligned}(P\circ u)_{d+j-1}(\zeta) &= \sum_\alpha P_\alpha \left[(u^1(\zeta))^{\alpha_1}\dots(u^n(\zeta))^{\alpha_n}\right]|_{d+j-1}\\ &= \sum_\alpha P_\alpha \sum_{\sum \delta^{r,s}=d+j-1} u^1_{\delta^{1,1}}(\zeta)\dots u^1_{\delta^{1,\alpha_1}}(\zeta)u^2_{\delta^{2,1}}(\zeta)\dots u^n_{\delta^{n,\alpha_n}}(\zeta).\end{aligned}$$

Hence, taking the norms we get

$$\begin{aligned}\|(P\circ u)_{d+j-1}\|_\Delta &= \max_{\zeta\in\Delta}|(P\circ u)_{d+j-1}(\zeta)|\\ &\le \sum_\alpha |P_\alpha| \sum_{\sum \delta^{r,s}=d+j-1} \|u^1_{\delta^{1,1}}\|_\Delta\dots\|u^1_{\delta^{1,\alpha_1}}\|_\Delta\|u^2_{\delta^{2,1}}\|_\Delta\dots\|u^n_{\delta^{n,\alpha_n}}\|_\Delta\\ &\le \sum_\alpha |P_\alpha| \sum_{\sum \delta^{r,s}=d+j-1} \|u_{\delta^{1,1}}\|_\Delta\dots\|u_{\delta^{1,\alpha_1}}\|_\Delta\|u_{\delta^{2,1}}\|_\Delta\dots\|u_{\delta^{n,\alpha_n}}\|_\Delta.\end{aligned}$$

But since the length of every multi-index $\alpha\in\mathbb{N}^N$ appearing in this equation is exactly d, we get

$$\begin{aligned}\|(P\circ u)_{d+j-1}\|_\Delta &\le \sum_\alpha |P_\alpha| \sum_{k_1+\dots+k_d=d+j-1}\|u_{k_1}\|_\Delta\dots\|u_{k_d}\|_\Delta\\ &\le \|P\|_1 \sum_{k_1+\dots+k_d=d+j-1}\|u_{k_1}\|_\Delta\dots\|u_{k_d}\|_\Delta\\ &\le \|P\|_1 \frac{1}{(d+j-1)!}\left[\frac{\mathrm{d}^{d+j-1}}{\mathrm{dt}^{d+j-1}}(v(t))^d\right]\Bigg|_{t=0}\\ &\le \|P\|_1 \sum \frac{d!}{k_1!\dots k_j!}\|u_1\|_\Delta^{k_1}\dots\|u_j\|_\Delta^{k_j},\end{aligned}$$

where the last sum goes over all $k_1,\dots,k_j$ such that $k_1+2k_2+\dots+jk_j=d+j-1$ and $k_1+\dots+k_j=d$, by Proposition 4.2. So we have proved (4.4).

The proof of the second estimate (4.5) is similar to what was done above for (4.4). Indeed note that one has

$$\begin{aligned}(4.6)\quad &(P\circ u)_{d+j-1}(\zeta)-(P'(u_1(\zeta)))\cdot u_j(\zeta)\\ &\qquad= \sum_\alpha P_\alpha \sum_{\substack{\sum \delta^{r,s}=d+j-1\\ 1\le\delta^{r,s}\le j-1}} u^1_{\delta^{1,1}}(\zeta)\dots u^1_{\delta^{1,\alpha_1}}(\zeta)u^2_{\delta^{2,1}}(\zeta)\dots u^n_{\delta^{n,\alpha_n}}(\zeta).\end{aligned}$$

Hence following the previous estimates, we get

$$\begin{aligned}\|(P\circ u)_{d+j-1}-(P'\circ u_1)\cdot u_j\|_\Delta &\le \|P\|_1 \sum_{\substack{k_1+\dots+k_d=d+j-1\\ 1\le k_\nu<j}}\|u_{k_1}\|_\Delta\dots\|u_{k_d}\|_\Delta\\ &\le \|P\|_1\sum\frac{d!}{m_1!\dots m_{j-1}!}\|u_1\|_\Delta^{m_1}\dots\|u_{j-1}\|_\Delta^{m_{j-1}},\end{aligned}$$

the last sum going over all nonnegative $m_1,\dots,m_{j-1}$ such that $m_1+2m_2+\dots+(j-1)m_{j-1}=d+j-1$ and $m_1+\dots+m_{j-1}=d$. The proof of Lemma 4.3 is complete. □

We will also need the following lemma, which, although a bit technical, is crucial for our treatment.

Lemma 4.4. *Let $s \in \mathbb{N}^*$, $a > 0$, $R > 0$, $C > 0$, $D > 0$ and $\gamma_1 > 0$. Define $\gamma_0 = 0$, and for $l \geq 2$,*

$$D\gamma_l := \sum \frac{(s+d)!}{k_1! \dots k_{l-1}!} Ra^d \gamma_1^{k_1} \dots \gamma_{l-1}^{k_{l-1}} + C^{l+s-1}; \tag{4.7}$$

the sum is over all nonnegative integers $k_1, \dots, k_{l-1}$ satisfying $\sum_{1 \leq \nu \leq l-1} k_\nu \geq s$ and $k_1 + 2k_2 + \dots + (l-1)k_{l-1} = s + l - 1$, and we set $s + d := k_1 + \dots + k_{l-1}$. Then the (*formal*) *power series $\sum_{l \geq 0} \gamma_l t^l$ converges in a neighbourhood of* 0.

Proof. We are going to show that Lemma 4.4 is a consequence of Proposition 4.2. For this, we consider the following analytic equation where the unknown is a complex-valued formal power series $\varphi(t)$ in one variable:

$$(Rs\gamma_1^{s-1} + D)t^{s-1}\varphi(t) - (R(s-1)\gamma_1^s + \gamma_1 D)t^s - \sum_{l>s} C^l t^l = \frac{R\varphi(t)^s}{1 - a\varphi(t)}. \tag{4.8}$$

We claim that there exists a unique formal power series solution $\varphi(t)$ satisfying (4.8) with $\varphi(0) = 0$ and $\varphi'(0) = \gamma_1$. For this, set $\varphi(t) := \sum_{l \geq 0} \gamma_l t^l$ and note that to prove the claim it suffices to show that (4.8) holds for a uniquely determined choice of the coefficients γ_l, $l \geq 2$. This can be checked by comparing homogeneous terms of the same order on both sides of (4.8). We first note that homogeneous terms of order s on both sides of (4.8) are identical. For $m \geq 2$, the homogeneous term of order $m + s - 1$ in the left-hand side of (4.8) is given by

$$\left((Rs\gamma_1^{s-1} + D)\gamma_m + C^{m+s-1}\right) t^{m+s-1}$$

while on the right-hand side such a term is of the form

$$\left(Rs\gamma_1^{s-1}\gamma_m + Q_m(\gamma_1, \dots, \gamma_{m-1})\right) t^{m+s-1}, \tag{4.9}$$

for some polynomial Q_m. Hence (4.8) is satisfied by $\varphi(t)$ if and only if for every $m \geq 2$, $D\gamma_m = Q_m(\gamma_1, \dots, \gamma_{m-1}) - C^{m+s}$, which shows the existence of a unique formal power series solution. Next, setting $\psi(t) := \varphi(t)/t$ and

$$F(x,t) := (Rs\gamma_1^{s-1} + D)x - (R(s-1)\gamma_1^s + \gamma_1 D) - \sum_{l>s} C^l t^{l-s} - \frac{Rx^s}{1 - atx},$$

it follows from (4.8) that ψ satisfies the equation $F(\psi(t), t) = 0$. Since F is convergent and $\dfrac{\partial F}{\partial x}(\gamma_1, 0) = D \neq 0$, from the implicit function theorem we get the convergence of ψ and hence that of φ. (We take the opportunity here to thank an anonymous referee for providing us a simple proof of the convergence of φ.) To complete the proof of the lemma, we are left to check that the coefficients γ_l, $l \geq 2$, satisfy (4.7). For this, we want a more precise formula for the coefficient given in (4.9). To the right-hand side of (4.8), we apply Faa di Bruno's formula (Proposition 4.2) with

$$h(t) = \varphi(t), \quad g(x) = \frac{Rx^s}{1 - ax},$$

to obtain that

$$(Rs\gamma_1^{s-1} + D)\gamma_m - C^{m+s-1} = \sum_{k_1, \dots, k_m} \frac{(k+d)!}{k_1! \dots k_m!} Ra^d \gamma_1^{k_1} \dots \gamma_m^{k_m}, \tag{4.10}$$

where the sum goes over all nonnegative integers $k_1, \dots, k_m$ such that $k_1 + \ldots + k_m \geq s$, $\sum_{1 \leq j \leq m} jk_j = s + m - 1$ and where $s + d := \sum_{1 \leq j \leq m} k_j$. (Indeed, one easily checks that since $g \circ h$ vanishes to order k, the k_{m+j} in (4.2) for $1 \leq j \leq s-1$ vanish, so we don't have to write them.) Moreover, in view of (4.9), the term on the right-hand side of (4.10) containing γ_m corresponds to $k_1 = s-1$, $k_2 = 0, \ldots, k_{m-1} = 0$, and $k_m = 1$, so that it is equal to $Rs\gamma_1^{s-1}\gamma_m$. Hence, (4.10) gives (4.7). The proof of Lemma 4.4 is complete. $\square$

4.2. **Reducing the initial system to a simpler one.** Let

$$A(w) = \big(A^1(w), \ldots, A^n(w)\big)$$

be a $\mathbb{C}^n$-valued holomorphic map defined in a neighbourhood of the origin in $\mathbb{C}^n$. Without loss of generality, we may assume that $A(0) = 0$. We decompose each A^j into homogeneous polynomials

$$A^j(w) = \sum_{k > k_j} A^j_k(w), \quad A^j_k(tw) = t^k A^k_j(w), \text{ for all } (t, w) \in \mathbb{C} \times C^n. \tag{4.11}$$

We will write $A_\nu = \big(A^1_{k_1+1+\nu}, \ldots, A^n_{k_n+1+\nu}\big)$ for all $\nu \in \mathbb{N}$. The first problem we have to face before proceeding to the proof of Theorem 3.1 is due to the fact that even if A is of generic full rank, its lowest-order homogeneous terms need not be, in general, of full rank too. This difficulty will be overcome by using the following trick contained in [34]. For the reader's convenience, we recall the proof.

Lemma 4.5. *Let A be as above and assume that A is generically of full rank. Then there exists a polynomial map $P : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ with the property that the holomorphic map $B\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ given by $B(w) := P(A(w))$ has its lowest-order homogeneous terms $B_0(w)$ of generic full rank.*

Proof. In what follows, for any local holomorphic map $h : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$, we denote by $h'(w)$ its Jacobian matrix at the point w.

Let e be the order of vanishing of $\det A'(w)$. We define $D = e - \sum_{1 \leq j \leq n} k_j$. Then clearly A satisfies $\det A'_0 \neq 0$ if and only if $D = 0$. If $D = 0$ we are, therefore, done and hence we may assume that $\det A'_0 = 0$. In this case, the polynomials $A^j_{k_j+1}$ for $j = 1, \ldots, n$ are algebraically dependent over $\mathbb{C}$ (see e.g. [26]), that is, there exists a nonzero polynomial $p \in \mathbb{C}[x_1, \ldots x_n]$ such that $p(A^1_{k_1+1}, \ldots, A^n_{k_n+1}) = 0$. Since each $A^j_{k_j+1}$ is homogeneous of degree $k_j + 1$, we may choose p to be weighted homogeneous (where x_j has weight $k_j + 1$) of lowest possible degree f. This means that $p(x_1, \ldots, x_n) = \sum_{\alpha \in \mathbb{N}^n} c_\alpha x^\alpha$ with $\alpha_1(k_1 + 1) + \ldots + \alpha_n(k_n + 1) = f$ and therefore for all $t \in \mathbb{C}$, $p(t^{k_1+1}x_1, \ldots, t^{k_n+1}x_n) = t^f p(x_1, \ldots, x_n)$. Without loss of generality, we may also assume that p_{x_1} is nonzero. We now consider the map $\tilde{A}\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ given by $\tilde{A}^1 = p(A)$, $\tilde{A}^j = A^j$, $j > 1$ and claim that for this map, the associated number $\tilde{D}$ is strictly smaller than that associated to A, namely D. Indeed, first note that we have

$$\det \tilde{A}'(w) = p_{x_1}(A^1(w), \ldots, A^n(w)) \cdot \det A'(w).$$

Moreover, by our choice of p, the order of vanishing of $p_{x_1}(A^1, \ldots, A^n)$ is exactly $f - (k_1 + 1)$ and, therefore, the order of vanishing of $\det \tilde{A}'$ is $e + f - (k_1 + 1)$. For $j = 1, \ldots, n$, denote by $\tilde{k}_j + 1$ the order of vanishing of $\tilde{A}^j$. Then by our choice of p, one has $\tilde{k}_1 + 1 > f$ while $\tilde{k}_j = k_j$ for $j > 1$. Thus we obtain $\tilde{D} = e + f - (k_1 + 1) - \tilde{k}_1 - \sum_{1 < j \leq n} k_j < e - \sum_{1 \leq j \leq n} k_j = D$, proving the claim. This in turn means that after a finite number of these procedures, we get at a map B satisfying the property claimed in the lemma. $\square$

4.3. **Completion of the proof.** By Lemma 4.5, there exists a polynomial map $P\colon \mathbb{C}^n \to \mathbb{C}^n$ (vanishing at the origin) such that $B(w) = P(A(w))$ has lowest-order homogeneous terms $B_0(w)$ of generic full rank n. Now note that if we have proved Theorem 3.1 for B, then Theorem 3.1 easily also holds for A. So we may assume that the lowest-order terms of A, denoted as above by $A_0(w) = A_0(w^1, \ldots, w^n)$, are generically independent. Furthermore, by taking every A^j as defined in (4.11) to some appropriate power, we may assume that $k_1 = \cdots = k_n =: \ell_0 \geq 0$. In addition, since after a linear change of coordinates $\widetilde{w} = (\widetilde{w}^1, \ldots, \widetilde{w}^n) = \mathcal{L}w$, $\det A_0$ may be normalized in the $\widetilde{w}^n$ direction, we claim that we may assume that such a property already holds in the original w-coordinates. Indeed, if in the $\widetilde{w}$-coordinates we have found a parametrization Ψ as given in (3.4) whose homogeneous parts satisfy all the conditions of Theorem 3.1, then it suffices to set $\psi(b, \lambda) := \mathcal{L}^{-1} \cdot \Psi(b, \mathcal{L} \cdot \lambda)$ for $(b, \lambda) \in \mathcal{H}(\mathbb{C}^N, 0) \times \mathsf{GL_n}(\mathbb{C})$ to get the right parametrization (whose homogeneous parts satisfy all the requirements of Theorem 3.1) in the original w-coordinates. This proves the claim.

After all these reductions, we may start with the formal part of the proof and first construct a sequence $\tilde{p}_\ell\colon \mathbb{C}^n \times J_0^{\ell_0+\ell}(\mathbb{C}^n) \to \mathbb{C}^n$ of polynomial maps, $\ell \geq 1$, such that for every germ of a holomorphic map $b\colon (\mathbb{C}^n, 0) \to \mathbb{C}^n$ if there exists a biholomorphism $u\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ with $u'(0)$ being the identity and $A(u(z)) = b(z)$, then $u_\ell(z) = \tilde{p}_\ell(z, j_0^{\ell_0+\ell} b)$ for all $\ell \geq 1$. Here we recall that u_ℓ denotes the homogeneous part of order ℓ of u. In what follows we also use the notation $\theta(z)|_\ell$ to denote the homogeneous part of order ℓ of a formal power series θ. Moreover, for every $k \in \mathbb{N}^*$, we introduce coordinates $\Lambda^k = (\Lambda_\alpha)_{|\alpha| \leq k}$, $\alpha \in \mathbb{N}^n$, in the jet space $J_0^k(\mathbb{C}^n)$ associated to a choice of coordinates $z = (z_1, \ldots, z_n)$ in $\mathbb{C}^n$. By Lemma 4.1, we may choose for each integer $\ell \geq 2$ a linear map $B_\ell\colon \mathcal{P}^n_{\ell_0+\ell} \to \mathcal{P}^n_\ell$, which is a left inverse to multiplication by A_0'. We set $\tilde{p}_1(z) := z$ and define inductively the $\tilde{p}_\ell$, $\ell \geq 2$, as follows:

(4.12)
$$\begin{cases} \tilde{p}_\ell(z, \Lambda^{\ell_0+\ell}) := B_\ell \left(\displaystyle\sum_{|\alpha| = \ell_0+\ell} \frac{\Lambda_\alpha}{\alpha!} z^\alpha - A(P^{\ell-1}(z, \Lambda^{\ell_0+\ell-1}))|_{\ell_0+\ell} \right), \\ P^{\ell-1}(z, \Lambda^{\ell_0+\ell-1}) := \displaystyle\sum_{1 \leq k \leq \ell-1} \tilde{p}_k(z, \Lambda^{\ell_0+k}). \end{cases}$$

It is easy to see that we also have

$$\tilde{p}_\ell(z, \Lambda^{\ell_0+\ell}) = B_\ell \left(\sum_{|\alpha| = \ell_0+\ell} \frac{\Lambda_\alpha}{\alpha!} z^\alpha - \sum_{\nu=0}^{\ell-1} A_\nu(P^{\ell-1}(z, \Lambda^{\ell_0+\ell-1}))|_{\ell_0+\ell} \right) \tag{4.13}$$

for $\ell \geq 2$, and hence by induction from (4.13) and (4.12) that the $\tilde{p}_\ell$ are indeed polynomial mappings of their arguments and homogeneous of degree ℓ in their first factor. Now assume that $b\colon (\mathbb{C}^n, 0) \to \mathbb{C}^n$ is a germ of a holomorphic map and that $u\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ is a germ of a biholomorphism satisfying $A(u(z)) = b(z)$ with $u'(0)$ being the identity. We claim that for every $\ell \geq 1$, $u_l(z) = \tilde{p}(z, j_0^{\ell_0+\ell} b)$. First note that $u_1 = \tilde{p}_1$. Suppose now that for $\ell \geq 2$, we have $u_k(z) = \tilde{p}_k(z, j_0^{\ell_0+k} b)$ for

$k < \ell$ and set $U_\ell(z) = \sum_{1\le k\le \ell} u_\ell(z)$. We then have

$$\begin{aligned} b_{\ell_0+\ell}(z) &= A(u(z))|_{\ell_0+\ell} \\ &= \sum_{\nu=0}^{\ell-1} A_\nu(u(z))|_{\ell_0+\ell} \\ &= \sum_{\nu=0}^{\ell-1} A_\nu(U_\ell(z))|_{\ell_0+\ell}. \end{aligned} \tag{4.14}$$

Noticing that $A_0(U_\ell(z))|_{\ell_0+\ell} = A_0'(z)\cdot u_\ell(z) + A_0(U_{\ell-1}(z))|_{\ell_0+\ell}$ and that for $1 \le \nu \le \ell - 1$, $A_\nu(U_\ell(z))|_{\ell_0+\ell} = A_\nu(U_{\ell-1}(z))|_{\ell_0+\ell}$, we obtain that

$$b_{\ell_0+\ell}(z) = A_0'(z)\cdot u_\ell(z) + \sum_{\nu=1}^{\ell-1} A_\nu(U_{\ell-1}(z))|_{\ell_0+\ell}, \tag{4.15}$$

and since $u_k(z) = \tilde p_k(z, j_0^{\ell_0+k} b)$ for $k<\ell$ by assumption, we get

$$A_0'(z)\cdot u_\ell(z) = b_{\ell_0+\ell}(z) - \sum_{\nu=0}^{\ell-1} A_\nu(P^{\ell-1}(z, j_0^{\ell_0+\ell-1}b))|_{\ell_0+\ell},$$

i.e.,

$$u_\ell = B_\ell\left(b_{\ell_0+\ell}(\cdot) - \sum_{\nu=0}^{\ell-1} A_\nu(P^{\ell-1}(\cdot, j_0^{\ell_0+\ell-1}b))|_{\ell_0+\ell}\right), \tag{4.16}$$

and consequently $u_\ell(z) = \tilde p_\ell(z, j_0^{\ell_0+\ell}b)$. This proves the claim.

We are now going to prove the convergence statement for the $\tilde p_\ell$. In this context, this means that there exists a polydisc $\Delta \subset \mathbb{C}^n$ centered at the origin such that given any $C>0$, there exists $K>0$ such that whenever $b\colon (\mathbb{C}^n,0)\to\mathbb{C}^n$ is a germ of a holomorphic map satisfying $\|b_k\|_\Delta \le C^k$ for all $k\in\mathbb{N}$ where $b(z) = \sum_{k\ge 0} b_k(z)$ is the decomposition of b into homogeneous terms, then $\left\|\tilde p_\ell(\cdot, j_0^{\ell_0+\ell}b)\right\|_\Delta \le K^\ell$ for all $\ell\in\mathbb{N}$. Let $\Delta = \Omega \subset \mathbb{C}^n$ be the polydisc given by Lemma 4.1 and $D>0$ also given by the same lemma (and depending only on A_0') such that the norm of each linear map $B_\ell\colon \mathcal{P}^n_{\ell_0+\ell} \to \mathcal{P}^n_\ell$ does not exceed $1/D$, for all $\ell \ge 2$. Let b be as above satisfying $\|b_k\|_\Delta \le C^k$ for all $k\in\mathbb{N}$. In what follows we assume b to be fixed and then suppress the dependence on b when writing $\tilde p_\ell$ and $P^{\ell-1}$. Since A is convergent, there exist $R>0$ and $a>0$ such that $\|A_j\|_1 \le Ra^j$ for all $j\in\mathbb{N}$. We also set $m_0 := \ell_0 + 1 \ge 1$. Let $\ell\ge 2$. By (4.16), we have

$$\|\tilde p_\ell\|_\Delta \le D^{-1}\left(\|b_{\ell_0+\ell}\|_\Delta + \left\|(A_0\circ P^{\ell-1})|_{\ell_0+\ell}\right\|_\Delta + \left\|\sum_{\nu=1}^{\ell-1}(A_\nu\circ P^{\ell-1})|_{\ell_0+\ell}\right\|_\Delta \right). \tag{4.17}$$

Since $A_0(P^\ell(z))|_{\ell_0+\ell} = A_0'(z)\cdot\tilde p_\ell(z) + A_0(P^{\ell-1}(z))$, (4.17) implies

$$\|\tilde p_\ell\|_\Delta \le D^{-1}\left(C^{\ell_0+\ell} + \left\|(A_0\circ P^{l})|_{\ell_0+\ell} - A_0'\cdot\tilde p_\ell\right\|_\Delta + \left\|\sum_{\nu=1}^{\ell-1}(A_\nu\circ P^{\ell-1})|_{\ell_0+\ell}\right\|_\Delta \right). \tag{4.18}$$

Applying (4.4) and (4.5) of Lemma 4.3 in our context and using the fact that $\tilde{p}_1(z) = z$ yields

$$\begin{aligned}\|\tilde{p}_\ell\|_\Delta &\leq D^{-1}\left(C^{\ell_0+\ell} + \sum \frac{(m_0+d)!}{k_1!\dots k_{\ell-1}!}\|A_d\|_1 \|\tilde{p}_1\|_\Delta^{k_1}\dots\|\tilde{p}_{\ell-1}\|_\Delta^{k_{\ell-1}}\right)\\ &\leq D^{-1}\left(C^{m_0+\ell-1} + \sum \frac{(m_0+d)!}{k_1!\dots k_{\ell-1}!}Ra^d \|\tilde{p}_1\|_\Delta^{k_1}\dots\|\tilde{p}_{\ell-1}\|_\Delta^{k_{\ell-1}}\right),\end{aligned} \tag{4.19}$$

where $k_1 + \cdots + k_{\ell-1} = m_0 + d$ and the sum goes over all nonnegative integers $k_1, \dots, k_{\ell-1}$ with $\sum_{1\leq j\leq \ell-1} jk_j = m_0 + \ell - 1$. We shall now use Lemma 4.4 with $\gamma_1 := \|\tilde{p}_1\|_\Delta$ (which is independent of b), R, a, C, D as above and $s = m_0 = \ell_0 + 1$. Let $(\gamma_\ell)_{\ell\geq 0}$ be given by Lemma 4.4 and associated to the above data. We claim that for all $\ell \geq 1$, $\|\tilde{p}_\ell\|_\Delta \leq \gamma_\ell$. Indeed, by construction this is the case for $\ell = 1$. Let us assume, therefore, that $\|\tilde{p}_k\|_\Delta \leq \gamma_k$ for $k < \ell$. Then (4.19) and (4.7) imply that

$$\|\tilde{p}_\ell\|_\Delta \leq D^{-1}\left(C^{m_0+\ell-1} + \sum \frac{(m_0+d)!}{k_1!\dots k_{\ell-1}!}Ra^d {\gamma_1}^{k_1}\dots\gamma_{\ell-1}^{k_{\ell-1}}\right) = \gamma_\ell, \tag{4.20}$$

which proves the claim by induction. Since the series $\sum \gamma_k t^k$ converges by Lemma 4.4, we see that there exists $K > 0$ such that $\|\tilde{p}_\ell\|_\Delta \leq K^\ell$ for $\ell \in \mathbb{N}^*$.

We shall now construct the p_ℓ from the $\tilde{p}_\ell$ for all $\ell \geq 1$. For any linear map $N\colon \mathbb{C}^n \to \mathbb{C}^n$ and for every positive integer k we denote by $N_*^k\colon J_0^k(\mathbb{C}^n) \to J_0^k(\mathbb{C}^n)$ the (unique linear) map satisfying $j_0^k(f\circ N) = N_*^k(j_0^k f)$ for every formal power series mapping $f\colon (\mathbb{C}^n, 0) \to \mathbb{C}^n$. (Note that by identifying jets of order k with Taylor polynomials of degree k, one may also view the mapping N_*^k as acting on such polynomials.) For every $\ell \geq 1$ and every $(z, \lambda, \Lambda^{\ell_0+\ell}) \in \mathbb{C}^n \times \mathsf{GL_n}(\mathbb{C}) \times J_0^{\ell_0+\ell}(\mathbb{C}^n)$, we set

$$r_\ell(z, \lambda, \Lambda^{\ell_0+\ell}) := \tilde{p}_\ell(\lambda z, (\lambda^{-1})_*^{\ell_0+\ell}(\Lambda^{\ell_0+\ell})). \tag{4.21}$$

It is easy to see that each r_ℓ is a homogeneous polynomial map of degree ℓ in z, whose components are rational functions of the other variables. In fact, even more can be said: by Cramer's rule we see that there exists a nonnegative integer $\kappa(\ell)$ such that $(\det\lambda)^{\kappa(\ell)} r_\ell(z, \lambda, \Lambda^{\ell_0+\ell}) =: p_\ell(z, \lambda, \Lambda^{\ell_0+\ell})$ is a polynomial in all its variables. Note that $r_1(z, \lambda, \Lambda^{\ell_0+1}) = \lambda z$ by construction, which proves (3.2). Moreover, for every germ of a holomorphic map $b\colon (\mathbb{C}^n, 0) \to \mathbb{C}^n$, we claim that if $u\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^n)$ is a germ of a biholomorphism satisfying $A(u(z)) = b(z)$, then necessarily for every $\ell \geq 1$, $u_\ell(z) = r_\ell(z, u'(0), j_0^{\ell_0+\ell} b)$. Indeed, set $\lambda := u'(0)$, and note that $v(z) := u(\lambda^{-1}z)$ is a biholomorphism tangent to the identity which satisfies $A(v(z)) = b(\lambda^{-1}z)$. Hence for every $\ell \geq 1$, the homogeneous part of order ℓ of v denoted v_ℓ satisfies $v_\ell(z) = \tilde{p}_\ell(z, j_0^{\ell_0+\ell}(b\circ\lambda^{-1})) = \tilde{p}_\ell(z, (\lambda^{-1})_*^{\ell_0+\ell}(j_0^{\ell_0+\ell} b))$ and therefore $v(\lambda z) = u(z)$ implies that $u_\ell(z) = \tilde{p}_\ell(\lambda z, (\lambda^{-1})_*^{\ell_0+\ell}(j_0^{\ell_0+\ell} b))$, which is the required conclusion in view of (4.21). To complete the proof of Theorem 3.1, it remains to check the convergence statement for the r_ℓ. So let $\Delta \subset \mathbb{C}^n$ be the polydisc given above and $L \subset \mathsf{GL_n}(\mathbb{C})$ be a given compact subset. Let $C > 0$ and $b\colon (\mathbb{C}^n, 0) \to \mathbb{C}^n$ be a germ of a holomorphic map satisfying $\|b_k\|_\Delta \leq C^k$ for all $k \in \mathbb{N}$. Since there exists $C_1 > 0$ such that $\left\|\lambda^{-1}\right\| \leq C_1$ for all $\lambda \in L$ and since $(\lambda^{-1})_*^k(b_k)$ is the homogeneous part of order k of $b\circ\lambda^{-1}$, we have $\left\|(\lambda^{-1})_*^k(b_k)\right\|_\Delta \leq \|b_k\|_{C_1\Delta}$, where $C_1\Delta$ is the polydisc of multiradius $C_1\rho$ if ρ is the multiradius of Δ. Using the homogeneity of each b_k we therefore get $\left\|(\lambda^{-1})_*^k(b_k)\right\|_\Delta \leq C_1^k \|b_k\|_\Delta \leq C_1^k C^k =: \tilde{C}^k$, for all $\lambda \in L$

and b as above. Hence $\left\|(b\circ\lambda^{-1})|_k\right\|_\Delta \le \tilde C^k$ for all $k\in\mathbb{N}$ and by the first part of the proof, there exists $\tilde K>0$ such that

$$\left\|\tilde p_\ell(\cdot, j_0^{\ell+\ell_0}(b\circ\lambda^{-1}))\right\|_\Delta = \left\|\tilde p_\ell(\cdot,(\lambda^{-1})_*^{\ell+\ell_0}(j_0^{\ell+\ell_0}b))\right\|_\Delta \le \tilde K^\ell.$$

Furthermore there exists $C_2>0$ such that $\|\lambda\|\le C_2$ for all $\lambda\in L$, so we have

$$\begin{aligned}\left\|r_\ell(\cdot,\lambda,j_0^{\ell_0+\ell}b)\right\|_\Delta &= \left\|\tilde p_\ell(\lambda\cdot,(\lambda^{-1})_*^{\ell+\ell_0}(j_0^{\ell+\ell_0}b))\right\|_\Delta\\ &\le \left\|\tilde p_\ell(\cdot,(\lambda^{-1})_*^{\ell+\ell_0}(j_0^{\ell+\ell_0}b))\right\|_{C_2\Delta}\\ &\le C_2^\ell\left\|\tilde p_\ell(\cdot,(\lambda^{-1})_*^{\ell+\ell_0}(j_0^{\ell+\ell_0}b))\right\|_\Delta \le K^\ell,\end{aligned}$$

where $K=\tilde K C_2$. The proof of Theorem 3.1 is complete.

5. Proof of Theorem 2.4

5.1. **Holomorphy of** $\Psi(b,\lambda)$**.** To each *fixed* choice of polynomial mappings (p_ℓ) and integers $(\kappa(\ell))$ satisfying the conclusions of Theorem 3.1, we consider the associated map $\Psi\colon \mathcal{H}(\mathbb{C}^n,0)\times \mathsf{GL_n}(\mathbb{C})\to\mathcal{B}^n$ given in (3.4) and we show here that Ψ is holomorphic, furnishing the proof of Theorem 2.4. Here we recall that holomorphy of a map $f\colon X\supset\Omega\to Y$ between locally convex complex vector spaces X,Y with Ω open in X means that f is holomorphic if it is continuous and its composition with any continuous linear functional on Y is holomorphic along finite-dimensional affine subspaces of X intersected with Ω (this last property being called *Gâteaux-holomorphy*). Note that the fact that Ψ takes its values indeed in $\mathcal{B}^n$ follows from Theorem 3.1 and the following well-known elementary fact.

Lemma 5.1. *Let $\phi=\phi(z)$ be a formal power series in $z=(z_1,\dots,z_n)$. Consider the decomposition of ϕ into homogeneous polynomial terms and in Taylor series $\phi(z)=\sum_{k\ge0}\phi_k(z)=\sum_{\alpha\in\mathbb{N}^n}\phi_\alpha z^\alpha$. Then the following are equivalent:*

- (i) *ϕ is convergent;*
- (ii) *there exists $C>0$ such that for all $\alpha\in\mathbb{N}^n$, $|\phi_\alpha|\le C^{|\alpha|}$;*
- (iii) *there exists a polydisc $\Delta\subset\mathbb{C}^n$ centered at the origin and $D>0$ such that for every positive integer k, $\|\phi_k\|_\Delta\le D^k$;*
- (iv) *for every polydisc $\Delta\subset\mathbb{C}^n$ centered at the origin, there exists $K>0$ such that for every positive integer k, $\|\phi_k\|_\Delta\le K^k$.*

All the power series ϕ satisfying either of the inequalities in (ii), (iii) *and* (iv) *have the property that their region of convergence contains some neighbourhood of the origin only depending on C or D or K, respectively.*

For $r>0$, we let $\mathcal{H}(\mathbb{C}^n,0)_r$ be the Fréchet space of holomorphic maps $b:\Delta_r\to\mathbb{C}^n$, where $\Delta_r=\{z\in\mathbb{C}^n\colon |z_j|<r \text{ for all } j\}$, endowed with the topology of uniform convergence on compact subsets of Δ_r. It is in our case generated by the seminorms

$$\forall f=(f^1,\dots,f^n)\in\mathcal{H}(\mathbb{C}^n,0)_r,\quad \|f\|_s=\max_{1\le j\le n}\sup_{z\in\Delta_s}|f^j(z)|,\quad s<r.$$

If we choose a sequence $r_j>0$ converging monotonically to 0, the space $\mathcal{H}(\mathbb{C}^n,0)$ is a locally convex space which is the inductive limit of the sequence of Fréchet spaces $\mathcal{H}(\mathbb{C}^n,0)_{r_j}$ or, equivalently, of the Banach spaces $H^\infty(\Delta_{r_j})^n$. We denote by $\mathcal{H}(\mathbb{C}^n,0)_r^0$ (resp. $\mathcal{H}(\mathbb{C}^n,0)^0$) the closed subspace of $\mathcal{H}(\mathbb{C}^n,0)_r$ (resp. of $\mathcal{H}(\mathbb{C}^n,0)$) consisting of those elements preserving the origin. We also let $\mathcal{B}_r^n$ be the open subset of $\mathcal{H}(\mathbb{C}^n,0)_r^0$ consisting of those maps b satisfying $\det b'(0) \neq 0$. For each $r > 0$, we have a continuous linear injection $\iota_r \colon \mathcal{H}(\mathbb{C}^n,0)_r \to \mathcal{H}(\mathbb{C}^n,0)$, through which we identify $\mathcal{H}(\mathbb{C}^n,0)_r$ with a subspace of $\mathcal{H}(\mathbb{C}^n,0)$. Note though that the image of ι_r in $\mathcal{H}(\mathbb{C}^n,0)_s$ is *not closed* for $s < r$; hence, $\mathcal{H}(\mathbb{C}^n,0)$ is not a strict inductive limit of the spaces $\mathcal{H}(\mathbb{C}^n,0)_{r_j}$. However, in our setting, the inclusion mappings for the inductive limit are *compact* (by Montel's Theorem). For this special case of an inductive limit some powerful results from functional analysis are available; the main one is that this inductive limit is *regular* (cited as Fact 1 below, which implies Fact 3) and that the topology induced on it as an inductive limit of locally convex spaces agrees with the topology induced on it as an inductive limit of topological spaces (this implies Fact 2 below). In what follows, when we identify subsets of $\mathcal{H}(\mathbb{C}^n,0)_r$ (resp. of $\mathcal{H}(\mathbb{C}^n,0)_r^0$) as subsets of $\mathcal{H}(\mathbb{C}^n,0)$ (resp. of $\mathcal{H}(\mathbb{C}^n,0)^0$), we always use the canonical embedding. These results can be found in e.g. [24, §23, p. 132] or [32, pp. 26–27] (and in this particular setting seem to originate from work by e Silva [40] and Raikov [38]). For the convenience of the reader, we summarize the facts we are going to use so we can refer to them later:

Fact 1: [24, 2.2. Satz, p.136] A subset $B \subset \mathcal{H}(\mathbb{C}^n,0)$ is bounded if and only if there exists $r > 0$ such that $B \subset \mathcal{H}(\mathbb{C}^n,0)_r$ and B is bounded in $\mathcal{H}(\mathbb{C}^n,0)_r$.

Fact 2: A mapping $f : \mathcal{H}(\mathbb{C}^n,0) \to Y$, where Y is a topological space, is continuous if and only if it is sequentially continuous.

Fact 3: [24, 2.10. Korollar 3, p.139] A sequence (y_n) converges to y in $\mathcal{H}(\mathbb{C}^n,0)$ if and only if there exists an $r > 0$ such that $y_j, y \in \mathcal{H}(\mathbb{C}^n,0)_r$ and the sequence converges there.

Fact 2 is an immediate consequence of Fact 1, Fact 3 and the fact that $\mathcal{H}(\mathbb{C}^N,0)_r$ is Fréchet and thus metrizable. We can now prove the following result concerning the parametrization Ψ, which essentially reduces the study of Ψ as a map between spaces of germs to that of a map between Fréchet spaces.

Lemma 5.2. *With the assumptions of Theorem* 3.1 *and in the above-mentioned setting, let* $\Psi \colon \mathcal{H}(\mathbb{C}^n,0) \times \mathsf{GL_n}(\mathbb{C}) \to \mathcal{B}^n$ *be the operator given by* (3.4). *Then there exists a family* $(\mathcal{V}_R)_{R>0}$, *where for each* $R > 0$, $\mathcal{V}_R$ *is an open subset of* $\mathcal{H}(\mathbb{C}^n,0)_R$, *satisfying the following properties:*

(i) *for any given bounded set* $B \subset \mathcal{H}(\mathbb{C}^n,0)$, *there exists* $S = S(B) > 0$ *such that* $B \subset \mathcal{V}_S$;

(ii) *for every* $R > 0$ *and for each relatively compact open subset* $\Omega \subset \mathsf{GL_n}(\mathbb{C})$ *there exists an* $\widetilde{R} > 0$ *such that*

$$\Psi|_{\mathcal{V}_R \times \Omega} \colon \mathcal{V}_R \times \Omega \to \mathcal{B}^n_{\widetilde{R}} \subset \mathcal{H}(\mathbb{C}^n,0)_{\widetilde{R}}$$

is holomorphic.

Proof. For every $R > 0$, let $\mathcal{V}_R$ be defined by

$$\mathcal{V}_R = \{b \in \mathcal{H}(\mathbb{C}^n,0)_R \colon \ \|b - b(0)\|_{R/2} < 1, \ \|b(0)\|_{R/2} < \frac{1}{R}\}.$$

This is clearly an open subset of $\mathcal{H}(\mathbb{C}^n,0)_R$. From the usual Cauchy estimates together with Fact 1, it is easy to see that every bounded set is contained in a set $\mathcal{V}_R$ for some $R > 0$.

Next, for every $R > 0$, by the Cauchy estimates, we easily have

$$\forall b \in \mathcal{V}_R, \ \forall d \in \mathbb{N}^*, \quad \|b_d\|_{R/2} \leq \left(\frac{nR}{2}\right)^d \ \left(\text{and } \ \|b_0\|_{R/2} < \frac{1}{R}\right).$$

(Recall here that b_d is the homogeneous part of order d of b.) By Theorem 3.1 (ii) (which holds automatically for all polydiscs centered at the origin), there exists $K > 0$ (depending on R) such that

$$\|\Psi(b,\lambda)|_d\|_{R/2} \le K^d, \quad \forall b \in \mathcal{V}_R, \quad \forall \lambda \in \Omega,\ \forall d \in \mathbb{N}^*.$$

Choosing $\widetilde{R} \le R/(4K)$, we see that for each $(b,\lambda) \in \mathcal{V}_R \times \Omega$, we have

$$\|\Psi(b,\lambda)\|_{\widetilde{R}} \le 1, \tag{5.1}$$

and $\Psi(b,\lambda) \in \mathcal{H}(\mathbb{C}^n,0)_{\widetilde{R}}$. Moreover, setting $\Psi^k(b,\lambda) := \sum_{\ell \le k} \Psi(b,\lambda)|_\ell$ for every $k \ge 1$ and $(b,\lambda) \in \mathcal{V}_R \times \Omega$, we have by the above choices that (Ψ^k) converges uniformly to Ψ in $\mathcal{H}_{\widetilde{R}}$, i.e., that for all $r \le \widetilde{R}$, $\sup_{(b,\lambda)\in\mathcal{V}_R\times\Omega} \left\|\Psi^k(b,\lambda) - \Psi(b,\lambda)\right\|_r \to 0$ when $k \to +\infty$. Denote by $\mathcal{M}_n(\mathbb{C})$ be the space of $n \times n$ matrices and let $E \subset \mathcal{H}(\mathbb{C}^n,0)_R \times \mathcal{M}_n(\mathbb{C})$ be any affine subspace and $L\colon \mathcal{H}(\mathbb{C}^n,0)_{\widetilde{R}} \to \mathbb{C}$ be any continuous linear functional. In view of the explicit form of the Ψ^k given by Theorem 3.1, we see that $L \circ \Psi^k$ is clearly holomorphic (and rational) along $E \cap (\mathcal{V}_R \times \Omega)$, and since $L \circ \Psi^k$ uniformly converges to $L \circ \Psi$ on $\mathcal{V}_R \times \Omega$, it follows that $L \circ \Psi$ is holomorphic along $E \cap (\mathcal{V}_R \times \Omega)$. Hence, $\Psi|_{\mathcal{V}_R\times\Omega}\colon \mathcal{V}_R \times \Omega \to \mathcal{H}(\mathbb{C}^n,0)_{\widetilde{R}}$ is Gâteaux-holomorphic. To conclude that it is holomorphic, it remains to check that it is continuous. In fact, in view of [20, Example 3.8(g)], since $\mathcal{H}(\mathbb{C}^n,0)_R$ is metrizable, we need only to show that for each continuous linear functional $L\colon \mathcal{H}(\mathbb{C}^n,0)_{\widetilde{R}} \to \mathbb{C}$ the map $L \circ \Psi$ is holomorphic. Since this latter is already Gâteaux-holomorphic, in view of [20, Proposition 3.7] it is enough to show that $L \circ \Psi$ is bounded on $\mathcal{V}_R \times \Omega$. Since $\Psi(\mathcal{V}_R \times \Omega)$ is clearly a bounded set in $\mathcal{H}(\mathbb{C}^n,0)_{\widetilde{R}}$ in view of (5.1), we reach the desired conclusion. The proof of the lemma is complete. □

Completion of the proof of Theorem 2.4. It is now not difficult to derive Theorem 2.4 from Lemma 5.2. Firstly, it follows from the lemma that $\Psi\colon \mathcal{H}(\mathbb{C}^n,0) \times \mathsf{GL_n}(\mathbb{C}) \to \mathcal{B}^n \subset \mathcal{H}(\mathbb{C}^n,0)$ is Gâteaux-holomorphic. It remains therefore to check that Ψ is continuous (between the right topological spaces). But Lemma 5.2 together with Fact 2 and Fact 3 from above implies that Ψ is indeed continuous: If $(y_k) \subset \mathcal{H}(\mathbb{C}^n,0) \times \mathsf{GL_n}(\mathbb{C})$ is a convergent sequence with limit y, there exists $R > 0$ and a relatively compact open set $\Omega \subset \mathsf{GL_n}(\mathbb{C})$ such that $(y_k) \subset \mathcal{V}_R \times \Omega$ (and $y \in \mathcal{V}_R \times \Omega$). The lemma then implies that $\Psi(y_k) \to \Psi(y)$ and we are done. □

5.2. **A further application of Theorems 2.4 and 3.1.** We want here to give the following continuous version of Corollary 3.2.

Theorem 5.3. *Let $A : (\mathbb{C}^n,0) \to \mathbb{C}^n$ be a germ of a holomorphic map of generic rank n, X a topological space and $b = b(z,\omega)$ be a $\mathbb{C}^n$-valued continuous map defined on an open neighbourhood V of $\{0\} \times X \subset \mathbb{C}^n \times X$ such that*

1. *$z \mapsto b(z,\omega)$ is holomorphic on $V_\omega = \{z \in \mathbb{C}^n \colon (z,\omega) \in V\}$ for each $\omega \in X$;*
2. *for each compact set $K \subset \mathbb{C}^n$ and for every point $\omega \in X$ with $K \times \{\omega\} \subset V$ there exists an open neighbourhood U of ω such that b is defined and uniformely bounded on $K \times U$ (this is satisfied if e.g. every point in X has a compact neighbourhood basis).*

Then there exists a continuous map $\Gamma = \Gamma(z,\lambda,\omega)\colon \mathbb{C}^n \times \mathsf{GL_n}(\mathbb{C}) \times X \to \mathbb{C}^n$, defined on an open neighbourhood Ω of $\{0\} \times \mathsf{GL_n}(\mathbb{C}) \times X$ (and holomorphic in the first two *factors) such that if* $u : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ *is a germ of a biholomorphism satisfying* $A(u(z)) = b(z, \omega_0)$ *for some* $\omega_0 \in X$*, then necessarily* $u(z) = \Gamma(z, u'(0), \omega_0)$*. Furthermore, the map* Γ *has the properties* (i) *and* (ii) *given in Corollary* 3.2.

Proof. We define the neighbourhood Ω in the following way: for each point $\omega \in X$ we choose subsets of the form $\Delta_R \times U$, where U and R are small enough so that for $\omega \in U$, $b(\cdot, \omega) \in \mathcal{V}_R$ (with the notation of §5). We claim that this is possible by Assumption (2): indeed, start by finding a neighbourhood U_0 of ω and an $R_0 > 0$ such that b is defined on $\Delta_{R_0} \times U_0$. Choose an $R_1 < R_0$ and consider the compact set $\partial\Delta_{R_1} \subset \mathbb{C}^n$. By assumption, there exists a neighbourhood $U_1 \subset U_0$ of ω such that $\|b(\cdot, \omega)\|_{\partial\Delta_{R_1}} \leq C$ for some constant C, for all $\omega \in U_1$. The Cauchy estimates imply that

$$\|b_\alpha(0, \omega)\| \leq \frac{C}{R_1^{|\alpha|}}, \quad \omega \in U_1.$$

From these estimates it is clear that if we choose $0 < R < R_1$ small enough, the desired claim follows.

Next, for every relatively compact open set $W \subset \mathsf{GL_n}(\mathbb{C})$ we may choose by Lemma 5.2 an $\widetilde{R} > 0$ such that $\Psi(b, \lambda) \in \mathcal{H}(\mathbb{C}^n, 0)_{\widetilde{R}}$ for every $(b, \lambda) \in \mathcal{V}_R \times W$. We let Ω be the union of the open sets $\Delta_{\widetilde{R}} \times W \times U$, as ω ranges over X. The mapping Γ is then defined by

$$\Gamma(z, \lambda, \omega) = \Psi(b(\cdot, \omega), \lambda)(z).$$

We will check that Γ is continuous by checking that it is continuous on sets of the form $\Delta_{\widetilde{R}} \times W \times U$. To do that, we will show the following two facts: firstly, the mapping $\omega \mapsto b(\cdot, \omega)$ is continuous from U to $\mathcal{H}(\mathbb{C}^n, 0)_R$; and secondly, the evaluation mapping $(f, z) \mapsto f(z)$ is holomorphic from $\mathcal{H}(\mathbb{C}^n, 0)_{\widetilde{R}} \times \Delta_{\widetilde{R}}$ to $\mathbb{C}^n$. These both follow from our assumptions using the Cauchy formula for polydiscs, as we will show; combining all these facts we get the result.

Let us start with the proof of the first claim. Let $\omega \in X$ and U $(= U_\omega)$ be as in the above construction of Ω. We check that $\omega \mapsto b(\cdot, \omega)$, as a map from U to $\mathcal{H}(\mathbb{C}^n, 0)_R$, is continuous at every point $\omega_0 \in U$. Let $\epsilon > 0$ and $0 < r < R$. We need to show that there exists a neighbourhood U_ϵ of ω_0 such that for $\omega \in U$,

$$\|b(\cdot, \omega) - b(\cdot, \omega_0)\|_r < \epsilon.$$

We choose s with $r < s < R$. By assumption (2), there exists a neighbourhood $V \subset U$ of ω_0 such that b is bounded on $\partial\Delta_s \times V$. The function

$$\rho(\omega) = \int_{\partial\Delta_s} \|b(z, \omega) - b(z, \omega_0)\| \; d|z|$$

is continuous on V, and $\rho(\omega_0) = 0$. The Cauchy formula implies that

$$b(z, \omega) - b(z, \omega_0) = \frac{1}{(2\pi i)^n} \int_{\partial\Delta_s} \frac{b(\zeta, \omega) - b(\zeta, \omega_0)}{\zeta - z} \, d\zeta, \quad z \in \Delta_r, \quad \omega \in V.$$

Estimating, we get

$$\|b(\cdot, \omega) - b(\cdot, \omega_0)\|_r \leq \frac{1}{(2\pi)^n} (s - r)^{-n} \rho(\omega),$$

and the desired estimate follows.

The proof of the second claim is similar to what we did for proving the holomorphy of the solution mapping in §5. It is clear that the evaluation mapping is holomorphic along finite-dimensional affine subspaces. To complete the proof, we just have to show that the mapping is locally bounded. We leave it to the reader to check that this latter fact also follows from the Cauchy formula. The proof of the corollary is, therefore, complete. □

6. Results for linear singular analytic systems

We consider here linear systems of the form $\Theta(z)\cdot w = b(z)$, $w\in\mathbb{C}^m$, where $\Theta(z)$ is an $m\times m$ matrix with holomorphic coefficients near the origin in $\mathbb{C}^n$, of generic rank m, and $b\colon(\mathbb{C}^n,0)\to\mathbb{C}^m$ is a germ of a holomorphic map. We are interested in getting a holomorphic parametrization of all solutions $w=w(z)$ of such systems, as well as other systems that are variations of the original one. In fact, for such systems and contrary to the nonlinear case, there is a unique potential obvious holomorphic solution of the system; such a solution has to be the quotient of the Weierstrass division of $\widehat{\Theta}\cdot b$ by $\det\Theta$, where $\widehat{\Theta}$ denotes the transpose of the comatrix of Θ as in the proof of Lemma 4.1. By analyzing the proof of the Weierstrass division theorem, it is very likely that such a quotient would be a suitable parametrization of the system satisfying the required conditions of Proposition 6.1 below.

Since we were not able to find in the literature the precise result given by Proposition 6.1 below, we choose instead to follow the approach developed in the previous section (for the nonlinear case) to get the parametrization needed for our purposes here. The obtained parametrization may be a priori different from the one given by Weierstrass division, since the algorithm proposed here to compute the parametrization is different from the algorithm given by Weierstrass division. In what follows we keep the notation defined in the previous sections.

We start with the following:

Proposition 6.1. *Let Θ be an $m\times m$ matrix with holomorphic coefficients near the origin in $\mathbb{C}^n$, $m,n\geq 1$, such that Θ is of generic rank m. Then there exists $\ell_0\in\mathbb{N}$ and for every nonnegative integer ℓ polynomial mappings $q_\ell\colon\mathbb{C}^n\times J_0^{\ell_0+\ell}(\mathbb{C}^n,\mathbb{C}^m)\to\mathbb{C}^m$ homogeneous of degree ℓ in their first variable such that for every germ of a holomorphic map $b\colon(\mathbb{C}^n,0)\to\mathbb{C}^m$, if $u\colon(\mathbb{C}^n,0)\to\mathbb{C}^m$ is a germ of a holomorphic map satisfying $\Theta(z)\cdot u(z)=b(z)$, then for every $\ell\geq 0$, $u_\ell(z)=q_\ell(z,j_0^{\ell_0+\ell}b)$, where u_ℓ denotes the homogeneous part of order ℓ of u. Furthermore, the q_ℓ can be chosen in such a way that there exists a polydisc $\Delta\subset\mathbb{C}^n$ centered at the origin for which the following holds: for all $C>0$ there exists $K>0$ with the property that whenever $b\colon(\mathbb{C}^n,0)\to\mathbb{C}^m$ is a germ of a holomorphic map written into homogeneous terms $b(z)=\sum_{k\geq 0}b_k(z)$ and satisfying $\|b_k\|_\Delta\leq C^k$ for all $k\in\mathbb{N}$, then $\left\|q_k(\cdot,j_0^{\ell_0+\ell}b)\right\|_\Delta\leq K^k$ for all $k\in\mathbb{N}$.*

The proof of Proposition 6.1 following our algorithm is given in §6.1. For our purposes, in addition to the preceding result, we need also the following refined version as well as a version with parameters given by Proposition 6.3 below, which follows immediately from Lemma 5.1 and Proposition 6.2.

Proposition 6.2. *Let Θ be an $m\times m$ matrix with holomorphic coefficients near the origin in $\mathbb{C}^n$, $m,n\geq 1$, such that Θ is of generic rank m. Then there exists $\ell_0\in\mathbb{N}$ and for every $\ell\in\mathbb{N}$ nonnegative integers $k(\ell)$ and polynomial mappings $p_\ell\colon\mathbb{C}^n\times J_0^{\ell_0+\ell}(\mathbb{C}^n)\times J_0^{\ell_0+\ell}(\mathbb{C}^n,\mathbb{C}^m)\to\mathbb{C}^m$ homogeneous of degree ℓ in their first variable such that for every germ of a holomorphic map $b\colon(\mathbb{C}^n,0)\to\mathbb{C}^m$ and every germ of a biholomorphic map $c\colon(\mathbb{C}^n,0)\to(\mathbb{C}^n,0)$, if $u\colon(\mathbb{C}^n,0)\to\mathbb{C}^m$ is a germ* *of a holomorphic map satisfying* $\Theta(c(z)) \cdot u(z) = b(z)$*, then for every* $\ell \geq 0$*,*

$$u_\ell(z) = \frac{p_\ell(z, j_0^{\ell_0+\ell}c, j_0^{\ell_0+\ell}b)}{(\det j_0^1 c)^{k(\ell)}},$$

where u_ℓ *denotes the homogeneous part of order* ℓ *of* u*. Furthermore, the* p_ℓ *can be chosen in such a way that there exists a polydisc* $\Delta \subset \mathbb{C}^n$ *centered at the origin for which the following holds: for all* $C > 0$ *and any compact subset* $L \subset \mathsf{GL_n}(\mathbb{C})$ *there exists* $D > 0$ *with the property that whenever* $b\colon (\mathbb{C}^n, 0) \to \mathbb{C}^m$ *is a germ of a holomorphic map and* $c\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ *is a germ of a biholomorphism (both written into homogeneous terms* $b(z) = \sum_{k\geq 0} b_k(z)$*,* $c(z) = \sum_{k\geq 0} c_k(z)$*) satisfying* $\|b_k\|_\Delta \leq C^k$*,* $\|c_k\|_\Delta \leq C^k$*, for all* $k \in \mathbb{N}$*, and* $c_1 \in L$*, then for every positive integer* ℓ*,*

$$\left\| \frac{p_\ell(\cdot, j_0^{\ell_0+\ell}c, j_0^{\ell_0+\ell}b)}{(\det j_0^1 c)^{k(\ell)}} \right\|_\Delta \leq D^\ell.$$

Proposition 6.3. *Let* Θ *be an* $m \times m$ *matrix with holomorphic coefficients near the origin in* $\mathbb{C}^n$*,* $m, n \geq 1$*, such that* Θ *is of generic rank* m*. Let* X *be a complex manifold, and assume that* $c\colon \mathbb{C}^n \times X \to \mathbb{C}^n$ *and* $b\colon \mathbb{C}^n \times X \to \mathbb{C}^m$ *are holomorphic maps defined on some neighbourhood of* $\{0\} \times X$*. Assume that* c *satisfies*

$$c(0,\omega) = 0, \quad \det c_z(0,\omega) \neq 0, \quad \text{for every } \omega \in X.$$

Then there exists a holomorphic map $\Gamma\colon \mathbb{C}^n \times X \to \mathbb{C}^m$ *defined on a neighbourhood of* $\{0\} \times X$ *such that if* $u\colon (\mathbb{C}^n, 0) \to \mathbb{C}^m$ *is a germ of a holomorphic map satisfying* $\Theta(c(z,\omega_0)) \cdot u(z) = b(z,\omega_0)$ *for some* $\omega_0 \in X$*, then* $u(z) = \Gamma(z,\omega_0)$*. Furthermore, if we write the Taylor expansions*

$$b(z,\omega) = \sum_{\alpha\in\mathbb{N}^n} b_\alpha(\omega) z^\alpha, \;\; c(z,\omega) = \sum_{\alpha\in\mathbb{N}^n} c_\alpha(\omega) z^\alpha,$$

then there exists an integer ℓ_0 *and for every* $\alpha \in \mathbb{N}^n$ *nonnegative integers* k_α *and polynomial mappings* $p_\alpha\colon J_0^{\ell_0+|\alpha|}(\mathbb{C}^n, \mathbb{C}^m) \times J_0^{\ell_0+|\alpha|}(\mathbb{C}^n) \to \mathbb{C}^n$ *such that*

$$\Gamma(z,\omega) = \sum_{\alpha\in\mathbb{N}^n} \frac{p_\alpha\big((b_\beta(\omega), c_\beta(\omega))_{|\beta|\leq \ell_0+|\alpha|}\big)}{(\det c_z(0,\omega))^{k_\alpha}}\, z^\alpha. \tag{6.1}$$

The proof of Proposition 6.2 is given in §6.2.

6.1. **Proof of Proposition 6.1.**

6.1.1. *Reducing the linear singular system to a simpler one.* A process similar to that used in §4.2 is done to bring the original $m \times m$ matrix Θ to another which has a convenient form. Given our $m\times m$ matrix $\Theta(z)$ with holomorphic coefficients near the origin in $\mathbb{C}^n$, for each $j = 1, \ldots, m$ we write $\Theta^j(z)$ for its j-th row and decompose the rows into homogeneous terms as $\Theta^j(z) = \sum_{k_j\leq k} \Theta^j_k(z)$ where Θ^j_k is a homogeneous polynomial map $\mathbb{C}^n \to \mathbb{C}^m$ of degree k. For $\nu \in \mathbb{N}$, we define Θ_ν to be the $m \times m$ matrix whose j-th row is given by $\Theta^j_{k_j+\nu}$. Hence the j-th row of Θ_ν is a homogeneous polynomial map of degree $k_j + \nu$. We say that Θ_0 is the matrix formed from Θ by taking its lowest-order homogeneous terms. We have the following reduction, analogous to that done in Lemma 4.5.

Lemma 6.4. *Let $\Theta(z)$ be an $m \times m$ matrix with holomorphic coefficients near the origin in $\mathbb{C}^n$ and assume that Θ is generically of full rank. Then there exists an $m \times m$ matrix $Q(z)$ of polynomials such that $N(z) := Q(z) \cdot \Theta(z)$ has the property that the matrix N_0 formed from N by taking the lowest-order homogeneous terms in each row is generically of full rank.*

Proof. Let e be the order of vanishing (at the origin) of $\det \Theta$ and define $D := e - \sum_{1\le j\le m} k_j$. First note that $\det \Theta_0$ is nonzero if and only if $D = 0$. In case $D = 0$, we are therefore done. Hence we may assume that $D > 0$. We claim that there is an $m \times m$ matrix $Q^{(1)}(z)$ of polynomials such that if we denote by $\tilde{k}_j$ the lowest-order homogeneous entry in the jth row of the matrix $\tilde{\Theta}(z) := Q^{(1)}(z)\Theta(z)$ and if we denote by $\tilde{e}$ the order of vanishing of $\det \tilde{\Theta}$, then $\tilde{D} := \tilde{e} - \sum_{1\le j\le m} \tilde{k}_j$ is strictly smaller than D. Indeed, if $\det \Theta_0 = 0$, there exist polynomials $p_1, \ldots, p_m \in \mathbb{C}[z]$ not all identically zero such that

$$p_1 \Theta^1_{k_1} + \cdots + p_m \Theta^m_{k_m} = 0. \tag{6.2}$$

Moreover, since each $\Theta^j_{k_j}$ is homogeneous of degree k_j, we may choose each polynomial p_j to be homogeneous of degree $\delta - k_j$ for some positive integer δ. Without loss of generality, we may also assume that $p_1 \neq 0$. Define

$$Q^{(1)} := \begin{pmatrix} p_1 & p_2 & \ldots & p_m \\ 0 & 1 & \ldots & 0 \\ & & \ddots & \\ 0 & \ldots & & 1 \end{pmatrix},$$

so that the order of vanishing of $\det \tilde{\Theta}$ is $e + \delta - k_1$ since $\det \tilde{\Theta} = p_1 \det \Theta$. Note that for $j = 2, \ldots, m$, $\tilde{k}_j = k + j$ and that it follows from (6.2) that the lowest-order homogeneous term of $\tilde{\Theta}^1$ is of order $\tilde{k}_1 > \delta$. Hence $\tilde{D} = e + \delta - k_1 - \sum_{1\le j\le m} \tilde{k}_j < e - \sum_{1\le j\le m} k_j = D$. This proves the claim. Therefore, after a finite number of these procedures, we arrive at a matrix N satisfying the property claimed in the lemma. $\square$

6.1.2. *Completion of the proof of Proposition* 6.1. Let Θ_0 be as defined in 6.1.1. By Lemma 6.4, there exists an $m \times m$ matrix $Q(z)$ of polynomials such that the matrix $N_0(z)$ formed from $N(z) := Q(z) \cdot \Theta(z)$ by taking the lowest-order homogeneous terms in each row is generically of full rank. Now observe that if we prove Proposition 6.1 for the matrix N the result easily follows for the original matrix Θ. So we may assume in what follows that the matrix Θ_0 is generically of full rank. Furthermore, by multiplying the rows with appropriate homogeneous polynomials, we may assume that the rows of Θ_0 are homogeneous of the same degree, say $\ell_0 \geq 0$. By Lemma 4.1, we may choose a polydisc Δ centered at the origin and a constant $D > 0$ and for all $\ell \geq 0$ a left inverse B_ℓ to the operator given by multiplication with Θ_0 (which takes $\mathcal{P}^m_\ell$ to $\mathcal{P}^m_{\ell_0+\ell}$), and which has norm at most D^{-1} (when $\mathcal{P}^m_\ell$ and $\mathcal{P}^m_{\ell_0+\ell}$ are equipped with $\|\cdot\|_\Delta$). We define inductively the polynomial maps

$(q_\ell)_{\ell\geq 0}$ as follows:
(6.3)
$$\begin{cases} q_0(\Lambda^{\ell_0}) := B_0\left(\sum_{|\alpha|=\ell_0}\dfrac{\Lambda_\alpha}{\alpha!}z^\alpha\right), \\ q_\ell(z,\Lambda^{\ell_0+\ell}) := B_\ell\left(\sum_{|\alpha|=\ell_0+\ell}\dfrac{\Lambda_\alpha}{\alpha!}z^\alpha - \displaystyle\sum_{j=0}^{\ell-1}\Theta_{\ell-j}(z)\cdot q_j(z,\Lambda^{\ell_0+j})\right),\ \ell\geq 1. \end{cases}$$

It is easy to show by induction on ℓ that each $p_\ell\colon \mathbb{C}^n\times J_0^{\ell_0+\ell}(\mathbb{C}^n,\mathbb{C}^m)\to\mathbb{C}^m$ is a polynomial map that is homogeneous of degree ℓ in its first variable z. We claim that given a germ of a holomorphic map $b\colon(\mathbb{C}^n,0)\to\mathbb{C}^m$ if $u\colon(\mathbb{C}^n,0)\to\mathbb{C}^m$ is a germ of a holomorphic map satisfying

$$\Theta(z)\cdot u(z) = b(z), \tag{6.4}$$

then necessarily $u_\ell(z) = q_\ell(z, j_0^{\ell_0+\ell}b)$. Indeed first note that by considering homogeneous terms of order ℓ_0 in (6.4), we get $\Theta_0(z)\cdot u_0(z) = b_{\ell_0}(z)$ and hence $u_0(z) = q_0(j_0^{\ell_0}b)$. Now assume that $u_k(z) = q_k(z, j_0^{\ell_0+k}b)$ for all $k<\ell$. Considering homogeneous terms of order $\ell+\ell_0$ in (6.4), we obtain

$$\Theta_0(z)\cdot u_\ell(z) + \sum_{\nu=1}^{\ell}\Theta_\nu(z)\cdot u_{\ell-\nu}(z) = b_{\ell_0+\ell}(z),$$

which easily gives the required conclusion by the induction assumption.

The convergence statement can now be derived by a similar but much simpler argument as in the proof of Theorem 3.1. Indeed, let $C>0$ and let $b\colon(\mathbb{C}^n,0)\to\mathbb{C}^m$ be a germ of a holomorphic map satisfying $\|b_k\|_\Delta \leq C^k$ for all $k\in\mathbb{N}$, where $b(z) = \sum_{k\geq 0} b_k(z)$ is the decomposition of b into homogeneous terms; without loss of generality we may assume that $\ell_0>0$. Since Θ is convergent, there exists $a>0$ such that $\|\Theta_j\|_\Delta \leq a^j$. We first claim that for any $\sigma_0\geq 0$, the series $\varphi(t) := \sum_{j\in\mathbb{N}}\sigma_j t^j$ converges in a neighbourhood of 0, where σ_j is defined for $j\geq 1$ by

$$D\sigma_j = \sum_{k<j} a^{j-k}\sigma_k + C^{j+\ell_0}. \tag{6.5}$$

This follows from the fact that $\varphi(t)$ is a solution of the equation

$$D\varphi(t) - D\sigma_0 = \frac{at}{1-at}\varphi(t) + \frac{tC^{\ell_0+1}}{1-Ct}.$$

Now, we set $\sigma_0 := C^{\ell_0}D^{-1}$ and let $(\sigma_j)_{j\geq 1}$ be as given by (6.5). Then by using (6.3) and (6.5) it is easy to show by induction that for all $\ell\in\mathbb{N}$, $\left\|q_\ell(\cdot, j_0^{\ell_0+\ell}b)\right\|_\Delta \leq \sigma_\ell$. Since $\sum_{j\geq 0}\sigma_j t^j$ is convergent, there exists $K>0$ (independent of b) such that $\sigma_\ell \leq K^\ell$ for all $\ell\in\mathbb{N}$. This completes the proof of Proposition 6.1.

6.2. **Proof of Proposition 6.2.** Actually, this proposition is a simple consequence of Proposition 6.1 and the Inverse Function Theorem. We will, of course, need a version of the latter which is compatible with the desired estimates. So we first turn to this. The reader will notice that the results we derive here are by no means new. Nevertheless we think it is important to put the proofs of the precise desired results here. We first have the following lemma.

Lemma 6.5. *For every positive integer ℓ, there exists a polynomial map $\rho_\ell\colon \mathbb{C}^n \times J_0^\ell(\mathbb{C}^n) \to \mathbb{C}^n$, homogeneous of degree ℓ in its first variable, such that for any germ of a biholomorphic map $c\colon (\mathbb{C}^n,0) \to (\mathbb{C}^n,0)$, written in homogeneous terms $c(z) = \sum_{j\geq 1} c_j(z)$, the inverse $c^{-1}(z)$ is given by*

$$c^{-1}(z) = \sum_{\ell>0} \frac{\rho_\ell(z, j_0^\ell c)}{(\det c_1)^\ell}.$$

Furthermore, there exists a polydisc $\Delta \subset \mathbb{C}^n$ centered at the origin such that for any $C > 0$ and any compact subset $L \subset \mathsf{GL_n}(\mathbb{C})$ there exists $K > 0$ such that for every $c\colon (\mathbb{C}^n,0) \to (\mathbb{C}^n,0)$ as above satisfying $c_1 \in L$ and $\|c_k\|_\Delta \leq C^k$ for every $k \in \mathbb{N}^*$, then for every $\ell \in \mathbb{N}^*$,

$$\left\| \frac{\rho_\ell(\cdot, j_0^\ell c)}{(\det c_1)^\ell} \right\|_\Delta \leq K^\ell.$$

Proof. We set $\widehat{\rho}_1(z,\Lambda^1) := (\Lambda^1)^{-1}\cdot z$ and define the polynomials ρ_ℓ, $\ell \geq 2$, inductively as follows (here we use again the notation $u_\ell(z) = u(z)|_\ell$):

$$\text{(6.6)} \qquad \begin{cases} \widehat{\rho}_\ell(z,\Lambda^\ell) := -(\Lambda^1)^{-1} \left(\sum_{1<j\leq d} \sum_{|\alpha|=j} \dfrac{\Lambda_\alpha}{\alpha!} \left((R^{\ell-1}(z,\Lambda^{\ell-1}))^\alpha \right)|_\ell \right), \\ R^{\ell-1}(z,\Lambda^{\ell-1}) := \sum_{1\leq \nu \leq \ell-1} \widehat{\rho}_\nu(z,\Lambda^{|\nu|}). \end{cases}$$

It is easy to see that for every $\ell \geq 1$, $\rho_\ell(z,\Lambda^\ell) := (\det c_1)^\ell\, \widehat{\rho}_\ell(z,\Lambda^\ell)$ satisfies the required conditions of Lemma 6.5. It remains therefore to check the estimates for the $\widehat{\rho}_\ell$. So let Δ be the unit polydisc in $\mathbb{C}^n$, L a compact subset of $\mathsf{GL_n}(\mathbb{C})$, $C > 0$ and suppose that $c\colon (\mathbb{C}^n,0) \to (\mathbb{C}^n,0)$ is a germ of a biholomorphic map satisfying $\|c_k\|_\Delta \leq C^k$ for all $k > 0$. We set

$$A := \min_{\lambda\in L} \frac{1}{\|\lambda^{-1}\|}, \quad B_1 := \max_{\lambda\in L} \left\|\lambda^{-1}\right\|.$$

(Here, as in Lemma 4.1, the matrix norm is adapted to the maximum norm on $\mathbb{C}^n$.) Let $\widetilde{C} > 0$ (independent of c) be such that $\|c_j\|_1 \leq \widetilde{C}^j$ for all $j > 0$. Consider the one-dimensional equation

$$\text{(6.7)} \qquad A\phi(t) - \frac{\widetilde{C}^2\phi(t)^2}{1-\widetilde{C}\phi(t)} = AB_1 t.$$

It is then easy to check that (6.7) has a unique convergent solution

$$\phi(t) = \sum_{\ell>0} B_\ell\, t^\ell$$

vanishing at the origin, which furthermore satisfies $B_\ell > 0$ for all $\ell > 0$. Moreover, Proposition 4.2 implies that the B_ℓ satisfy the following relation for $\ell \geq 2$:

$$\text{(6.8)} \qquad AB_\ell = \sum \frac{d!C^d}{k_1!\ldots k_{\ell-1}!} B_1^{k_1}\ldots B_{\ell-1}^{k_{\ell-1}},$$

where $d = k_1 + \cdots + k_{\ell-1} \geq 2$, and the sum goes over all nonnegative integers $k_1,\ldots,k_{\ell-1}$ with $k_1 + 2k_2 + \cdots + (\ell-1)k_{\ell-1} = \ell$. We claim that for all $\ell \geq 1$, $\left\|\widehat{\rho}_\ell(\cdot, j_0^\ell c)\right\|_\Delta \leq B_j$. For $\ell = 1$ this is clear by the choice of B_1 and the definition of

$\widehat{\rho}_1$. Suppose now that $\left\|\widehat{\rho}_d(\cdot, j_0^d c)\right\|_\Delta \leq B_d$ for $d < \ell$. Using (6.6) and Lemma 4.3, we get the following estimates:

$$\begin{aligned}
\left\|\widehat{\rho}_\ell(\cdot, j_0^\ell c)\right\|_\Delta &\leq \left\|c_1^{-1}\right\| \left\| \sum_{1<j\leq \ell} \left(c_j(R^{\ell-1}(z, j_0^{\ell-1}c))\right)|_\ell \right\|_\Delta \\
&\leq \left\|c_1^{-1}\right\| \sum_{2\leq j\leq \ell} \frac{j!}{k_1 \dots k_{\ell-1}!} \|c_j\|_1 \|\widehat{\rho}_1\|_\Delta^{k_1} \dots \|\widehat{\rho}_{\ell-1}\|_\Delta^{k_{\ell-1}},
\end{aligned}$$

where $j = k_1 + \cdots + k_{\ell-1}$, and the sum goes over all nonnegative integers $k_1, \dots, k_{\ell-1}$ with $k_1 + 2k_2 + \cdots + (\ell-1)k_{\ell-1} = \ell$. Hence, by (6.8) and the induction assumption we get

$$\begin{aligned}
\left\|\widehat{\rho}_\ell(\cdot, j_0^\ell c)\right\|_\Delta &\leq \left\|c_1^{-1}\right\| \sum_{2\leq j\leq \ell} \frac{j!}{k_1 \dots k_{\ell-1}!} C^d B_1^{k_1} \dots B_{\ell-1}^{k_{\ell-1}} \\
&= \left\|c_1^{-1}\right\| A B_\ell \leq B_\ell,
\end{aligned}$$

by our choice of A, which proves the claim and also completes the proof of the lemma. □

Remark 6.6. Note that Lemma 6.5 also holds for *every* polydisc $\Delta \subset \mathbb{C}^n$ centered at the origin.

We also need the following lemma about the composition of analytic functions with estimates.

Lemma 6.7. *Let $K > 0$ and assume that $\Delta \subset \mathbb{C}^N$ and $\Delta' \subset \mathbb{C}^n$ are polydiscs centered at* 0*. There exists $C > 0$ such that whenever $f\colon (\mathbb{C}^n, 0) \to \mathbb{C}^m$ and $g\colon (\mathbb{C}^N, 0) \to (\mathbb{C}^n, 0)$ are germs of holomorphic mappings written into homogeneous terms $f = \sum_{j\geq 0} f_j$, $g = \sum_{j\geq 1} g_j$ and satisfying $\|g_j\|_\Delta \leq K^j$ and $\|f_j\|_{\Delta'} \leq K^j$ for every positive integer j, then the composition $h = f \circ g = \sum_{j\geq 0} h_j$ satisfies $\|h_j\|_\Delta \leq C^j$ for $j > 0$.*

Before we begin with the proof, let us point out that there are easier proofs for this lemma than the one we give; we prefer to give a proof using majorization by a conveniently chosen power series because this ties in nicely with the proofs of the other convergence estimates given before.

Proof. We use the geometric sum to compute

$$\frac{K_1 \frac{Kt}{1-Kt}}{1 - K_1 \frac{Kt}{1-Kt}} = \frac{K_1 K t}{1 - K(1+K_1)t} = \sum_{j>0} K_1 K^j (1+K_1)^{j-1} t^j. \tag{6.9}$$

Now choose K_1 large enough (depending only on K and Δ') so that $\|f_j\|_1 \leq K_1^j$ for all $j > 0$, for all f satisfying $\|f_j\|_{\Delta'} \leq K^j$. We claim that $\|h_j\|_\Delta \leq K_1 K^j (1+K_1)^{j-1}$ for $j > 0$. This is proved using Lemma 4.3 and Faa di Bruno's formula (Proposition 4.2). For every $j > 0$, we have the following estimates:

$$\begin{aligned}
\|h_j\|_\Delta &\leq \sum_{d=1}^{j} \|(f_d \circ g)|_j\|_\Delta \\
&\leq \sum \frac{d!\,\|f_d\|_1}{k_1! \dots k_j!} \|g_1\|_\Delta^{k_1} \dots \|g_j\|_\Delta^{k_j},
\end{aligned}$$

where $d = k_1 + \ldots + k_j \geq 1$ and the sum goes over all nonnegative integers $k_1, \ldots, k_j$ such that $k_1 + 2k_2 + \ldots + jk_j = j$. Using (6.9), we get

$$\begin{aligned} \|h_j\|_\Delta &\leq \sum \frac{d!\, K_1^d}{k_1! \ldots k_j!} K^{k_1+2k_2+\ldots jk_j} \\ &= K_1 K^j (1+K_1)^{j-1}. \end{aligned}$$

Setting $C := K(1+K_1)$ yields the desired result. □

Completion of the proof of Proposition 6.2. We can now prove Proposition 6.2. Let ℓ_0 be given by Proposition 6.2 and for every $k \in \mathbb{N}$, let q_k be the polynomial map given by the same proposition. Define for every nonnegative integer ℓ the map $\tilde{p}_\ell$ by setting

$$\tilde{p}_\ell(z, j_0^{\ell+\ell_0} c, j_0^{\ell+\ell_0} b) = \left(\sum_{k \in \mathbb{N}} q_k(c(z), j_0^{\ell+\ell_0}(b \circ c^{-1})) \right) |_\ell, \tag{6.10}$$

for all germs of holomorphic maps b and c as in Proposition 6.2, where, as used before, the notation $|_\ell$ on the right-hand side of (6.10) denotes the homogeneous part of order ℓ of the corresponding power series mapping. It is clear from Lemma 6.5 and the chain rule that for every $\ell \in \mathbb{N}$, there exists $k(\ell) \in \mathbb{N}$ such that for every b and c as above we have

$$(\det j_0^1 c)^{k(\ell)}\, \tilde{p}_\ell(z, j_0^{\ell+\ell_0} c, j_0^{\ell+\ell_0} b) =: p_\ell(z, j_0^{\ell+\ell_0} c, j_0^{\ell+\ell_0} b)$$

for some polynomial map satisfying the required properties. Finally, we leave the reader to check that the needed estimates follow from the estimates given in Lemma 6.5 and Lemma 6.7 and from Remark 6.6. The proof of Proposition 6.2 is complete. □

Remark 6.8. Let $\mathcal{H}(\mathbb{C}^n, \mathbb{C}^m)$ be the topological vector space of germs of holomorphic maps $(\mathbb{C}^n, 0) \to \mathbb{C}^m$ (endowed with the topology of uniform convergence on compact neighbourhoods of the origin in $\mathbb{C}^n$) and let $\Theta = \Theta(z)$ be an $m \times m$ matrix with entries in $\mathcal{H}(\mathbb{C}^n, \mathbb{C})$ with Θ of generic rank m. Define the holomorphic map $\Theta_* \colon \mathcal{B}^n \times \mathcal{H}(\mathbb{C}^n, \mathbb{C}^m) \to \mathcal{B}^n \times \mathcal{H}(\mathbb{C}^n, \mathbb{C}^m)$ by setting $\Theta_*(c, u) = (c, (\Theta \circ c) \cdot u)$ for all $u \in \mathcal{H}(\mathbb{C}^n, \mathbb{C}^m)$ and $c \in \mathcal{B}^n$. Then we leave it to the reader to check that, following the lines of the proof of Theorem 2.4 given in §5, the statement given by Proposition 6.2 yields the existence of a holomorphic left inverse $\Phi \colon \mathcal{B}^n \times \mathcal{H}(\mathbb{C}^n, \mathbb{C}^m) \to \mathcal{B}^n \times \mathcal{H}(\mathbb{C}^n, \mathbb{C}^m)$ to the map Θ_*.

7. A class of real-analytic generic submanifolds

We introduce here a class of real-analytic generic submanifolds, more general than the class of essentially finite generic submanifolds, and state the most general parametrization theorem of this paper (Theorem 7.3) for such a class; all the results stated in §2 will then follow from that theorem. We also discuss a few properties of this class of manifolds, compare it to some well-known other ones and also give several examples of manifolds in this class that are not essentially finite.

7.1. **Nondegeneracy conditions for real-analytic generic submanifolds.** We start by introducing a nondegeneracy condition for real-analytic CR submanifolds of $\mathbb{C}^N$ and then compare it to some well-known ones. For more background concerning CR structures we refer the reader, e.g., to the books [8, 10].

Let (M,p) be a germ of a real-analytic generic submanifold of $\mathbb{C}^N$, i.e., satisfying $T_pM + J(T_pM) = T_p\mathbb{C}^N$, where J is the complex structure map of $\mathbb{C}^N$ and T_pM (resp. $T_p\mathbb{C}^N$) denotes the tangent space of M (resp. of $\mathbb{C}^N$) at p. Let n and d be respectively the CR dimension and the codimension of the manifold M so that $N = n + d$. Let $\rho = (\rho_1, \ldots, \rho_d)$ be a real-analytic vector-valued defining equation for M in some neighbourhood U of p in $\mathbb{C}^N$ satisfying $\partial\rho_1 \wedge \ldots \wedge \partial\rho_d \neq 0$ in U. For every point $q \in \mathbb{C}^N$ sufficiently close to p, recall that the Segre variety attached to q is the n-dimensional complex submanifold of $\mathbb{C}^N$ given by

$$S_q := \{Z \in U : \rho(Z, \overline{q}) = 0\},$$

where the original real-analytic map ρ has been complexified and U may have been shrunk so that $\rho(Z,\zeta)$ is now convergent on $U \times \overline{U}$. On the other hand, the complexification of M is the germ through the point $(p,\bar{p}) \in \mathbb{C}^{2N}$ of the $(2n+d)$-dimensional complex submanifold of $\mathbb{C}^{2N}$ given by

$$\mathcal{M} := \{(Z,\zeta) \in (\mathbb{C}^{2N}, (p,\bar{p})) : \rho(Z,\zeta) = 0\}. \tag{7.1}$$

Here and throughout the paper, given any holomorphic map η defined on some open subset Ω of $\mathbb{C}^k_t$ and given any point $a \in \Omega$, we denote by $\{t \in (\mathbb{C}^k, a) : \eta(t) = 0\}$ the germ at a of the complex-analytic set $\{t \in \Omega : \eta(t) = 0\}$.

For every integer k and for $u \in \mathbb{C}^N$, let $E^{k,n}_u(\mathbb{C}^N)$ be the jet space of order k at q of n-dimensional complex submanifolds of $\mathbb{C}^N$ passing through u. Then $\bigcup_{u\in\mathbb{C}^N} E^{k,n}_u(\mathbb{C}^N)$ carries a natural fiber bundle structure over $\mathbb{C}^N$. Following [18, 7], for every $q \in M$ sufficiently close to p, we consider the antiholomorphic map π^k_q defined as follows:

$$\pi^k_q \colon S_q \to E^{k,n}_q(\mathbb{C}^N), \quad \pi^k_q(\xi) = j^k_q S_\xi, \tag{7.2}$$

where $j^k_q S_\xi$ denotes the k-jet at q of the submanifold S_ξ (see e.g. [25]).

Definition 7.1. *We say that the germ* (M,p) *belongs to the class* $\mathcal{C}$ *if the antiholomorphic map* π^k_p *is of generic rank* n *for* k *large enough* (*or, equivalently, is generically immersive for* k *large enough*).

For a germ $(M,p) \in \mathcal{C}$, we denote by $\kappa_M(p)$ the smallest integer k for which the map π^k_p is of generic rank n. Clearly, the integer $\kappa_M(p)$ is a biholomorphic invariant attached to the germ (M,p). Furthermore, for $q \in M$ sufficiently close to p, we see that, shrinking the neighbourhood U if necessary, that S_q depends antiholomorphically on q (i.e., can be parametrized by a holomorphic map depending anti-holomorphically on q; see e.g. §7.3 below for more details if needed). By unique continuation it follows that if $(M,p) \in \mathcal{C}$, then the map $\pi^{\kappa_M(p)}_q$ is still generically immersive for $q \in M$ close enough to p, i.e., that (M,q) is in $\mathcal{C}$. Hence, there exists a neighbourhood V of p in M such that κ_M is well defined on V and from the definition we have that κ_M is *upper-semicontinuous*.

It is interesting to compare the above local open condition with other well-known ones. We recall the following nondegeneracy conditions for a germ (M,p) of a real-analytic generic submanifold of $\mathbb{C}^N$:

(I) the submanifold M is *Levi-nondegenerate* at p if the map π_p^1 is an immersion at p;

(II) the submanifold M is said to be *finitely nondegenerate* at p if the map π_p^k is an immersion at p for k sufficiently large; in this case, M is said to be k_p-nondegenerate at p if k_p is the smallest k for which the above condition holds (see [6, 8]);

(III) the submanifold M is said to be *essentially finite* at p if the antiholomorphic map π_p^k is finite for k large enough; in this case, the smallest of such k's is called the *essential type* of M at p and denoted $\mathrm{esstype}_M(p)$ (see [7]);

(IV) the submanifold M is said to be *holomorphically nondegenerate* at p (in the sense of Stanton [41]) if for a generic point $q \in M$ sufficiently close to p the antiholomorphic map π_q^k is generically immersive for k large enough. (Equivalently, for a generic point $q \in M$ sufficiently close to p, $(M, q) \in \mathcal{C}$.)

The definition of holomorphic nondegeneracy given in (IV) does not correspond here to the original one, but it is not difficult to show by using Stanton's criterion of holomorphic nondegeneracy (see Proposition 7.6 below) that it is in fact equivalent to the original one. It is well known that (I)$\Rightarrow$(II)$\Rightarrow$(III)$\Rightarrow$(IV) (but converses do not hold). It is also clear from the above definitions that if $(M, p) \in \mathcal{C}$, then (M, p) is holomorphically nondegenerate and that if (M, p) is essentially finite, then $(M, p) \in \mathcal{C}$. Hence, the essentially finite submanifolds form one general subclass of submanifolds that are in the class $\mathcal{C}$. Another subclass of the class $\mathcal{C}$ that is worth pointing out consists of those germs (M, p) of generic real-analytic submanifolds that are rigid and holomorphically nondegenerate (see §7.3 for the definition and details). We summarize the above in the following:

Proposition 7.2. *Consider the following conditions on a germ (M, p) of a real-analytic generic submanifold of $\mathbb{C}^N$:*

(i) *(M, p) is essentially finite;*

(ii) *(M, p) does not contain any nontrivial complex analytic subvariety;*

(iii) *(M, p) is rigid and holomorphically nondegenerate.*

If (M, p) satisfies one of the above three conditions, then $(M, p) \in \mathcal{C}$.

In §7.3, we shall provide the reader with an elementary example of a germ of a submanifold belonging to the class $\mathcal{C}$ that is not essentially finite nor rigid holomorphically nondegenerate and also give a more explicit criterion (Proposition 7.5) that allows one to decide when a given germ of a submanifold is in $\mathcal{C}$ or not.

Another nondegeneracy condition on a germ (M, p) we also need to recall is the minimality condition due to Tumanov [43]: (M, p) is called minimal if there is no proper CR submanifold $S \subset M$ passing through p with the same CR dimension as that of M. As is well known (see [8]), for real-analytic generic submanifolds, this condition is equivalent to the finite type condition of Kohn [31] and Bloom-Graham [9] on the CR vector fields of M.

7.2. **The main result of the paper.** The most general parametrization theorem we shall prove in this paper is the following.

Theorem 7.3. *Let M be a real-analytic generic submanifold of $\mathbb{C}^N$ of codimension d. Let $p \in M$ and assume that (M, p) is minimal and belongs to the class $\mathcal{C}$ and set $\ell_p := (d+1)\kappa_M(p)$. Then there exists an open subset $\Omega \subset \mathbb{C}^N \times G_{p,p}^{\ell_p}(\mathbb{C}^N)$ and a* *real-analytic map* $\Psi(Z,\Lambda)\colon \Omega \to \mathbb{C}^N$ *holomorphic in the first factor, such that the following hold:*

(i) *for any* $H \in \mathrm{Aut}(M,p)$ *the point* $(p, j_p^{\ell_p} H)$ *belongs to* Ω *and the following identity holds:*

$$H(Z) = \Psi(Z, j_p^{\ell_p} H) \text{ for all } Z \in \mathbb{C}^N \text{ near } p;$$

(ii) *the map* Ψ *has the following formal Taylor expansion*

$$\Psi(Z,\Lambda) = \sum_{\alpha \in \mathbb{N}^N} \frac{P_\alpha(\Lambda, \overline{\Lambda})}{(D(\Lambda^1)^{s_\alpha} (\overline{D(\Lambda^1)})^{r_\alpha}} (Z-p)^\alpha, \tag{7.3}$$

where for every $\alpha \in \mathbb{N}^N$, s_α *and* r_α *are nonnegative integers,* P_α *and* D *are polynomials in their arguments,* Λ^1 *denotes the linear part of the jet* Λ *and* $D(j_0^1 H) \neq 0$ *for every* $H \in \mathrm{Aut}(M,p)$.

Let us remark here that the set Ω appearing in the statement of Theorem 7.3 is only determined by some algebraic conditions on the first-order jets, and these conditions are automatically fulfilled for jets of CR automorphisms of M.

Quite analogously to §2, we obtain by following some arguments from [8, 3] the following result from Theorem 7.3.

Theorem 7.4. *Let* M *be a real-analytic generic submanifold of* $\mathbb{C}^N$ *of codimension* d. *Let* $p \in M$ *and assume that* (M,p) *is minimal and belongs to the class* $\mathcal{C}$ *and set* $\ell_p := (d+1)\kappa_M(p)$. *Then the jet mapping*

$$j_p^{\ell_p}\colon \mathrm{Aut}(M,p) \to G_{p,p}^{\ell_p}(\mathbb{C}^N) \tag{7.4}$$

is a continuous group homomorphism that is a homeomorphism onto a real-algebraic Lie subgroup of $G_{p,p}^{\ell_p}(\mathbb{C}^N)$.

Another special class of real-analytic generic submanifolds of $\mathbb{C}^N$ that has been much studied in recent years is that of finitely nondegenerate ones. For a germ of a k_p-nondegenerate real-analytic generic submanifold (M,p), Theorem 7.3 (and hence Theorem 7.4) has been established by Baouendi, Ebenfelt and Rothschild in [3] (see also [47] for a previous weaker version) with a bound $\ell_p' = (d+1)k_p$. We would like to point out that it follows from the discussion in §7.1 that we always have $\ell_p \le \ell_p'$ with ℓ_p given by Theorem 7.3. Furthermore, we give in §7.3 a number of examples of finitely nondegenerate submanifolds for which the above inequality is in fact strict (see Example 7.8). This shows that Theorem 7.3 improves also the known results for finitely nondegenerate submanifolds.

The proof of Theorem 7.3 is given throughout §8–§10 and is completed in §11 together with the proof of Theorem 7.4.

7.3. **Some properties of the class $\mathcal{C}$ and comparison between known invariants.** Our goal here is to give a characterization (Proposition 7.5) by using local holomorphic coordinates of germs of real-analytic generic submanifolds that belongs to $\mathcal{C}$.

Let (M,p) be a germ of a real-analytic generic submanifold of $\mathbb{C}^N$. Following [8], we use a system of *normal coordinates* $(z,w) \in \mathbb{C}^n \times \mathbb{C}^d$ for (M,p), which means that in these coordinates $p=0$ and that there is a defining equation for M near the origin of the form

$$\mathrm{Im}\, w = \varphi(z,\bar z, \mathrm{Re}\, w), \quad \varphi(z,0,s) = \varphi(0,\bar z, s) = 0, \tag{7.5}$$

where $\varphi\colon \mathbb{C}^n \times \mathbb{R}^d \to \mathbb{R}^d$ is a germ of a real-analytic map at the origin. By the implicit function theorem, (7.5) is also equivalent to a defining equation of the following form:

$$w = Q(z, \bar{z}, \bar{w}), \tag{7.6}$$

with $Q(z, 0, \bar{w}) = Q(0, \bar{z}, \bar{w}) = \bar{w}$; here $Q\colon \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}^d \to \mathbb{C}^d$ is a germ of a holomorphic map at the origin. We write and use the following expansions:

$$\bar{Q}(\chi, z, w) = \sum_{\alpha \in \mathbb{N}^n} \bar{q}_\alpha(z, w)\chi^\alpha,\ \bar{Q} = (\bar{Q}^1, \dots, \bar{Q}^d),\ \bar{q}_\alpha = (\bar{q}^1_\alpha, \dots, \bar{q}^d_\alpha). \tag{7.7}$$

Here and throughout the rest of the paper, for any holomorphic function $h\colon \mathbb{C}^r \to \mathbb{C}$ defined near the origin in $\mathbb{C}^r$, we denote by $\overline{h}\colon \mathbb{C}^r \to \mathbb{C}$ the holomorphic function obtained from h by taking the complex conjugates of the coefficients of its Taylor series. In the above normal coordinates, the complexification $\mathcal{M}$ is given by

$$\mathcal{M} := \{((z, w), (\chi, \tau)) \in (\mathbb{C}^N \times \mathbb{C}^N, 0) : \tau = \bar{Q}(\chi, z, w)\}, \tag{7.8}$$

and for every integer k, the mapping π_p^k defined in (7.2) may be identified with the conjugate of the local holomorphic map $z \mapsto (\alpha!\, \overline{q}_\alpha(z, 0))_{|\alpha| \le k}$. As a consequence, we have the following characterization.

Proposition 7.5. *Let (M, p) be a germ of a real-analytic generic submanifold in $\mathbb{C}^N$ and let (z, w) be normal coordinates for (M, p) so that M is given by* (7.6). *Then the following are equivalent:*

(i) $(M, p) \in \mathcal{C}$*;*
(ii) *for some integer k, the local holomorphic map $z \mapsto (\bar{q}_\alpha(z, 0)\colon |\alpha| \le k)$ is generically of full rank.*

Furthermore, if any of these conditions is satisfied, the integer $\kappa_M(p)$ is the smallest integer k for which condition (ii) *holds.*

As already mentioned in §7.1, the first natural class of manifolds that belongs to $\mathcal{C}$ is given by the class of essentially finite ones. Another one is given by the class of rigid holomorphically nondegenerate submanifolds. Recall that (M, p) is called *rigid* if there exist normal coordinates $(z, w) \in \mathbb{C}^n \times \mathbb{C}^d$ such that M is given by an equation of the form (7.5) with the additional requirement that φ does not depend on s, i.e., $\varphi(z, \bar{z}, s) = \varphi(z, \bar{z})$ (see [8]). The fact that the class of rigid holomorphically nondegenerate real-analytic generic submanifolds is contained in $\mathcal{C}$ follows from Proposition 7.5 and Stanton's criterion of holomorphic nondegeneracy whose statement we now recall.

Proposition 7.6 ([41, 8])**.** *Let (M, p) be a germ of a real-analytic generic submanifold in $\mathbb{C}^N$ and let (z, w) be normal coordinates for (M, p) so that M is given by* (7.6)*. Then the following are equivalent:*

(i) *(M, p) is holomorphically nondegenerate;*
(ii) *for some integer k, the local holomorphic map $(z, w) \mapsto (\bar{q}_\alpha(z, w)\colon |\alpha| \le k)$ is of generic rank N.*

We now exhibit examples of generic submanifolds that belong to the class $\mathcal{C}$ without being essentially finite nor rigid holomorphically nondegenerate.

Example 7.7. Let $M \subset \mathbb{C}^2_z \times \mathbb{C}_w$ be defined by

$$\operatorname{Im} w = |z_1|^2 + (\operatorname{Re} w)\,|z_2|^2.$$

Let us check that M is neither essentially finite nor rigid; by using Proposition 7.5 one sees that it does belong to $\mathcal{C}$, though. Moreover, we have $\kappa_M(0) = 2$. Indeed, rewriting in complex form, the defining equation of M is given by

$$w = \bar{w}\frac{1 + i|z_2|^2}{1 - i|z_2|^2} + 2i\frac{|z_1|^2}{1 - i|z_2|^2} = Q(z, \bar{z}, \bar{w}).$$

Thus, $Q(z, \bar{z}, 0)$ is given by

$$Q(z, \bar{z}, 0) = 2i\sum_{j=0}^{\infty} i^j |z_1|^2 |z_2|^{2j}.$$

Using this expansion, we can now apply Proposition 7.5 to see that $M \in \mathcal{C}$; however, M is not essentially finite since the map $\chi \mapsto (\bar{q}_\alpha(\chi))$ is, up to constant factors and uninteresting components, just the map

$$\chi \mapsto (\chi_1, \chi_1\chi_2, \chi_1\chi_2^2, \dots),$$

which is not finite. To check that M is not rigid, note that if M is rigid, there exists a (germ of a real-analytic) curve $(\mathbb{R}, 0) \ni r \mapsto C(r) = (z(r), w(r)) = (z(r), s(r)+it(r))$ in M with $C(0) = 0$ and $s'(0) \neq 0$ such that the germs $(M, 0)$ and $(M, C(r))$ are biholomorphically equivalent for all r close to 0. Such a curve is obtained as follows: since M is assumed to be rigid, in suitable local holomorphic coordinates $(\zeta, \eta) \in \mathbb{C}^2 \times \mathbb{C}$, M is given by an equation of the form $\operatorname{Im} \eta = \varphi(\zeta, \bar{\zeta})$ with φ real-analytic near the origin; in these coordinates the claimed curve is just the line $(0, r)$, $r \in (\mathbb{R}, 0)$. On the other hand, $(M, 0)$ contains the complex line $(0, z_2, 0)$ while for $(z, s + it) \in \mathbb{C}^2 \times \mathbb{C}$ close to 0 with $s \neq 0$ the germ $(M, (z, s + it))$ does not contain any complex curves through $(z, s + it)$ since a calculation in the given coordinates shows that M is actually strongly pseudoconvex at all such points. The germs $(M, 0)$ and $(M, C(r))$ can therefore not be biholomorphically equivalent for $r \neq 0$ close to 0, which shows that M is not rigid.

Suppose now that $(M, 0)$ is a germ of a real-analytic k_0-nondegenerate generic submanifold (as defined in §7.1). Then obviously $(M, 0) \in \mathcal{C}$ and one has $\kappa_M(0) \leq k_0$. In the following example, we show that this inequality may be strict. In fact, this example shows for every $k \geq 2$ there exists a k-nondegenerate real-analytic hypersurface $(M_k, 0)$ in $\mathbb{C}^3$ for which $\kappa_{M_k}(0) = 1 < k$.

Example 7.8. Let $(M_k, 0) \subset \mathbb{C}^3_{(z_1,z_2,w)}$ be defined by

$$\operatorname{Im} w = \operatorname{Re}\left((z_1 + z_2)\bar{z}_2^{\,k}\right) + |z_1|^2.$$

Moreover, analogous examples of real-analytic hypersurfaces or generic submanifolds may be constructed in $\mathbb{C}^N$ for arbitrary N; this is left to the reader as an easy exercise. Similarly, for essentially finite submanifolds, we have the following.

Example 7.9. For every $k \geq 3$, let $(\widetilde{M}_k, 0) \subset \mathbb{C}^3_{(z_1,z_2,w)}$ be defined by

$$\operatorname{Im} w = |z_1|^2 + |z_1 z_2|^2 + |z_2|^{2k}.$$

Then $\widetilde{M}_k$ is essentially finite at the origin (but not finitely nondegenerate) and $\operatorname{esstype}_{\widetilde{M}_k}(0) = k$ while $\kappa_{\widetilde{M}_k}(0) = 2 < \operatorname{esstype}_{\widetilde{M}_k}(0)$.

8. Spaces of holomorphic maps in jet spaces

Recall that for any positive integer k, we denote the space of jets at the origin of order k of holomorphic mappings from $\mathbb{C}^N$ into itself and fixing the origin by $J^k_{0,0}(\mathbb{C}^N)$. Given coordinates $Z = (Z_1, \ldots, Z_N)$ in $\mathbb{C}^N$, we identify a jet $\mathcal{J} \in J^k_{0,0}(\mathbb{C}^N)$ with a polynomial map of the form

$$\mathcal{J} = \mathcal{J}(Z) = \sum_{\alpha \in \mathbb{N}^r,\ 1 \le |\alpha| \le k} \frac{\Lambda^k_\alpha}{\alpha!} Z^\alpha, \tag{8.1}$$

where $\Lambda^k_\alpha \in \mathbb{C}^N$. We thus have for a jet $\mathcal{J} \in J^k_{0,0}(\mathbb{C}^N)$ coordinates

$$\Lambda^k := (\Lambda^k_\alpha)_{1 \le |\alpha| \le k} \tag{8.2}$$

as given in (8.1). Given a germ of a holomorphic map $h\colon (\mathbb{C}^N, 0) \to (\mathbb{C}^N, 0)$, $h = h(t)$, we denote by $j^k_t h$ the k-jet of h at t for t sufficiently small and also use the following splitting $j^k_t h =: (h(t), \widehat{j}^k_t h)$. Moreover, since $h(0) = 0$, we may also identify $j^k_0 h$ with $\widehat{j}^k_0 h$, which we will freely do in the sequel.

For later purposes, given a splitting $(z, w) \in \mathbb{C}^n \times \mathbb{C}^d$ (where $N = n + d$) of coordinates in $\mathbb{C}^N$ (as for example induced by a choice of normal coordinates (7.6)), we introduce a special component of any jet $\Lambda^k \in J^k_{0,0}(\mathbb{C}^N)$. We denote the set of multi-indices of length one having 0 from the $(n+1)$-th to the N-th digit by S, and the projection onto the first n coordinates by $\mathrm{pr}_1\colon \mathbb{C}^N \to \mathbb{C}^n$ (that is, $\mathrm{pr}_1(z, w) = z$). Then

$$\widetilde{\Lambda}^1 := (\mathrm{pr}_1(\Lambda_\alpha))_{\alpha \in S}. \tag{8.3}$$

Note that for any local holomorphic map $(\mathbb{C}^n \times \mathbb{C}^d, 0) \ni (z, w) \mapsto H(z, w) = (f(z, w), g(z, w)) \in (\mathbb{C}^n \times \mathbb{C}^d, 0)$, if $j^k_0 H = \Lambda^k$, then $\widetilde{\Lambda}^1 = (\frac{\partial f}{\partial z}(0))$. We can therefore identify $\widetilde{\Lambda}^1$ with an $n \times n$ matrix or equivalently with an element of $J^1_{0,0}(\mathbb{C}^n)$. Throughout the paper, given any other notation for a jet $\lambda^k \in J^k_{0,0}(\mathbb{C}^N)$, $\widetilde{\lambda}^1$ will always denote the component of λ^k defined above. We denote by $\mathcal{G}^k_0(\mathbb{C}^N)$ the connected open subset of $J^k_{0,0}(\mathbb{C}^N)$ consisting of those jets Λ^k for which $\det \widetilde{\Lambda}^1 \neq 0$, i.e., for which $\widetilde{\Lambda}^1 \in \mathsf{GL_n}(\mathbb{C})$.

For later purposes, it is convenient to introduce the following spaces of functions. For any positive integers k, s, let $\mathcal{F}^k_s = \mathcal{O}(\{0\} \times \mathcal{G}^1_0(\mathbb{C}^N) \times \mathcal{G}^k_0(\mathbb{C}^N))$ be the ring of germs of holomorphic functions on $\{0\} \times \mathcal{G}^1_0(\mathbb{C}^N) \times \mathcal{G}^k_0(\mathbb{C}^N) \subset \mathbb{C}^s \times \mathcal{G}^1_0(\mathbb{C}^N) \times \mathcal{G}^k_0(\mathbb{C}^N)$; recall that this is just the space of holomorphic functions which are defined on some connected open subset (which may depend on the function) of $\mathbb{C}^s \times \mathcal{G}^1_0(\mathbb{C}^N) \times \mathcal{G}^k_0(\mathbb{C}^N)$ containing $\{0\} \times \mathcal{G}^1_0(\mathbb{C}^N) \times \mathcal{G}^k_0(\mathbb{C}^N)$, where we identify two such functions if there exists an open connected subset containing $\{0\} \times \mathcal{G}^1_0(\mathbb{C}^N) \times \mathcal{G}^k_0(\mathbb{C}^N)$ on which they are both defined and agree. For any $\Theta = \Theta(t, \lambda^1, \Lambda^k) \in \mathcal{F}^k_s$, we write the Taylor expansion

$$\Theta(t, \lambda^1, \Lambda^k) = \sum_{\nu \in \mathbb{N}^s} \Theta_\nu(\lambda^1, \Lambda^k) t^\nu.$$

Let $\mathcal{E}^k_s$ denote the subspace of $\mathcal{F}^k_s$ consisting of those germs $\Theta \in \mathcal{F}^k_s$ for which $\Theta_\nu(\lambda^1, \Lambda^k)$ can be written, for any $\nu \in \mathbb{N}^s$, in the following form:

$$\frac{B_\nu(\lambda^1, \Lambda^k)}{(\det \widetilde{\Lambda}^1)^{c_\nu} (\det \widetilde{\lambda}^1)^{a_\nu}},$$

for some polynomial B_ν and some nonnegative integers c_ν, a_ν.

In what follows, for $x \in \mathbb{C}^q$ and $y \in \mathbb{C}^r$, we denote by $\mathbb{C}\{x\}[y]$ the ring of polynomials in y with coefficients that are convergent power series in x.

Throughout the paper, we will need the following elementary facts about the space $\mathcal{E}_s^k$ that we shall freely use.

Lemma 8.1. *Let $\Theta_1, \ldots, \Theta_r, \Theta_{r+1}, \ldots, \Theta_q$ be q holomorphic functions belonging to $\mathcal{E}_s^k$ such that $\Theta_{r+1}(0, \lambda^1, \Lambda^k) = \ldots = \Theta_q(0, \lambda^1, \Lambda^k) = 0$ for all $(\lambda^1, \Lambda^k) \in \mathcal{G}_0^1(\mathbb{C}^N) \times \mathcal{G}_0^k(\mathbb{C}^N)$. Then the following hold.*

(i) *If $V(t, T; y) \in \mathbb{C}\{t, T\}[y]$, $(t, T, y) \in \mathbb{C}^s \times \mathbb{C}^{q-r} \times \mathbb{C}^r$, then the holomorphic function*

$$V(t, \Theta_{r+1}(t, \lambda^1, \Lambda^k), \ldots, \Theta_q(t, \lambda^1, \Lambda^k); \Theta_1(t, \lambda^1, \Lambda^k), \ldots, \Theta_r(t, \lambda^1, \Lambda^k))$$

also belongs to $\mathcal{E}_s^k$.

(ii) *If $C(t; y) \in \mathbb{C}\{t\}[y]$, $D(t; y) \in \mathbb{C}\{t\}[y]$, $(t, y) \in \mathbb{C}^s \times \mathbb{C}^r$, and for some nonnegative integers a, b, $D(0; \lambda^1, \Lambda^k) = (\det \widetilde{\Lambda}^1)^a\ (\det \widetilde{\lambda}^1)^b$, then for every nonnegative integer m the ratio*

$$\frac{C(t; \Theta_1(t, \lambda^1, \Lambda^k), \ldots, \Theta_r(t, \lambda^1, \Lambda^k))}{(D(t; \Theta_1(t, \lambda^1, \Lambda^k), \ldots, \Theta_r(t, \lambda^1, \Lambda^k)))^m}$$

defines an element of $\mathcal{E}_s^k$.

9. Reflection identities

Throughout the rest of this paper, we let $(M, 0)$ denote a germ of a real-analytic generic submanifold of $\mathbb{C}^N$ through the origin. This section is the starting point of the proof of Theorem 7.3. We shall assume, without loss of generality, that M is given in normal coordinates $Z = (z, w) \in \mathbb{C}^n \times \mathbb{C}^d$ as in (7.5) and (7.6). Recall also that the complexification $\mathcal{M} \subset \mathbb{C}^N \times \mathbb{C}^N$ is the complex submanifold given through the origin by (7.8). In what follows, we shall keep the notation and terminology introduced in §7–§8.

This section is devoted to the first of the three main steps of the proof of Theorem 7.3. We establish here some general identities on the complexification $\mathcal{M}$ of M (see Propositions 9.1 and 9.4), which are valid without any condition on the submanifold M.

Recall that for any positive integer k, Λ^k denotes the coordinates in $J_{0,0}^k(\mathbb{C}^N)$ as explained in §8, and $\widetilde{\Lambda}^1$ is the component of Λ^1 given by (8.3), corresponding to our choice of normal coordinates (z, w). In what follows, for every $H \in \operatorname{Aut}(M, 0)$, following the splitting of the coordinates $Z = (z, w)$, we split the map H as $H = (f, g) \in \mathbb{C}^n \times \mathbb{C}^d$.

For the map $\bar{Q}$ previously defined, we will use the following notation to simplify our formulas:

$$\bar{Q}_Z = \begin{pmatrix} \bar{Q}^1_{Z_1} & \cdots & \bar{Q}^d_{Z_1} \\ \vdots & \ddots & \vdots \\ \bar{Q}^1_{Z_N} & \cdots & \bar{Q}^d_{Z_N} \end{pmatrix}, \quad \bar{Q}_{\chi^\alpha, Z} = \begin{pmatrix} \bar{Q}^1_{\chi^\alpha, Z_1} & \cdots & \bar{Q}^d_{\chi^\alpha, Z_1} \\ \vdots & \ddots & \vdots \\ \bar{Q}^1_{\chi^\alpha, Z_N} & \cdots & \bar{Q}^d_{\chi^\alpha, Z_N} \end{pmatrix},$$

where $\alpha \in \mathbb{N}^n$ and $\bar{Q} = (\bar{Q}^1, \ldots, \bar{Q}^d)$.

We start with the following well-known computational lemma (see, e.g., [3, Equation (3.4.14)]).

Proposition 9.1. *There exists a polynomial $\mathcal{D}=\mathcal{D}(Z,\zeta,\Lambda^1)\in\mathbb{C}\{Z,\zeta\}[\Lambda^1]$ which is universal (i.e., only depends on M), and, for every $\alpha\in\mathbb{N}^n\setminus\{0\}$, another universal $\mathbb{C}^d$-valued polynomial map $\mathcal{P}_\alpha=\mathcal{P}_\alpha(Z,\zeta,\Lambda^{|\alpha|})$ whose components are in the ring $\mathbb{C}\{Z,\zeta\}[\Lambda^{|\alpha|}]$ such that for any $H\in\operatorname{Aut}(M,0)$, the following holds:*

(i) $\mathcal{D}(0,0,\Lambda^1)=\det\widetilde{\Lambda}^1$;

(ii) $\mathcal{D}(0,0,j_0^1H)\neq 0$;

(iii) *for all $(Z,\zeta)\in\mathcal{M}$ near 0,*

$$(\mathcal{D}(Z,\zeta,\widehat{j}_\zeta^1\overline{H}))^{2|\alpha|-1}\,\bar{Q}_{\chi^\alpha}(\bar{f}(\zeta),H(Z))=\mathcal{P}_\alpha(Z,\zeta,\widehat{j}_\zeta^{|\alpha|}\overline{H}). \tag{9.1}$$

Our next identity is concerned with pure transversal derivatives of every element of $\operatorname{Aut}(M,0)$.

Proposition 9.2. *For any $\mu\in\mathbb{N}^d\setminus\{0\}$ and $\alpha\in\mathbb{N}^n\setminus\{0\}$, there exist a universal $\mathbb{C}^d$-valued polynomial map $\mathcal{T}_{\mu,\alpha}(Z,\zeta,Z',\zeta',\lambda^{|\mu|-1},\Lambda^{|\mu|})$ whose components belong to the ring $\mathbb{C}\{Z,\zeta,Z',\zeta'\}[\lambda^{|\mu|-1},\Lambda^{|\mu|}]$ and another universal $\mathbb{C}^d$-valued polynomial map $\mathcal{Q}_{\mu,\alpha}(Z,\zeta,\Lambda^{|\alpha|+|\mu|})$ whose components are in the ring $\mathbb{C}\{Z,\zeta\}[\Lambda^{|\alpha|+|\mu|}]$, such that for any $H\in\operatorname{Aut}(M,0)$ and for any $(Z,\zeta)\in\mathcal{M}$ close to the origin, the following relation holds:*

$$\frac{\partial^{|\mu|}H}{\partial w^\mu}(Z)\cdot\bar{Q}_{\chi^\alpha,Z}(\bar{f}(\zeta),H(Z))=(*)_1+(*)_2, \tag{9.2}$$

where

$$(*)_1:=\mathcal{T}_{\mu,\alpha}\left(Z,\zeta,H(Z),\overline{H}(\zeta),\widehat{j}_Z^{|\mu|-1}H,\widehat{j}_\zeta^{|\mu|}\overline{H}\right) \tag{9.3}$$

and

$$(*)_2:=\frac{\mathcal{Q}_{\mu,\alpha}(Z,\zeta,\widehat{j}_\zeta^{|\alpha|+|\mu|}\overline{H})}{(\mathcal{D}(Z,\zeta,\widehat{j}_\zeta^1\overline{H}))^{2|\alpha|+|\mu|-1}}. \tag{9.4}$$

Proof. Consider the following holomorphic vector fields tangent to $\mathcal{M}$:

$$R_j=\frac{\partial}{\partial w_j}+\bar{Q}_{w_j}(\chi,z,w)\cdot\frac{\partial}{\partial\tau}=\frac{\partial}{\partial w_j}+\sum_{k=1}^d\bar{Q}_{w_j}^k(\chi,z,w)\frac{\partial}{\partial\tau_k},\ j=1,\dots,d. \tag{9.5}$$

We prove the proposition by induction on the length of μ. For $\mu\in\mathbb{N}^d$ of length one, say, without loss of generality, for $\mu=(1,0,\dots,0)$, (9.2) follows easily after applying R_1 to (9.1). Suppose now that (9.2) holds for all multi-indices μ of a certain length k. Then if $\mu'\in\mathbb{N}^d$ is of length $k+1$, say again $\mu'=(1,0,\dots,0)+\mu$ for some $\mu\in\mathbb{N}^d$ of length k, we apply again R_1 to (9.2), and one easily sees from the chain rule that (9.2) is satisfied with μ replaced by μ'. We leave the details to the reader. □

The next lemma deals with the transversal derivatives of the normal component g of any $H=(f,g)\in\operatorname{Aut}(M,0)$.

Lemma 9.3. *For any $\mu\in\mathbb{N}^d\setminus\{0\}$, there exist a universal $\mathbb{C}^d$-valued polynomial map $W_\mu=W_\mu(Z,\zeta,Z',\zeta',\lambda^{|\mu|-1},\Lambda^{|\mu|})$ whose components belong to the ring $\mathbb{C}\{Z,\zeta,Z',\zeta'\}[\lambda^{|\mu|-1},\Lambda^{|\mu|}]$ such that for every $H=(f,g)\in\operatorname{Aut}(M,0)$ and for any $(Z,\zeta)\in\mathcal{M}$ sufficiently close to the origin,*

$$\frac{\partial^{|\mu|}g}{\partial w^\mu}(Z)=\frac{\partial^{|\mu|}f}{\partial w^\mu}(Z)\cdot Q_z(f(Z),\overline{H}(\zeta))+(*)_3 \tag{9.6}$$

with

$$(*)_3 := W_\mu \left(Z, \zeta, H(Z), \overline{H}(\zeta), \widehat{j}_Z^{|\mu|-1} H, \widehat{j}_\zeta^{|\mu|} \overline{H}\right). \tag{9.7}$$

Proof. We again prove the lemma by induction on $|\mu|$.

Let $\mu \in \mathbb{N}^d$ be of length one, say $\mu = (1, 0, \dots, 0)$. Applying the vector field R_1 (as defined in (9.5)) to the equation $g(Z) = Q(f(Z), \overline{H}(\zeta))$, for $(Z, \zeta) \in \mathcal{M}$, one easily sees that the existence of W_μ satisfying (9.6) follows from the chain rule.

Suppose now that (9.6) holds for all multi-indices μ of a certain length k. Then if $\mu' \in \mathbb{N}^d$ is of length $k+1$, say again $\mu' = (1, 0, \dots, 0) + \mu$ for some $\mu \in \mathbb{N}^d$ of length k, we apply again R_1 to (9.6), and the existence of $W_{\mu'}$ satisfying the desired equality follows again from the chain rule. We leave the details to the reader. □

Combining Proposition 9.2 and Lemma 9.3, one gets the following.

Proposition 9.4. *For any $\mu \in \mathbb{N}^d \setminus \{0\}$ and $\alpha \in \mathbb{N}^n \setminus \{0\}$, there exists a universal $\mathbb{C}^d$-valued polynomial map $\mathcal{T}'_{\mu,\alpha}(Z, \zeta, Z', \zeta', \lambda^{|\mu|-1}, \Lambda^{|\mu|})$ whose components belong to the ring $\mathbb{C}\{Z, \zeta, Z', \zeta'\}[\lambda^{|\mu|-1}, \Lambda^{|\mu|}]$ such that for any $H = (f, g) \in \mathrm{Aut}(M, 0)$ and for any $(Z, \zeta) \in \mathcal{M}$ close to the origin, the following relation holds:*

$$\frac{\partial^{|\mu|} f}{\partial w^\mu}(Z) \cdot \left(\bar{Q}_{\chi^\alpha, z}(\bar{f}(\zeta), H(Z)) + Q_z(f(Z), \overline{H}(\zeta)) \cdot \bar{Q}_{\chi^\alpha, w}(\bar{f}(\zeta), H(Z))\right) = (*)_1' + (*)_2, \tag{9.8}$$

where $(*)_2$ is given by (9.4) *and $(*)_1'$ is given by*

$$(*)_1' := \mathcal{T}'_{\mu,\alpha} \left(Z, \zeta, H(Z), \overline{H}(\zeta), \widehat{j}_Z^{|\mu|-1} H, \widehat{j}_\zeta^{|\mu|} \overline{H}\right). \tag{9.9}$$

Remark 9.5. In (9.8), one should see $\dfrac{\partial^{|\mu|} f}{\partial w^\mu}(Z)$ as a $1 \times n$ matrix, $\bar{Q}_{\chi^\alpha, z}(\bar{f}(\zeta), H(Z))$ as an $n \times d$ matrix, $Q_z(f(Z), \overline{H}(\zeta))$ as an $n \times d$ matrix and $\bar{Q}_{\chi^\alpha, w}(\bar{f}(\zeta), H(Z)) = (\bar{Q}^j_{\chi^\alpha, w_i}(\bar{f}(\zeta), H(Z)))_{1 \le i, j \le d}$ as a $d \times d$ matrix.

10. Parametrization of $\mathrm{Aut}(M, 0)$ along Segre sets

This section is devoted to the second part of the proof of Theorem 7.3, which involves two main ingredients: the Segre sets and mappings introduced by Baouendi, Ebenfelt and Rothschild [1] and the parametrization theorems for singular analytic systems developed in the first part of the paper. We shall now prove that for a real-analytic generic submanifold $M \in \mathcal{C}$ (the class $\mathcal{C}$ is defined in §7), the elements of $\mathrm{Aut}(M, 0)$, restricted to any Segre set, are parametrized by their jets at the origin. Such a property is in fact first shown to be true on the Segre variety attached to the origin (in Proposition 10.1) and then we establish that this parametrization property "propagates" to higher-order Segre sets (this is done in Proposition 10.2). For both of these results, we now use the fact that M belongs to the class $\mathcal{C}$.

10.1. **Segre sets and mappings.** We start by recalling the Segre mappings as introduced in [1]. They play a fundamental role in the study of mappings between real-analytic or real-algebraic CR-manifolds of arbitrary codimension [8, 5, 39]. In what follows, for any positive integer s, $t^s := (t^s_1, \dots, t^s_n)$ will denote an n-dimensional complex variable.

For $j \in \mathbb{N}^*$, we define the Segre mappings $v^j \colon (\mathbb{C}^{nj}, 0) \to (\mathbb{C}^N, 0)$ inductively: for $j = 1$, $v^1(t^1) := (t^1, 0)$; and for $j \ge 1$,

$$v^{j+1}(t^1, \dots, t^j) := (t^{j+1}, Q(t^{j+1}, \bar{v}^j(t^1, \dots, t^j))). \tag{10.1}$$

By definition, given a sufficiently small neighbourhood U_j of the origin in $\mathbb{C}^{nj}$, the Segre set is the image of the neighbourhood U_j under the map v^j (for a thorough discussion, see e.g. [8]). Note that for any $j \geq 1$, the map

$$(\mathbb{C}^{n(j+1)}, 0) \ni (t^1, \ldots, t^j, t^{j+1}) \mapsto (v^{j+1}(t^1, \ldots, t^{j+1}), \bar{v}^j(t^1, \ldots, t^j)) \in (\mathbb{C}^N \times \mathbb{C}^N, 0)$$

takes in fact its values on the complexification $\mathcal{M}$. In what follows, for any $j \geq 1$, we shall use the following useful notation:

$$t^{[j]} := (t^1, \ldots, t^j). \tag{10.2}$$

10.2. **Jet parametrization of CR automorphisms along the first Segre set.** In what follows, we keep the notation introduced at the beginning of this section as well as in §9.

The next proposition gives a precise parametrization property by jets at the origin of all elements of $\mathrm{Aut}(M, 0)$ and of their derivatives along the first Segre set. Note that for any $H \in \mathrm{Aut}(M, 0)$ and for any positive integers k, s, $(0, j_0^1\overline{H}, j_0^k H)$ and $(0, j_0^1 H, j_0^k\overline{H})$ are in the domain of definition of any holomorphic function belonging to the space $\mathcal{F}_s^k$ as defined in §8.

Proposition 10.1. *In the above setting and with the above notation, assume furthermore that $(M, 0)$ belongs to the class $\mathcal{C}$ and denote by $\kappa_0 := \kappa(M, 0)$ the invariant integer attached to $(M, 0)$ as defined in §7. Then for any multi-index $\beta \in \mathbb{N}^N$, there exists a $\mathbb{C}^N$-valued holomorphic map*

$$\begin{aligned} \Psi_1^\beta &= \Psi_1^\beta\left(t^1, \lambda^1, \Lambda^{\kappa_0+|\beta|}\right) \\ &:= \left(\widetilde{\Psi}_1^\beta\left(t^1, \lambda^1, \Lambda^{\kappa_0+|\beta|}\right), \widehat{\Psi}_1^\beta\left(t^1, \lambda^1, \Lambda^{\kappa_0+|\beta|}\right)\right) \in \mathbb{C}^n \times \mathbb{C}^d, \end{aligned} \tag{10.3}$$

whose components belong to $\mathcal{E}_n^{\kappa_0+|\beta|}$, satisfying the following properties:

(i) *If we denote by $S \subset \mathbb{N}^N$ the set of multi-indices of length one having 0 from the $(n+1)$-th to the N-th component, then for any $(\lambda^1, \Lambda^{\kappa_0}) \in \mathcal{G}_0^1(\mathbb{C}^N) \times \mathcal{G}_0^{\kappa_0}(\mathbb{C}^N)$ we have*

$$\Psi_1^0(0, \lambda^1, \Lambda^{\kappa_0}) = 0, \ (\widetilde{\Psi}_1^\beta(0, \lambda^1, \Lambda^{\kappa_0}))_{\beta \in S} = \frac{\partial \widetilde{\Psi}_1^0}{\partial t^1}(0, \lambda^1, \Lambda^{\kappa_0}) = \widetilde{\lambda}^1. \tag{10.4}$$

(ii) *For every $H \in \mathrm{Aut}(M, 0)$, we have*

$$((\partial^\beta H) \circ v^1)(t^1) = \Psi_1^\beta(t^1, j_0^1 H, j_0^{\kappa_0+|\beta|}\overline{H}), \tag{10.5}$$

for all t^1 sufficiently close to the origin in $\mathbb{C}^n$.

We prove the proposition by induction on $|\beta|$. The induction start $\beta = 0$ is treated in §10.2.1 and the induction step is carried out in §10.2.2.

10.2.1. *Proof of Proposition* 10.1 *for* $|\beta| = 0$. For $\beta = 0$, we first note that it follows from the normality of the chosen coordinates (z, w) that for any $H = (f, g) \in \mathrm{Aut}(M, 0)$, one has $(g \circ v^1)(t^1) \equiv 0$. We may therefore define $\widehat{\Psi}_1^0(t^1, \lambda^1, \Lambda^{\kappa_0}) := 0$, and it is enough to construct $\widetilde{\Psi}_1^0(t^1, \lambda^1, \Lambda^{\kappa_0})$. Setting $Z = v^1(t^1)$ and $\zeta = 0$ in (9.1) implies that for any $H \in \mathrm{Aut}(M, 0)$ and for any $\alpha \in \mathbb{N}^n \setminus \{0\}$,

$$\bar{Q}_{\chi^\alpha}(0, (H \circ v^1)(t^1)) = \frac{\mathcal{P}_\alpha(v^1(t^1), 0, \widehat{j}_0^{|\alpha|}\overline{H})}{(\mathcal{D}(v^1(t^1), 0, \widehat{j}_0^1\overline{H}))^{2|\alpha|-1}}, \tag{10.6}$$

for all t^1 sufficiently close to the origin. Using (7.7) and the fact that $g \circ v^1 \equiv 0$, we obtain equivalently

$$\alpha!\, \bar{q}_\alpha(f \circ v^1(t^1), 0) = \frac{\mathcal{P}_\alpha(v^1(t^1), 0, \widehat{j}_0^{|\alpha|}\overline{H})}{(\mathcal{D}(v^1(t^1), 0, \widehat{j}_0^1\overline{H}))^{2|\alpha|-1}}. \tag{10.7}$$

In what follows, we use (10.7) only for multi-indices $\alpha \in \mathbb{N}^n \setminus \{0\}$ with $|\alpha| \leq \kappa_0$. For such α's and for $\sigma \in \mathbb{N}$, we set

$$p_\alpha(t^1, \Lambda^{\kappa_0}) := \mathcal{P}_\alpha(v^1(t^1), 0, \Lambda^{|\alpha|}), \quad \Delta_\sigma(t^1, \Lambda^1) := (\mathcal{D}(v^1(t^1), 0, \Lambda^1))^\sigma, \tag{10.8}$$

and

$$r_\alpha(t^1, \Lambda^{\kappa_0}) := \frac{p_\alpha(t^1, \Lambda^{\kappa_0})}{\Delta_{2|\alpha|-1}(t^1, \Lambda^1)}. \tag{10.9}$$

It follows from Lemma 8.1 and Proposition 9.1 that the map $r_\alpha = (r_\alpha^1, \ldots, r_\alpha^d)$ defines a $\mathbb{C}^d$-valued holomorphic map in a neighbourhood of $\{0\} \times \mathcal{G}_0^{\kappa_0}(\mathbb{C}^N)$ whose components belong to $\mathcal{E}_n^{\kappa_0}$. Moreover, since $M \in \mathcal{C}$, Proposition 7.5 implies that we may choose n integers $i_1, \ldots, i_n \in \{1, \ldots, d\}$ and n multi-indices $\alpha^{(1)}, \ldots, \alpha^{(n)} \in \mathbb{N}^n \setminus \{0\}$ of length $\leq \kappa_0$ so that $z \mapsto (\bar{q}^{i_1}_{\alpha^{(1)}}(z, 0), \ldots, \bar{q}^{i_n}_{\alpha^{(n)}}(z, 0))$ is of generic rank n. Consider the following system of complex-analytic equations in the unknown $T^1 \in \mathbb{C}^{n-1}$:

$$\alpha^{(\nu)}!\, \bar{q}^{i_\nu}_{\alpha^{(\nu)}}(T^1, 0) = r^{i_\nu}_{\alpha^{(\nu)}}(t^1, \Lambda^{\kappa_0}), \ \nu = 1, \ldots, n. \tag{10.10}$$

By Corollary 3.2, there exists a holomorphic map

$$\Gamma_1^0 = \Gamma_1^0(t^1, \widetilde{\lambda}^1, \Lambda^{\kappa_0}) \colon \mathbb{C}^n \times \mathsf{GL_n}(\mathbb{C}) \times \mathcal{G}_0^{\kappa_0}(\mathbb{C}^N) \to \mathbb{C}^n \tag{10.11}$$

defined in a neighbourhood of $\{0\} \times \mathsf{GL_n}(\mathbb{C}) \times \mathcal{G}_0^{\kappa_0}(\mathbb{C}^N)$, satisfying

$$\Gamma_1^0(0, \widetilde{\lambda}^1, \Lambda^{\kappa_0}) = 0, \quad \frac{\partial \Gamma_1^0}{\partial t^1}(0, \widetilde{\lambda}^1, \Lambda^{\kappa_0}) = \widetilde{\lambda}^1$$

such that if $u \colon (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ is a germ of a biholomorphism satisfying (10.10) for some $\Lambda^{\kappa_0} \in \mathcal{G}_0^{\kappa_0}(\mathbb{C}^N)$, then $u(t^1) = \Gamma_1^0(t^1, j_0^1 u, \Lambda^{\kappa_0})$. Hence, since for any $H \in \mathrm{Aut}(M, 0)$, $t^1 \mapsto (f \circ v^1)(t^1)$ is a germ at 0 of a biholomorphism satisfying (10.10) with $\Lambda^{\kappa_0} = j_0^{\kappa_0}\overline{H}$, in view of (10.7), (10.8) and (10.9), we have

$$(f \circ v^1)(t^1) = \Gamma_1^0(t^1, j_0^1(f \circ v^1), j_0^{\kappa_0}\overline{H}) = \Gamma_1^0\left(t^1, \left(\frac{\partial f}{\partial z}(0)\right), j_0^{\kappa_0}\overline{H}\right). \tag{10.12}$$

Therefore if we set $\widetilde{\Psi}_1^0(t^1, \lambda^1, \Lambda^{\kappa_0}) := \Gamma_1^0(t^1, \widetilde{\lambda}^1, \Lambda^{\kappa_0})$, we are left to prove that the components of $\widetilde{\Psi}_1^0(t^1, \lambda^1, \Lambda^{\kappa_0})$ belong to $\mathcal{E}_n^{\kappa_0}$. For this, we write the Taylor expansion

$$\widetilde{\Psi}_1^0(t^1, \lambda^1, \Lambda^{\kappa_0}) = \Gamma_1^0(t^1, \widetilde{\lambda}^1, \Lambda^{\kappa_0}) = \sum_{\gamma \in \mathbb{N}^n} \Gamma^0_{1,\gamma}(\widetilde{\lambda}^1, \Lambda^{\kappa_0})(t^1)^\gamma. \tag{10.13}$$

From Corollary 3.2 (ii) we know that for each $\gamma \in \mathbb{N}^n$, there exist positive integers e_γ and s_γ such that $\Gamma^0_{1,\gamma}(\widetilde{\lambda}^1, \Lambda^{\kappa_0})$ can be written in the form

$$\frac{\rho_\gamma\left(\widetilde{\lambda}^1, \left((\partial^\nu_{t^1} r_\alpha)(0, \Lambda^{\kappa_0})\right)_{\substack{1 \leq |\alpha| \leq \kappa_0 \\ |\nu| \leq s_\gamma}}\right)}{(\det \widetilde{\lambda}^1)^{e_\gamma}}, \tag{10.14}$$

where ρ_γ is a $\mathbb{C}^n$-valued polynomial map of its arguments. Since each component of the map r_α, $|\alpha| \leq \kappa_0$, belongs to $\mathcal{E}_n^{\kappa_0}$, it follows from (10.14) and (10.13) that the same fact holds for every component of $\widetilde{\Psi}_1^0$. This completes the proof of Proposition 10.1 in the case $\beta = 0$.

10.2.2. *Induction for Proposition* 10.1. Assume now that Proposition 10.1 holds for all multi-indices $\beta \in \mathbb{N}^N$ of length $|\alpha| \le k$, $k \in \mathbb{N}$, and let $\delta = (\delta_1, \ldots, \delta_N) \in \mathbb{N}^N$ with $|\delta| = k+1$. Suppose first that there exists $m \in \{1, \ldots, n\}$ such that $\delta_m > 0$. Then by setting

$$\Psi_1^\delta(t^1, \lambda^1, \Lambda^{\kappa_0+k+1}) := \frac{\partial \Psi_1^\beta}{\partial t_m^1}(t^1, \lambda^1, \Lambda^{\kappa_0+k}), \tag{10.15}$$

where $t^1 = (t_1^1, \ldots, t_n^1) \in \mathbb{C}^n$ and $\beta = (\beta_1, \ldots, \beta_N)$, $\beta_i = \delta_i$ for $i \neq m$ and $\beta_m = \delta_m - 1$, we see that the derivative of order δ of every $H \in \mathrm{Aut}(M, 0)$ is parametrized in the right way and we are done in this case. Equation (10.15) also shows that the parametrizations Ψ_1^δ with $\delta \in S$ are constructed in such a way that the second part of (10.4) holds. We may now therefore assume that $\delta_1 = \ldots = \delta_n = 0$ (and $|\delta| = k+1$). We set $\mu = (\delta_{n+1}, \ldots, \delta_N) \in \mathbb{N}^d$ and we note that $|\mu| = |\delta| = k+1$. Using (9.8) and (9.9) in which we set $Z = v^1(t^1)$ and $\zeta = 0$ and using the fact that $Q(z, 0, 0) = 0$ (normality of the coordinates), we obtain the following equality for every $\alpha \in \mathbb{N}^n \setminus \{0\}$ and every $H = (f, g) \in \mathrm{Aut}(M, 0)$ and every $t^1 \in \mathbb{C}^n$ sufficiently close to the origin:

$$\begin{aligned}\alpha!\, ((\partial_w^\mu f) \circ v^1) \cdot \frac{\partial \bar{q}_\alpha}{\partial z}(f \circ v^1, 0) = {} & \frac{\mathcal{Q}_{\mu,\alpha}(v^1, 0, \widehat{j}_0^{|\alpha|+k+1}\overline{H})}{\Delta_{2|\alpha|+k}(t^1, \widehat{j}_0^1\overline{H})} \\ & + \mathcal{T}'_{\mu,\alpha}\left(v^1, 0, H \circ v^1, 0, ((\partial^\beta H) \circ v^1)_{1 \le |\beta| \le k}, j_0^{k+1}\overline{H}\right).\end{aligned} \tag{10.16}$$

We have deliberately dropped the variable t^1 in (10.16) to simplify a bit the formula. In what follows, we restrict ourselves to multi-indices $\alpha \in \mathbb{N}^d$ such that $|\alpha| \le \kappa_0$. For such α's, we set

$$\Xi_{\mu,\alpha}(t^1, \Lambda^{\kappa_0+k+1}) := \frac{\mathcal{Q}_{\mu,\alpha}(v^1(t^1), 0, \Lambda^{|\alpha|+k+1})}{\Delta_{2|\alpha|+k}(t^1, \Lambda^1)}, \tag{10.17}$$

$$\begin{aligned}& \Upsilon_{\mu,\alpha}(t^1, \lambda^1, \Lambda^{\kappa_0+k+1}) \\ & := \mathcal{T}'_{\mu,\alpha}\left(v^1(t^1), 0, \Psi_1^0(t^1, \lambda^1, \Lambda^{\kappa_0}), 0, \Psi_1^{(k)}(t^1, \lambda^1, \Lambda^{\kappa_0+k}), \Lambda^{k+1}\right).\end{aligned} \tag{10.18}$$

In the last equation we use the notation

$$\Psi_1^{(k)}(t^1, \lambda^1, \Lambda^{\kappa_0+k}) := \left(\Psi_1^\beta(t^1, \lambda^1, \Lambda^{\kappa_0+|\beta|})\right)_{1 \le |\beta| \le k}.$$

It follows from (10.8), (10.4), Lemma 8.1 and Proposition 9.1 that the mappings $\Xi_{\mu,\alpha}$ and $\Upsilon_{\mu,\alpha}$ define holomorphic mappings in an open neighbourhood of $\{0\} \times \mathcal{G}_0^1(\mathbb{C}^N) \times \mathcal{G}_0^{\kappa_0+k+1}(\mathbb{C}^N)$ in $\mathbb{C}^n \times J_{0,0}^1(\mathbb{C}^N) \times J_{0,0}^{\kappa_0+k+1}(\mathbb{C}^N)$, whose components belong to $\mathcal{E}_n^{\kappa_0+k+1}$. Consider the following linear singular system in $X \in \mathbb{C}^n$:

$$\alpha^{(\nu)}!\, X \cdot \frac{\partial \bar{q}^{i_\nu}_{\alpha^{(\nu)}}}{\partial z}(\widetilde{\Psi}_1^0(t^1, \lambda^1, \Lambda^{\kappa_0}), 0) = \Xi_{\mu,\alpha^{(\nu)}}(t^1, \Lambda^{\kappa_0+k+1}) + \Upsilon_{\mu,\alpha^{(\nu)}}(t^1, \lambda^1, \Lambda^{\kappa_0+k+1}), \tag{10.19}$$

for $\nu = 1, \ldots, n$. By (10.16) and the induction assumption, for any $H \in \mathrm{Aut}(M, 0)$, $X = (\partial_w^\mu f) \circ v^1$ satisfies the above system with $\lambda^1 = j_0^1 H$ and $\Lambda^{\kappa_0+k+1} = j_0^{\kappa_0+k+1}\overline{H}$. Now, the holomorphic map $z \mapsto (\bar{q}^{i_1}_{\alpha^{(1)}}(z, 0), \ldots, \bar{q}^{i_n}_{\alpha^{(n)}}(z, 0))$ is of generic rank n,

$\Psi_1^0(0,\lambda^1,\Lambda^{\kappa_0})=0$, and $\dfrac{\partial \widetilde{\Psi}_1^0}{\partial t^1}(0,\lambda^1,\Lambda^{\kappa_0})$ is invertible for all $(\lambda^1,\Lambda^{\kappa_0}) \in \mathcal{G}_0^1(\mathbb{C}^N)\times \mathcal{G}_0^{\kappa_0}(\mathbb{C}^N)$ by (10.4). Hence we may apply Proposition 6.3 to conclude that there exists a holomorphic map

$$\Gamma_1^\mu = \Gamma_1^\mu(t^1,\lambda^1,\Lambda^{\kappa_0+k+1})\colon \mathbb{C}^n \times \mathcal{G}_0^1(\mathbb{C}^N)\times \mathcal{G}_0^{\kappa_0+k+1}(\mathbb{C}^N)\to \mathbb{C}^n \tag{10.20}$$

defined in a neighbourhood of $\{0\}\times \mathcal{G}_0^1(\mathbb{C}^N)\times \mathcal{G}_0^{\kappa_0+k+1}(\mathbb{C}^N)$ which satisfies that if $X\colon(\mathbb{C}^n,0)\to(\mathbb{C}^n,0)$ is a solution of (10.19) for some $(\lambda^1,\Lambda^{\kappa_0+k+1})\in \mathcal{G}_0^1(\mathbb{C}^N)\times \mathcal{G}_0^{\kappa_0+k+1}(\mathbb{C}^N)$, then $X=X(t^1)=\Gamma_1^\mu(t^1,\lambda^1,\Lambda^{\kappa_0+k+1})$. We set

$$\widetilde{\Psi}_1^\delta(t^1,\lambda^1,\Lambda^{\kappa_0+k+1}) := \Gamma_1^\mu(t^1,\lambda^1,\Lambda^{\kappa_0+k+1}), \tag{10.21}$$

(10.22)

$$\widehat{\Psi}_1^\delta(t^1,\lambda^1,\Lambda^{\kappa_0+k+1}) := W_\mu\left(v^1(t^1),0,\Psi_1^0(t^1,\lambda^1,\Lambda^{\kappa_0}),0,\Psi_1^{(k)}(t^1,\lambda^1,\Lambda^{\kappa_0+k}),\Lambda_0^{k+1}\right),$$

where W_μ is the $\mathbb{C}^d$-valued polynomial map given by Lemma 9.3. In view of Lemma 9.3, the constructed map Ψ_1^δ is the desired parametrization for the δ-th derivative of all elements of $\mathrm{Aut}(M,0)$. It remains to check that its components belong to the space $\mathcal{E}_n^{\kappa_0+k+1}$. In view of the induction assumption, Lemma 8.1 (i) and (10.22), this fact is clear for all the components of $\widehat{\Psi}_1^\delta$. To deal with the components of $\widetilde{\Psi}_1^\delta$, we first expand $\widetilde{\Psi}_1^\delta$ in a Taylor series:

$$\widetilde{\Psi}_1^\delta(t^1,\lambda^1,\Lambda^{\kappa_0+k+1}) = \Gamma_1^\mu(t^1,\lambda^1,\Lambda^{\kappa_0+k+1}) = \sum_{\gamma\in\mathbb{N}^n}\Gamma_{1,\gamma}^\mu(\lambda^1,\Lambda^{\kappa_0+k+1})(t^1)^\gamma. \tag{10.23}$$

Note that from Proposition 6.3, we know that each $\Gamma_{1,\gamma}^\mu(\lambda^1,\Lambda^{\kappa_0+k+1})$ is of the following form:

$$\frac{C_\gamma\left[\left((\partial_{t^1}^\nu \Xi_{\mu,\alpha}),(\partial_{t^1}^\nu \Upsilon_{\mu,\alpha}),(\partial_{t^1}^\nu \widetilde{\Psi}_1^0)\right)_{\substack{1\le|\alpha|\le\kappa_0\\ |\nu|\le s_\gamma}}\right]}{\left[\det\left(\dfrac{\partial\widetilde{\Psi}_1^0}{\partial t^1}(0,\lambda^1,\Lambda^{\kappa_0})\right)\right]^{e_\gamma}}, \tag{10.24}$$

where C_γ is a $\mathbb{C}^n$-valued polynomial map of its arguments, s_γ and e_γ are positive integers, and the numerator is evaluated at $(0,\lambda^1,\Lambda^{\kappa_0+k+1})$. Since each component of the $\widetilde{\Psi}_1^0$, $\Xi_{\mu,\alpha}$ and $\Upsilon_{\mu,\alpha}$ for $|\alpha|\le\kappa_0$ belongs to $\mathcal{E}_n^{\kappa_0+k+1}$ and since $\dfrac{\partial\widetilde{\Psi}_1^0}{\partial t^1}(0,\lambda^1,\Lambda^{\kappa_0}) = \widetilde{\lambda}^1$ by (10.4), it follows clearly from (10.23) and (10.24) that each component of $\widetilde{\Psi}_1^0$ belongs to the space $\mathcal{E}_n^{\kappa_0+k+1}$. This completes the proof of Proposition 10.1.

10.3. **Iteration along higher-order Segre sets.** We now turn to constructing a parametrization for the elements of the stability group $\mathrm{Aut}(M,0)$ along higher-order Segre sets analogous to that obtained along the first Segre set in §10.2. For any positive integer k, we recall the notation $t^{[k]}=(t^1,\dots,t^k)\in\mathbb{C}^{nk}$ as introduced in (10.2). We will prove the following.

Proposition 10.2. *With the previous notation and under the assumptions of Proposition* 10.1*, the following holds. For any $k\in\mathbb{N}^*$ and any multi-index $\beta\in\mathbb{N}^N$,* *there exists a $\mathbb{C}^N$-valued holomorphic map*

$$\Psi_k^\beta = \Psi_k^\beta(t^{[k]}, \lambda^1, \Lambda^{k\kappa_0+|\beta|}) := (\widetilde{\Psi}_k^\beta(t^{[k]}, \lambda^1, \Lambda^{k\kappa_0+|\beta|}), \widehat{\Psi}_k^\beta(t^{[k]}, \lambda^1, \Lambda^{k\kappa_0+|\beta|})) \in \mathbb{C}^n \times \mathbb{C}^d, \tag{10.25}$$

whose components belong to $\mathcal{E}_{kn}^{k\kappa_0+|\beta|}$, satisfying the following properties:

(i) *If we denote by $S \subset \mathbb{N}^N$ the set of multi-indices of length one having* 0 *from the $(n+1)$-th to the N-th component, then for any $(\lambda^1, \Lambda^{k\kappa_0}) \in \mathcal{G}_0^1(\mathbb{C}^N) \times \mathcal{G}_0^{k\kappa_0}(\mathbb{C}^N)$ we have*

$$\begin{gathered} \Psi_k^0(0, \lambda^1, \Lambda^{k\kappa_0}) = 0, \\ (\widetilde{\Psi}_k^\beta(0, \lambda^1, \Lambda^{k\kappa_0+1}))_{\beta\in S} = \frac{\partial \widetilde{\Psi}_k^0}{\partial t^k}(0, \lambda^1, \Lambda^{k\kappa_0}) = \begin{cases} \widetilde{\lambda}^1, & \text{if } k \text{ is odd,} \\ \widetilde{\Lambda}^1, & \text{if } k \text{ is even.} \end{cases} \end{gathered} \tag{10.26}$$

(ii) *For every $H \in \mathrm{Aut}(M, 0)$,*

$$((\partial^\beta H) \circ v^k)(t^{[k]}) = \Psi_k^\beta(t^{[k]}, j_0^1 H, j_0^{k\kappa_0+|\beta|}\overline{H}), \quad \text{if } k \text{ is odd,} \tag{10.27}$$

$$((\partial^\beta H) \circ v^k)(t^{[k]}) = \Psi_k^\beta(t^{[k]}, j_0^1 \overline{H}, j_0^{k\kappa_0+|\beta|}H), \quad \text{if } k \text{ is even,} \tag{10.28}$$

for all $t^{[k]}$ sufficiently close to the origin in $\mathbb{C}^{kn}$.

We prove Proposition 10.2 by induction on k. For $k = 1$, the result is exactly the content of Proposition 10.1. Suppose now that the result holds for all $k \in \{1, \ldots, m\}$ and let us prove the proposition for $k = m+1$. We assume that $m+1$ is even, and leave the odd case to the reader. We shall construct the maps $(\Psi_{m+1}^\beta)_{\beta\in\mathbb{N}^N}$ by induction on $|\beta|$.

10.3.1. *Construction of the parametrization Ψ_{m+1}^0.* We use (9.1) in which we set $Z = v^{m+1} = v^{m+1}\left(t^{[m+1]}\right)$ and $\zeta = \bar{v}^m = \bar{v}^m\left(t^{[m]}\right)$. We get for every $\alpha \in \mathbb{N}^n \setminus \{0\}$,

$$\bar{Q}_{\chi^\alpha}(\bar{f} \circ \bar{v}^m, H \circ v^{m+1}) = \frac{\mathcal{P}_\alpha(v^{m+1}, \bar{v}^m, ((\partial^\beta \overline{H}) \circ \bar{v}^m)_{1\le|\beta|\le|\alpha|})}{(\mathcal{D}(v^{m+1}, \bar{v}^m, ((\partial^\beta \overline{H}) \circ \bar{v}^m)_{|\beta|=1}))^{2|\alpha|-1}}, \tag{10.29}$$

for all $t^{[m+1]}$ sufficiently close to the origin in $\mathbb{C}^{(m+1)n}$. In what follows, we will only use (10.29) for multi-indices $\alpha \in \mathbb{N}^n \setminus \{0\}$ such that $|\alpha| \le \kappa_0$. For such α's, we set

$$\varrho_\alpha(t^{[m+1]}, \lambda^1, \Lambda^{(m+1)\kappa_0}) := \frac{\mathcal{P}_\alpha\left(v^{m+1}\left(t^{[m+1]}\right), \bar{v}^m\left(t^{[m]}\right), \overline{\Psi_m^{(|\alpha|)}}(t^{[m]}, \lambda^1, \Lambda^{m\kappa_0+|\alpha|})\right)}{\left(\mathcal{D}(v^{m+1}\left(t^{[m+1]}\right), \bar{v}^m\left(t^{[m]}\right), \overline{\Psi_m^{(1)}}(t^{[m]}, \lambda^1, \Lambda^{m\kappa_0+1})\right)^{2|\alpha|-1}}. \tag{10.30}$$

As before, for any integer k we set

$$\Psi_m^{(k)}(t^{[m]}, \lambda^1, \Lambda^{m\kappa_0+k}) = \left(\Psi_m^\beta(t^{[m]}, \lambda^1, \Lambda^{m\kappa_0+|\beta|})\right)_{1\le|\beta|\le k}$$

and note that in (10.30) the bar is applied to the parametrizations Ψ_m^β viewed as functions of the variable $t^{[m]}$ only. It follows from Proposition 9.1 (i) and (10.26) (with $k = m$) that

$$\mathcal{D}(0,0,\overline{\Psi^{(1)}_m}(0,\lambda^1,\Lambda^{m\kappa_0+1})) = \det\left((\overline{\Psi^{\beta}_m}(0,\lambda^1,\Lambda^{m\kappa_0+1}))_{\beta\in S}\right) = \det\widetilde{\lambda}^1. \tag{10.31}$$

This shows that for each $\alpha\in\mathbb{N}^n\setminus\{0\}$ with $|\alpha|\le\kappa_0$, $\varrho_\alpha := (\varrho^1_\alpha,\ldots,\varrho^d_\alpha)$ defines a $\mathbb{C}^d$-valued holomorphic map near $\{0\}\times\mathcal{G}^1_0(\mathbb{C}^N)\times\mathcal{G}^{m\kappa_0+1}_0(\mathbb{C}^N)\subset\mathbb{C}^{(m+1)n}\times\mathcal{G}^1_0(\mathbb{C}^N)\times\mathcal{G}^{m\kappa_0+1}_0(\mathbb{C}^N)$. Moreover, it also follows from Lemma 8.1 (ii) and the induction assumption that the components of each ϱ_α belong to the space $\mathcal{E}^{(m+1)\kappa_0}_{(m+1)n}$. Next, we notice that we have

$$H\circ v^{m+1} = v^{m+1}(f\circ v^{m+1},\bar f\circ\bar v^m,\ldots,\bar f\circ\bar v^1). \tag{10.32}$$

Since M is assumed to be in $\mathcal{C}$, by Proposition 7.5, we may choose n integers $i_1,\ldots,i_n\in\{1,\ldots,d\}$ and n multi-indices $\alpha^{(1)},\ldots,\alpha^{(n)}\in\mathbb{N}^n\setminus\{0\}$ of length $\le\kappa_0$ so that $z\mapsto(\bar q^{i_1}_{\alpha^{(1)}}(z,0),\ldots,\bar q^{i_n}_{\alpha^{(n)}}(z,0))$ is of generic rank n. Consider the following singular analytic system in the unknowns $(T^1,\ldots,T^{m+1})\in\mathbb{C}^n\times\mathbb{C}^n\ldots\times\mathbb{C}^n = \mathbb{C}^{(m+1)n}$:

$$\begin{cases} \bar Q^{i_\nu}_{\chi^{\alpha^{(\nu)}}}(T^m, v^{m+1}(T^1,\ldots,T^{m+1})) = \varrho^{i_\nu}_{\alpha^{(\nu)}}(t^{[m+1]},\lambda^1,\Lambda^{(m+1)\kappa_0}), & \nu=1,\ldots,n,\\ T^k = \overline{\widetilde{\Psi}^0_k}(t^{[k]},\lambda^1,\Lambda^{k\kappa_0}), & k\in\{1,\ldots,m\},\ k\text{ odd},\\ T^k = \widetilde{\Psi}^0_k(t^{[k]},\lambda^1,\Lambda^{k\kappa_0}), & k\in\{1,\ldots,m\},\ k\text{ even}.\end{cases} \tag{10.33}$$

Note that by (10.29), (10.30), (10.32) and the induction assumption, for any $H = (f,g)\in\mathrm{Aut}(M,0)$, the map

$$(T^1,T^2,\ldots,T^m,T^{m+1}) = \vartheta_f\left(t^{[m+1]}\right) := (\bar f\circ\bar v^1, f\circ v^2,\ldots,\bar f\circ\bar v^m, f\circ v^{m+1}) \tag{10.34}$$

is a solution of the system (10.33) with $\lambda^1 := j^1_0\overline{H}$ and $\Lambda^{(m+1)\kappa_0} := j^{(m+1)\kappa_0}_0 H$. Moreover, it follows from the normality of the chosen coordinates (z,w) that the Jacobian matrix of the map ϑ_f is the following $(m+1)n\times(m+1)n$ block triangular matrix

$$\begin{pmatrix} \frac{\partial\bar f}{\partial\chi}(0) & 0 & 0 & \ldots & 0\\ 0 & \frac{\partial f}{\partial z}(0) & 0 & \ldots & 0\\ 0 & 0 & \frac{\partial\bar f}{\partial\chi}(0) & 0 & 0\\ \vdots & \vdots & 0 & \ddots & 0\\ 0 & 0 & 0 & 0 & \frac{\partial f}{\partial z}(0)\end{pmatrix}. \tag{10.35}$$

Hence the map ϑ_f is a local biholomorphism of $\mathbb{C}^{(m+1)n}$ fixing the origin. In order to apply Corollary 3.2 so as to get a parametrization of the invertible solutions of the system (10.33) we need to check that the holomorphic map

$$(T^1,\ldots,T^m,T^{m+1})\mapsto\left(T^1,\ldots,T^m,\left(\bar Q^{i_\nu}_{\chi^{\alpha^{(\nu)}}}(T^m,v^{m+1}(T^1,\ldots,T^{m+1}))\right)_{\nu=1,\ldots,n}\right)$$

is of generic rank $(m+1)n$. Since the Jacobian of this system with respect to $(T^1,\ldots,T^{m+1})$ is equal to the Jacobian of the map

$$(T^1,\ldots,T^m,T^{m+1})\mapsto\left(\bar Q^{i_\nu}_{\chi^{\alpha^{(\nu)}}}(T^m,v^{m+1}(T^1,\ldots,T^{m+1}))\right)_{\nu=1,\ldots,n}$$

with respect to T^{m+1}, it is enough to check that this map is of generic rank n for generic $T^1,\dots,T^m$ (or, equivalently, is of generic rank n for one particular point $T_0^1,\dots T_0^m$). This is indeed the case since, by the above choice, the holomorphic map

$$T^{m+1}\mapsto \left(\bar{Q}^{i_\nu}_{\chi^{\alpha^{(\nu)}}}(0,T^{m+1},0)\right)_{\nu=1,\dots,n}=\left(\bar{q}^{\,i_1}_{\alpha^{(1)}}(T^{m+1},0),\dots,\bar{q}^{\,i_n}_{\alpha^{(n)}}(T^{m+1},0)\right)$$

is of generic rank n. We may, therefore, apply Corollary 3.2 to conclude that there exists a holomorphic map

$$\Gamma^0_{m+1}\colon \mathbb{C}^{(m+1)n}\times \mathsf{GL}_{\mathsf{(m+1)n}}(\mathbb{C})\times \mathcal{G}^1_0(\mathbb{C}^N)\times \mathcal{G}^{(m+1)\kappa_0}_0(\mathbb{C}^N)\to \mathbb{C}^{(m+1)n} \tag{10.36}$$

defined in a neighbourhood of $\{0\}\times \mathsf{GL}_{\mathsf{(m+1)n}}(\mathbb{C})\times \mathcal{G}^1_0(\mathbb{C}^N)\times \mathcal{G}^{(m+1)\kappa_0}_0(\mathbb{C}^N)$, satisfying

$$\Gamma^0_{m+1}(0,A,\lambda^1,\Lambda^{(m+1)\kappa_0})=0,\quad \frac{\partial \Gamma^0_{m+1}}{\partial t^{[m+1]}}(0,A,\lambda^1,\Lambda^{(m+1)\kappa_0})=A \tag{10.37}$$

for all $(A,\lambda^1,\Lambda^{(m+1)\kappa_0})\in \mathsf{GL}_{\mathsf{(m+1)n}}(\mathbb{C})\times \mathcal{G}^1_0(\mathbb{C}^N)\times \mathcal{G}^{(m+1)\kappa_0}_0(\mathbb{C}^N)$ such that if $u\colon(\mathbb{C}^{(m+1)n},0)\to(\mathbb{C}^{(m+1)n},0)$ is a germ of a biholomorphism satisfying (10.33) for some $(\lambda^1,\Lambda^{(m+1)\kappa_0})\in \mathcal{G}^1_0(\mathbb{C}^N)\times \mathcal{G}^{\kappa_0}_0(\mathbb{C}^N)$, then

$$u\left(t^{[m+1]}\right)=\Gamma^0_{m+1}(t^{[m+1]},j^1_0u,\lambda^1,\Lambda^{(m+1)\kappa_0}).$$

Consider the following map $\varpi\colon \mathcal{G}^1_0(\mathbb{C}^N)\times \mathcal{G}^1_0(\mathbb{C}^N)\to \mathsf{GL}_{\mathsf{(m+1)n}}(\mathbb{C})$ given by

$$(\lambda^1,\Lambda^1)\mapsto \varpi(\lambda^1,\Lambda^1):=\begin{pmatrix} \widetilde{\lambda}^1 & 0 & 0 & \dots & 0\\ 0 & \widetilde{\Lambda}^1 & 0 & \dots & 0\\ 0 & 0 & \widetilde{\lambda}^1 & 0 & 0\\ \vdots & \vdots & 0 & \ddots & 0\\ 0 & 0 & 0 & 0 & \widetilde{\Lambda}^1\end{pmatrix}. \tag{10.38}$$

We write $\Gamma^0_{m+1}=(\Gamma^{0,1}_{m+1},\dots,\Gamma^{0,m+1}_{m+1})\in\mathbb{C}^n\times\mathbb{C}^n\times\dots\times\mathbb{C}^n=\mathbb{C}^{(m+1)n}$ and set

$$\widetilde{\Psi}^0_{m+1}(t^{[m+1]},\lambda^1,\Lambda^{(m+1)\kappa_0}):=\Gamma^{0,m+1}_{m+1}(t^{[m+1]},\varpi(\lambda^1,\Lambda^1),\lambda^1,\Lambda^{(m+1)\kappa_0}). \tag{10.39}$$

Since for every $(\lambda^1,\Lambda^{(m+1)\kappa_0})\in\mathcal{G}^1_0(\mathbb{C}^N)\times\mathcal{G}^{(m+1)\kappa_0}_0$, $\widetilde{\Psi}^0_{m+1}(0,\lambda^1,\Lambda^{(m+1)\kappa_0})=0$ and $\Psi^0_m(0,\lambda^1,\Lambda^{m\kappa_0})=0$ by (10.37), (10.39) and the induction assumption, we may also set

$$\begin{aligned}\widehat{\Psi}^0_{m+1}&(t^{[m+1]},\lambda^1,\Lambda^{(m+1)\kappa_0})\\ &:=Q(\widetilde{\Psi}^0_{m+1}(t^{[m+1]},\lambda^1,\Lambda^{(m+1)\kappa_0}),\overline{\Psi^0_m}(t^{[m]},\lambda^1,\Lambda^{m\kappa_0})).\end{aligned} \tag{10.40}$$

Then $\Psi^0_{m+1}=(\widetilde{\Psi}^0_{m+1},\widehat{\Psi}^0_{m+1})$ defines a $\mathbb{C}^N$-valued holomorphic map in a neighbourhood of $\{0\}\times\mathcal{G}^1_0(\mathbb{C}^N)\times\mathcal{G}^{(m+1)\kappa_0}_0(\mathbb{C}^N)$ in $\mathbb{C}^{(m+1)n}\times\mathcal{G}^1_0(\mathbb{C}^N)\times\mathcal{G}^{(m+1)\kappa_0}_0(\mathbb{C}^N)$ and satisfies $\Psi^0_{m+1}(0,\lambda^1,\Lambda^{(m+1)\kappa_0})=0$ for all $(\lambda^1,\Lambda^{(m+1)\kappa_0})\in\mathcal{G}^1_0(\mathbb{C}^N)\times\mathcal{G}^{(m+1)\kappa_0}_0(\mathbb{C}^N)$. Furthermore, in view of (10.35), (10.38) and the above construction, for any $H=(f,g)\in \operatorname{Aut}(M,0)$, we have that for $t^{[m+1]}$ sufficiently small in $\mathbb{C}^{(m+1)n}$,

$$(H\circ v^{m+1})\left(t^{[m+1]}\right)=\Psi^0_{m+1}(t^{[m+1]},j^1_0\overline{H},j^{(m+1)\kappa_0}_0H). \tag{10.41}$$

From (10.37), we have $\dfrac{\partial \Gamma^0_{m+1}}{\partial t^{[m+1]}}(0,\varpi(\lambda^1,\Lambda^1),\lambda^1,\Lambda^{(m+1)\kappa_0}) = \varpi(\lambda^1,\Lambda^1)$ and hence from (10.38) and (10.39), we get that

$$\frac{\partial \widetilde{\Psi}^0_{m+1}}{\partial t^{m+1}}(0,\lambda^1,\Lambda^{(m+1)\kappa_0}) = \widetilde{\Lambda}^1$$

for all $(\lambda^1,\Lambda^{(m+1)\kappa_0}) \in \mathcal{G}^1_0(\mathbb{C}^N)\times\mathcal{G}^{(m+1)\kappa_0}_0(\mathbb{C}^N)$. This proves part of (10.26). To finish the case $\beta=0$, we need to check that each of the components of the constructed parametrization Ψ^0_{m+1} belongs to the space $\mathcal{E}^{(m+1)\kappa_0}_{(m+1)n}$. It is actually enough to prove that each component of $\widetilde{\Psi}^0_{m+1}$ is in $\mathcal{E}^{(m+1)\kappa_0}_{(m+1)n}$ by (10.40), Lemma 8.1 (i) and the induction assumption. Consider the map Γ^0_{m+1} given in (10.36) and obtained from Corollary 3.2 applied to the system (10.33). Then from that result, we know that if we write the Taylor expansion

$$\Gamma^0_{m+1}(t^{[m+1]},A,\lambda^1,\Lambda^{(m+1)\kappa_0}) := \sum_{\gamma\in\mathbb{N}^{(m+1)n}} \Gamma^0_{m+1,\gamma}(A,\lambda^1,\Lambda^{(m+1)\kappa_0})\left(t^{[m+1]}\right)^{\gamma},$$

then each $\Gamma^0_{m+1,\gamma}(A,\lambda^1,\Lambda^{(m+1)\kappa_0})$ can be written in the following form:

$$\frac{\Omega_\gamma(A,\lambda^1,\Lambda^{(m+1)\kappa_0})}{(\det A)^{e_\gamma}}, \tag{10.42}$$

where e_γ is a nonnegative integer and $\Omega_\gamma(A,\lambda^1,\Lambda^{(m+1)\kappa_0})$ is a polynomial map in the following arguments:

$$\left((\partial^{s_\gamma}_{t^{[m+1]}}\varrho_\alpha)(0,\lambda^1,\Lambda^{(m+1)\kappa_0})\right)_{|\alpha|\le\kappa_0}, \tag{10.43}$$

$$\left((\partial^{s_\gamma}_{t^{[m+1]}}\Psi^0_k)(0,\lambda^1,\Lambda^{m\kappa_0})\right)_{1\le k\le m}, \tag{10.44}$$

$$\left((\partial^{s_\gamma}_{t^{[m+1]}}\overline{\Psi^0_k})(0,\lambda^1,\Lambda^{m\kappa_0})\right)_{1\le k\le m} \tag{10.45}$$

and in A. Here s_γ denotes some positive integer. By the induction assumption and the discussion after (10.31), each component of each term in (10.43), (10.44) and in (10.45) can be written as a polynomial in λ^1 and $\Lambda^{m\kappa_0}$ over a product of powers of $\det\widetilde{\lambda}^1$ and $\det\widetilde{\Lambda}^1$. Now, if we write the Taylor expansion

$$\widetilde{\Psi}^0_{m+1}(t^{[m+1]},\lambda^1,\Lambda^{(m+1)\kappa_0}) := \sum_{\gamma\in\mathbb{N}^{(m+1)n}} \widetilde{\Psi}^0_{m+1,\gamma}(\lambda^1,\Lambda^{(m+1)\kappa_0})\left(t^{[m+1]}\right)^{\gamma},$$

then in view of (10.39), each component of $\widetilde{\Psi}^0_{m+1,\gamma}(\lambda^1,\Lambda^{(m+1)\kappa_0})$ is a component of (10.42) with $A=\varpi(\lambda^1,\Lambda^1)$. From (10.38) and the above discussion, we see that each component $\widetilde{\Psi}^0_{m+1}$ indeed belongs to the space $\mathcal{E}^{(m+1)\kappa_0}_{(m+1)n}$. This completes the proof of Proposition 10.2 in the case $\beta=0$.

10.3.2. *Induction for the construction of the parametrizations* $(\Psi^\beta_{m+1})_{\beta\in\mathbb{N}^N}$. We assume here that the desired parametrizations Ψ^β_j for all $j\in\{1,\ldots,m\}$ and all $\beta\in\mathbb{N}^N$ as well as the parametrizations Ψ^β_{m+1} for $|\beta|\le r$ have already been constructed. We will construct the parametrization Ψ^δ_{m+1} for $\delta\in\mathbb{N}^N$ of length $r+1$ with the desired properties. We first reduce this to the construction for δ corresponding to purely transversal derivatives in Lemma 10.4, and start with the following computational lemma.

Lemma 10.3. *With the above notation and assumptions, the following hold.*

(i) *For any $H \in \mathrm{Aut}(M,0)$, we have*

$$\frac{\partial \Psi^0_{m+1}}{\partial t^{m+1}}(t^{[m+1]}, j_0^1\overline{H}, j_0^{(m+1)\kappa_0}H) = \left(\frac{\partial H}{\partial z}\circ v^{m+1}\right)\left(t^{[m+1]}\right) + \left(\frac{\partial H}{\partial w}\circ v^{m+1}\right)\left(t^{[m+1]}\right)\cdot Q_z\left(t^{m+1}, \bar v^m\left(t^{[m]}\right)\right), \tag{10.46}$$

for all $t^{[m+1]}$ sufficiently close to the origin in $\mathbb{C}^{(m+1)n}$.

(ii) *Let*

$$I_r = \left\{\mu \in \mathbb{N}^N : |\mu| = r\right\}, \quad J_r = \left\{\nu \in \mathbb{N}^d : |\nu| = r+1\right\}.$$

For every multi-index $\delta \in \mathbb{N}^N$ of length $r+1$, there exist holomorphic $n\times 1$ matrices $(h_{\mu,\delta})_{\mu\in I_r}$ and holomorphic functions $(\xi_{\nu,\delta})_{\nu\in J_r}$ defined near the origin in $\mathbb{C}^{(m+1)n}$, such that for any $H \in \mathrm{Aut}(M,0)$,

$$\begin{aligned}\left((\partial^\delta H)\circ v^{m+1}\right)\left(t^{[m+1]}\right) &= \sum_{\mu\in I_r} \frac{\partial \Psi^\mu_{m+1}}{\partial t^{m+1}}\left(t^{[m+1]}, j_0^1\overline{H}, j_0^{(m+1)\kappa_0+r}H\right)\cdot h_{\mu,\delta}\left(t^{[m+1]}\right)\\ &\quad + \sum_{\nu\in J_r} \xi_{\nu,\delta}\left(t^{[m+1]}\right)\left(\frac{\partial^{|\nu|}H}{\partial w^\nu}\circ v^m\right)\left(t^{[m+1]}\right).\end{aligned} \tag{10.47}$$

Proof. Part (i) of Lemma 10.3 follows by differentiating the identity

$$\begin{aligned}\left(H\circ v^{m+1}\right)\left(t^{[m+1]}\right) &= H\left(t^{m+1}, Q\left(t^{m+1}, \bar v^m\left(t^{[m]}\right)\right)\right)\\ &= \Psi^0_{m+1}\left(t^{[m+1]}, j_0^1\overline{H}, j_0^{(m+1)\kappa_0}H\right),\end{aligned}$$

which holds for all $H \in \mathrm{Aut}(M,0)$, with respect to t^{m+1}.

To prove (ii) for every $\delta = (\delta_1, \ldots, \delta_N) \in \mathbb{N}^N$ of length r, we set $q_\delta := \sum_{i=1}^n \delta_i$, and we prove (10.47) by induction on q_δ. If $q_\delta = 0$, (10.47) is a trivial statement. Suppose now that $q_\delta > 0$ with $\delta \in \mathbb{N}^N$ of length $r+1$. Then we may write $\delta = e+\omega$ with $e \in S$ (S denotes the set of multi-indices of length 1 having 0 from the $(n+1)$-th component to the N-th component, as defined in Proposition 10.2 (i)), $q_\omega < q_\delta$ and $|\omega| = r$. Differentiating the identity

$$\begin{aligned}\Psi^\omega_{m+1}(t^{[m+1]}, j_0^1\overline{H}, j_0^{(m+1)\kappa_0+r}H) &= \left((\partial^\omega H)\circ v^{m+1}\right)\left(t^{[m+1]}\right)\\ &= \left((\partial^\omega H)(t^{m+1}, Q(t^{m+1}, \bar v^m(t^{[m]})))\right),\end{aligned}$$

which holds for all $H \in \mathrm{Aut}(M,0)$ with respect to t^{m+1}, we get (similarly to what was done for the proof of (i)) that

$$\begin{aligned}\frac{\partial \Psi^\omega_{m+1}}{\partial t^{m+1}}\left(t^{[m+1]}, j_0^1\overline{H}, j_0^{(m+1)\kappa_0+r}H\right) &= \left(\left(\frac{\partial}{\partial z}(\partial^\omega H)\right)\circ v^{m+1}\right)\left(t^{[m+1]}\right)\\ &\quad + Q_z\left(t^{m+1}, \bar v^m\left(t^{[m]}\right)\right)\cdot\left(\left(\frac{\partial}{\partial w}(\partial^\omega H)\right)\circ v^{m+1}\right)\left(t^{[m+1]}\right).\end{aligned} \tag{10.48}$$

Hence (10.48) and the induction hypothesis show that $(\partial^\delta H)\circ v^{m+1}$ may be written in the form (10.47). The proof of the lemma is complete. $\square$

We let O_{r+1} denote the set of multi-indices $\delta \in \mathbb{N}^N$ of length $r+1$ with 0 from the first to the n-th component. Note that the set of multi-indices in $\mathbb{N}^d$ of length $r+1$ can be identified with O_{r+1} via the map $\mathbb{N}^d \ni \nu \mapsto (0,\nu) \in O_{r+1}$. The purpose of Lemma 10.4 is to show that it is enough to find the parametrizations Ψ^δ_{m+1} with $\delta \in O_{r+1}$ to get to the parametrizations Ψ^δ_{m+1} for all $\delta \in \mathbb{N}^N$ of length $r+1$.

Lemma 10.4. *Assume that one has constructed the parametrizations* (Ψ^β_{m+1}) *for all multi-indices* $\beta \in O_{r+1}$ *satisfying the required properties of Proposition* 10.2. *Then one may construct all the parametrizations* (Ψ^δ_{m+1}) *for all multi-indices* $\delta \in \mathbb{N}^N$ *of length* $r+1$ *satisfying the required properties of Proposition* 10.2.

Proof. If $r>0$, for any multi-index $\delta \in \mathbb{N}^N \setminus O_{r+1}$, we set (writing $t = t^{[m+1]}$)

$$\begin{aligned}(10.49)\qquad &\Psi^\delta_{m+1}(t,\lambda^1,\Lambda^{(m+1)\kappa_0+r+1}) \\ &:= \sum_{\mu\in I_r} \frac{\partial \Psi^\mu_{m+1}}{\partial t^{m+1}}(t,\lambda^1,\Lambda^{(m+1)\kappa_0+r})\cdot h_{\mu,\delta}(\cdot) \\ &\qquad\qquad + \sum_{\nu\in J_r} \xi_{\nu,\delta}(t)\cdot \Psi^{(0,\nu)}_{m+1}(t,\lambda^1,\Lambda^{(m+1)\kappa_0+r+1}).\end{aligned}$$

Then it follows from Lemma 10.3 (ii) that the desired parametrizations are constructed for all δ of length $r+1$. Moreover, it is clear that since each component of Ψ^β_{m+1} for $\beta \in O_{r+1}$ or for $|\beta| = r$ belongs to the space $\mathcal{E}^{(m+1)\kappa_0+r+1}_{(m+1)n}$ by assumption, the same fact also holds for each component of $\Psi^\delta_{m+1}(t^{[m+1]},\lambda^1,\Lambda^{(m+1)\kappa_0+r+1})$ with $\delta \notin O_{r+1}$.

In case $r=0$, note that $\mathbb{N}^d \setminus O_{r+1}$ coincides with S as defined in Proposition 10.2 (i). If the parametrizations Ψ^δ_{m+1} for $\delta \in O_{r+1}$ have been constructed, then we may set

$$\begin{aligned}(10.50)\quad \left(\Psi^\delta_{m+1}\left(t^{[m+1]},\lambda^1,\Lambda^{(m+1)\kappa_0+1}\right)\right)_{\delta\in S} &:= \frac{\partial \Psi^0_{m+1}}{\partial t^{m+1}}(t^{[m+1]},\lambda^1,\Lambda^{(m+1)\kappa_0}) \\ -\left(\Psi^\delta_{m+1}\left(t^{[m+1]},\lambda^1,\Lambda^{(m+1)\kappa_0+1}\right)\right)_{\delta\in O_{r+1}} &\cdot Q_z\left(t^{m+1},\bar v^m\left(t^{[m]}\right)\right),\end{aligned}$$

where we see $(\Psi^\delta_{m+1})_{\delta\in S}$ as an $N\times n$ matrix and $(\Psi^\delta_{m+1})_{\delta\in O_{r+1}}$ as an $N\times d$ matrix. It follows from Lemma 10.3 (i) that the parametrizations are constructed for all δ of length 1. It also follows from (10.50) that our construction gives

$$(\widetilde\Psi^\delta_{m+1}(0,\lambda^1,\Lambda^{(m+1)\kappa_0}))_{\delta\in S} = \frac{\partial \widetilde\Psi^0_{m+1}}{\partial t^{m+1}}(0,\lambda^1,\Lambda^{(m+1)\kappa_0}),$$

which proves the required condition in (10.26). Finally, since each component of Ψ^μ_{m+1} for $\mu = 0$ or for $\mu \in O_{r+1}$ belongs to the space $\mathcal{E}^{(m+1)\kappa_0+1}_{(m+1)n}$ by assumption, the same fact also holds for each component of $\Psi^\delta_{m+1}(t^{[m+1]},\lambda^1,\Lambda^{(m+1)\kappa_0+1})$ with $\delta \in S$ in view of (10.50). This completes the proof of Lemma 10.4. $\square$

By Lemma 10.4, it is now enough to construct the parametrization Ψ^δ_{m+1} for $\delta \in \mathbb{N}^N$ with $|\delta| = r+1$ and $\delta \in O_{r+1}$. In what follows, we fix such a multi-index δ, and we, therefore, write $\delta = (0,\mu)$ with $\mu \in \mathbb{N}^d$ and $|\mu| = r+1$.

We start by using (9.8) in which we set $Z = v^{m+1} = v^{m+1}\left(t^{[m+1]}\right)$ and $\zeta = \bar v^m = \bar v^m\left(t^{[m]}\right)$. We obtain that for every $\alpha \in \mathbb{N}^n \setminus \{0\}$ and for every $H=(f,g) \in \mathrm{Aut}(M,0)$ the expression

$$\left(\frac{\partial^{|\mu|} f}{\partial w^\mu} \circ v^{m+1}\right) \cdot \left(\bar Q_{\chi^\alpha,z}(\bar f \circ \bar v^m, H \circ v^{m+1}) + Q_z(f \circ v^{m+1}, \overline{H} \circ \bar v^m) \cdot \bar Q_{\chi^\alpha,w}(\bar f \circ \bar v^m, H \circ v^{m+1})\right) \tag{10.51}$$

is equal to (10.52)+(10.53), where

$$\mathcal{T}'_{\mu,\alpha}\big(v^{m+1}, \bar v^m, H \circ v^{m+1}, \overline{H} \circ \bar v^m, ((\partial^\beta H) \circ v^{m+1})_{1 \le |\beta| \le r}, ((\partial^\beta \overline{H}) \circ \bar v^m)_{1 \le |\beta| \le r+1}\big)\,, \tag{10.52}$$

$$\frac{\mathcal{Q}_{\mu,\alpha}\left(v^{m+1}, \bar v^m, ((\partial^\beta \overline{H}) \circ \bar v^m)_{1 \le |\beta| \le |\alpha|+r+1}\right)}{\left(\mathcal{D}(v^{m+1}, \bar v^m, ((\partial^\beta \overline{H}) \circ \bar v^m)_{|\beta|=1})\right)^{2|\alpha|+r}}. \tag{10.53}$$

As in all previous analogous situations, we are only interested in multi-indices $\alpha \in \mathbb{N}^n \setminus \{0\}$ such that $|\alpha| \le \kappa_0$. For such α's, we define the following $n \times d$ matrix with holomorphic coefficients near the origin in $\mathbb{C}^{(m+1)n}$:

$$\begin{aligned} K_\alpha(T^1, T^2, \ldots, T^{m+1}) := {} & \bar Q_{\chi^\alpha,z}(T^m, v^{m+1}(T^1, \ldots, T^{m+1})) + \\ & Q_z(T^{m+1}, \bar v^m(T^1, \ldots, T^m)) \cdot \bar Q_{\chi^\alpha,w}(T^m, v^{m+1}(T^1, \ldots, T^{m+1})). \end{aligned} \tag{10.54}$$

Here each $T^j \in \mathbb{C}^n$, $j = 1, \ldots, m+1$. We also write $K_\alpha := (K_\alpha^1, \ldots, K_\alpha^d)$ and set

$$\begin{aligned} & \phi_{\mu,\alpha}(t^{[m+1]}, \lambda^1, \Lambda^{(m+1)\kappa_0+r+1}) \\ & := \mathcal{T}'_{\mu,\alpha}\left(v^{m+1}, \bar v^m, \Psi^0_{m+1}, \overline{\Psi^0_m}, \Psi^{(r)}_{m+1}\overline{\Psi^{(r+1)}_m}\right) + \frac{\mathcal{Q}_{\mu,\alpha}\left(v^{m+1}, \bar v^m, \overline{\Psi^{(r+1)}_m}\right)}{\left(\mathcal{D}(v^{m+1}, \bar v^m, \overline{\Psi^{(1)}_m})\right)^{2|\alpha|+r}}. \end{aligned} \tag{10.55}$$

By the induction assumption, (10.31) and Lemma 8.1, $\phi_{\mu,\alpha} := (\phi^1_{\mu,\alpha}, \ldots, \phi^d_{\mu,\alpha})$ defines a $\mathbb{C}^d$-valued holomorphic map defined in a neighbourhood of $\{0\} \times \mathcal{G}^1_0(\mathbb{C}^N) \times \mathcal{G}_0^{(m+1)\kappa_0+r+1}(\mathbb{C}^N)$ in $\mathbb{C}^{(m+1)n} \times \mathcal{G}^1_0(\mathbb{C}^N) \times \mathcal{G}_0^{(m+1)\kappa_0+r+1}(\mathbb{C}^N)$, whose components belong to $\mathcal{E}^{(m+1)\kappa_0+r+1}_{(m+1)n}$. Since M is assumed to be in $\mathcal{C}$, we may again, by Proposition 7.5, choose n integers $i_1, \ldots, i_n \in \{1, \ldots, d\}$ and n multi-indices $\alpha^{(1)}, \ldots, \alpha^{(n)} \in \mathbb{N}^n \setminus \{0\}$ of length $\le \kappa_0$ so that $z \mapsto (\bar q^{i_1}_{\alpha^{(1)}}(z,0), \ldots, \bar q^{i_n}_{\alpha^{(n)}}(z,0))$ is of generic rank n. In what follows, we write $Y^k = (Y^k_1, \ldots, Y^k_n) \in \mathbb{C}^n$, $k \in \{1, \ldots, m\}$ and for $j = 1, \ldots, n$ we denote by $\widetilde\Psi^{0,j}_k$ (resp. $\overline{\widetilde\Psi^{0,j}_k}$) the j-th component of $\widetilde\Psi^0_k$ (resp. $\overline{\widetilde\Psi^0_k}$). Consider the following linear singular analytic system in the unknowns $(X, Y^1, \ldots, Y^m) \in \mathbb{C}^n \times \mathbb{C}^n \times \ldots \times \mathbb{C}^n = \mathbb{C}^{(m+1)n}$ given by

$$\begin{cases} X \cdot K^{i_\nu}_{\alpha^{(\nu)}}(T^1, \ldots, T^{m+1}) = \phi^{i_\nu}_{\mu,\alpha^{(\nu)}}(t^{[m+1]}, \lambda^1, \Lambda^{(m+1)\kappa_0+r+1}), \quad 1 \le \nu \le n, \\ \qquad Y^k_j \cdot T^k_j = \overline{\widetilde\Psi^{0,j}_k}(t^{[k]}, \lambda^1, \Lambda^{k\kappa_0}), \quad 1 \le j \le n,\ 1 \le k \le m,\ k \text{ odd}, \\ \qquad Y^k_j \cdot T^k_j = \widetilde\Psi^{0,j}_k(t^{[k]}, \lambda^1, \Lambda^{k\kappa_0}), \quad 1 \le j \le n,\ 1 \le k \le m,\ k \text{ even}, \end{cases} \tag{10.56}$$

in which for every $k \in \{1, \ldots, m+1\}$ we have set

$$T^k := \begin{cases} \widetilde\Psi^0_k(t^{[k]}, \lambda^1, \Lambda^{k\kappa_0}) \text{ if } k \text{ is even}, \\ \overline{\widetilde\Psi^0_k}(t^{[k]}, \lambda^1, \Lambda^{k\kappa_0}) \text{ if } k \text{ is odd}. \end{cases} \tag{10.57}$$

In view of (10.51), (10.52), (10.32), (10.54), (10.55) and the induction assumption, we know that the vector $((\partial_w^\mu f)\circ v^{m+1},1,1,\ldots,1)$ is a solution of the system (10.56) with $\lambda^1=j_0^1\overline{H}$ and $\Lambda^{(m+1)\kappa_0+r+1}=j_0^{(m+1)\kappa_0+r+1}H$ for any $H\in \mathrm{Aut}(M,0)$. To apply Proposition 6.3 to get a parametrization of the solutions of the system (10.56)–(10.57), we need to check that the $(m+1)n\times(m+1)n$ matrix associated to this linear system has maximal generic rank and that for all $(\lambda^1,\Lambda^{(m+1)\kappa_0+r+1})\in \mathcal{G}_0^1(\mathbb{C}^N)\times\mathcal{G}_0^{(m+1)\kappa_0+r+1}(\mathbb{C}^N)$, the mapping $\mathcal{I}\colon(\mathbb{C}^{(m+1)n},0)\to(\mathbb{C}^{(m+1)n},0)$ given by

$$(10.58)\qquad t^{[m+1]}\mapsto\left(\overline{\widetilde{\Psi}_1^0}(t^{[1]},\cdot),\widetilde{\Psi}_2^0(t^{[2]},\cdot),\ldots,\overline{\widetilde{\Psi}_m^0}(t^{[m]},\cdot),\widetilde{\Psi}_{m+1}^0(t^{[m+1]},\cdot)\right)$$

is a local biholomorphism at the origin. Let us check the first condition. The $(m+1)n\times(m+1)n$ matrix associated to the linear singular system (10.56) is given by the following:

$$(10.59)\qquad \begin{pmatrix} K(T^1,\ldots,T^{m+1}) & 0 & \ldots & \ldots & \ldots & \ldots & \ldots & 0\\ 0 & T_1^1 & 0 & \ldots & \ldots & \ldots & \ldots & 0\\ \vdots & 0 & \ddots & 0 & \ldots & \ldots & \ldots & 0\\ \vdots & \vdots & 0 & T_n^1 & 0 & \ldots & \ldots & 0\\ \vdots & \vdots & \ldots & 0 & T_1^2 & 0 & \ldots & 0\\ \vdots & \vdots & \ldots & \ldots & 0 & \ddots & 0 & 0\\ \vdots & \vdots & \ldots & \ldots & \ldots & 0 & \ddots & 0\\ 0 & 0 & \ldots & \ldots & \ldots & \ldots & 0 & T_n^m \end{pmatrix},$$

where $T^k=(T_1^k,\ldots,T_n^k)$ for $k=1,\ldots,m$ and K is the $n\times n$ matrix whose columns are given by the vectors $K_{\alpha^{(\nu)}}^{i_\nu}$, $\nu=1,\ldots,n$. Hence it is enough to show that K has generic rank n. In fact, we claim that K restricted to the subspace $T^1=\ldots=T^m=0$ already has generic rank n. Indeed, in view of (10.54) and the normality of the chosen coordinates (z,w), $K(0,\ldots,0,T^{m+1})$ is the $n\times n$ matrix whose columns are the vectors $\alpha^{(\nu)}!\,\dfrac{\partial\bar{q}_{\alpha^{(\nu)}}^{i_\nu}}{\partial z}(T^{m+1},0)$ for $\nu=1,\ldots,n$, which proves the above claim and the fact that the matrix (10.59) has generic rank $(m+1)n$. We now prove that the local holomorphic map given by (10.58) is invertible at the origin. For this, we note that its Jacobian matrix at the origin is a $(m+1)n$ lower triangular matrix by blocks with $(m+1)$ blocks of size $n\times n$ in the diagonal. Moreover, in view of (10.26), we obtain that this matrix is of the following form:

$$(10.60)\qquad \begin{pmatrix} \widetilde{\lambda}^1 & 0 & 0 & \ldots & 0\\ * & \widetilde{\Lambda}^1 & 0 & \ldots & 0\\ * & * & \widetilde{\lambda}^1 & 0 & 0\\ * & * & * & \ddots & 0\\ * & * & * & * & \widetilde{\Lambda}^1 \end{pmatrix}$$

and hence is invertible. This shows that the map given by (10.58) is a local biholomorphism. We may now apply Proposition 6.3 to conclude that there exists a holomorphic map

$$(10.61)\qquad \Gamma_{m+1}^\mu\colon\mathbb{C}^{(m+1)n}\times\mathcal{G}_0^1(\mathbb{C}^N)\times\mathcal{G}_0^{(m+1)\kappa_0+r+1}(\mathbb{C}^N)\to\mathbb{C}^n\times\mathbb{C}^{mn}$$

defined in a neighbourhood of $\{0\}\times\mathcal{G}_0^1(\mathbb{C}^N)\times\mathcal{G}_0^{(m+1)\kappa_0+r+1}(\mathbb{C}^N)$ such that if

$$(X,Y)\colon(\mathbb{C}^{(m+1)n},0)\to(\mathbb{C}^{(m+1)n},0)$$

solves (10.56)–(10.57) for some $(\lambda^1,\Lambda^{(m+1)\kappa_0+r+1})\in\mathcal{G}_0^1(\mathbb{C}^N)\times\mathcal{G}_0^{(m+1)\kappa_0+r+r}(\mathbb{C}^N)$, then $(X,Y)=(X\left(t^{[m+1]}\right),Y(t^{[m+1]}))=\Gamma^{\mu}_{m+1}(t^{[m+1]},\lambda^1,\Lambda^{(m+1)\kappa_0+r+1})$. We write

$$\Gamma^{\mu}_{m+1}:=(\Gamma^{\mu,1}_{m+1},\Gamma^{\mu,2}_{m+1},\ldots,\Gamma^{\mu,m+1}_{m+1})\in\mathbb{C}^n\times\mathbb{C}^n\times\ldots\times\mathbb{C}^n$$

and set

$$\widetilde{\Psi}^{\delta}_{m+1}(t^{[m+1]},\lambda^1,\Lambda^{(m+1)\kappa_0+r+1}):=\Gamma^{\mu,1}_{m+1}(t^{[m+1]},\lambda^1,\Lambda^{(m+1)\kappa_0+r+1}),\tag{10.62}$$

$$\begin{aligned}&\widehat{\Psi}^{\delta}_{m+1}(t^{[m+1]},\lambda^1,\Lambda^{(m+1)\kappa_0+r+1})\\&:=\widetilde{\Psi}^{\delta}_{m+1}\cdot Q_z(\widetilde{\Psi}^0_{m+1},\overline{\Psi^0_m})+W_\mu\left(v^{m+1},\bar v^m,\Psi^0_{m+1},\overline{\Psi^0_m},\Psi^{(r)}_{m+1},\overline{\Psi^{(r+1)}_m}\right),\end{aligned}\tag{10.63}$$

where W_μ is the $\mathbb{C}^d$-valued polynomial map given by Lemma 9.3. In view of the above construction and Lemma 9.3, the map Ψ^{δ}_{m+1} is the desired parametrization for the δ-th derivative of all elements of $\mathrm{Aut}(M,0)$. It remains to check that its components belong to the space $\mathcal{E}^{(m+1)\kappa_0+r+1}_{(m+1)n}$. In view of Lemma 8.1 (i), (10.62), (10.63) and the induction assumption, it is enough to check that the components of the map Γ^{μ}_{m+1} given by (10.61) belong to the above-mentioned space. For this we write the Taylor expansion

$$\Gamma^{\mu}_{m+1}(t^{[m+1]},\lambda^1,\Lambda^{(m+1)\kappa_0+r+1}):=\sum_{\gamma\in\mathbb{N}^{(m+1)n}}\Gamma^{\mu}_{m+1,\gamma}(\lambda^1,\Lambda^{(m+1)\kappa_0+r+1})\left(t^{[m+1]}\right)^{\gamma}.\tag{10.64}$$

From Proposition 6.3 we know that $\Gamma^{\mu}_{m+1,\gamma}(\lambda^1,\Lambda^{(m+1)\kappa_0+r+1})$ may be written in the following form:

$$\frac{B_\gamma\left[\left((\partial^{\nu}_{t^{[m+1]}}\phi_{\mu,\alpha})(0,\lambda^1,\Lambda^{(m+1)\kappa_0+r+1}),(\partial^{\nu}_{t^{[k]}}\widetilde{\Psi}^0_k),(\partial^{\nu}_{t^{[k]}}\overline{\widetilde{\Psi}^0_k})\right)_{\substack{1\le|\alpha|\le\kappa_0\\|\nu|\le d_\gamma;1\le k\le m+1}}\right]}{\left[\det\left(\dfrac{\partial\mathcal{I}}{\partial t^{[m+1]}}(0,\lambda^1,\Lambda^{(m+1)\kappa_0+r+1})\right)\right]^{c_\gamma}},\tag{10.65}$$

where B_γ is a $\mathbb{C}^{(m+1)n}$-valued polynomial map of its arguments, c_γ and d_γ are positive integers, and the last two arguments of B_γ are evaluated at $(0,\lambda^1,\Lambda^{k\kappa_0})$. Since each component of the $\phi_{\mu,\alpha}$, $\widetilde{\Psi}^0_k$ and $\overline{\widetilde{\Psi}^0_k}$ for $1\le|\alpha|\le\kappa_0$, $1\le k\le m+1$ belongs to the space $\mathcal{E}^{(m+1)\kappa_0+r+1}_{(m+1)n}$ by the above construction and the induction assumption and since the Jacobian matrix of $\mathcal{I}$ is given by (10.60), it is clear that each component of Γ^{μ}_{m+1} belongs to the space $\mathcal{E}^{(m+1)\kappa_0+r+1}_{(m+1)n}$. This completes the induction part of the proof of Proposition 10.2 and hence the proof of that proposition.

11. Proofs of Theorems 7.3, 7.4, 2.1, 2.2, 2.3, 1.1, 1.2

In this section, we keep the notation introduced in §10.

Proof of Theorem 7.3. By the minimality criterion of Baouendi, Ebenfelt and Rothschild (see e.g. [3]), there exists $1\le k_1\le d+1$ (where d is the codimension of M in $\mathbb{C}^N$) such that the generic rank of the Segre set map v^{k_1} is N. Applying firstly Proposition 10.2 with $k = 2k_1$, we obtain that there exists a map $\Psi^0_{2k_1}\colon \mathbb{C}^{2nk_1} \times \mathcal{G}^1_0(\mathbb{C}^N) \times \mathcal{G}^{2k_1\kappa_0}_0(\mathbb{C}^N) \to \mathbb{C}^N$ holomorphic in a neighbourhood of $\{0\} \times \mathcal{G}^1_0(\mathbb{C}^N) \times \mathcal{G}^{2k_1\kappa_0}_0(\mathbb{C}^N)$ whose components belong to the space $\mathcal{E}^{2k_1\kappa_0}_{2k_1n}$ such that for every $H \in \mathrm{Aut}(M,0)$ and for every $t^{[2k_1]}$ sufficiently close to the origin in $\mathbb{C}^{2k_1n}$ the following identity holds:

$$(H \circ v^{2k_1})(t^{[2k_1]}) = \Psi^0_{2k_1}(t^{[2k_1]}, j^1_0\overline{H}, j^{2k_1\kappa_0}_0 H).$$

Here we recall that $\kappa_0 = \kappa(M,0)$. By mimicing the proofs of [3, §4.1], one may construct a map $\Psi^0\colon \mathbb{C}^N \times \mathcal{G}^1_0(\mathbb{C}^N) \times \mathcal{G}^{2k_1\kappa_0}_0(\mathbb{C}^N) \to \mathbb{C}^N$ holomorphic in a neighbourhood of $\{0\} \times \mathcal{G}^1_0(\mathbb{C}^N) \times \mathcal{G}^{2k_1\kappa_0}_0(\mathbb{C}^N)$ whose components belong to the space $\mathcal{E}^{2k_1\kappa_0}_N$ such that for every $H \in \mathrm{Aut}(M,0)$ and for every Z sufficiently close to the origin in $\mathbb{C}^N$ the following identity holds:

$$H(Z) = \Psi^0(Z, j^1_0\overline{H}, j^{2k_1\kappa_0}_0 H).$$

Now the map $\psi(Z, \Lambda^{2k_1\kappa_0}) := \Psi^0(Z, \overline{\Lambda^1}, \Lambda^{2k_1\kappa_0})$ satisfies all the conclusions of Theorem 7.3 except that ψ is a function of $\Lambda^{2k_1\kappa_0} \in G^{2(d+1)\kappa_0}_0(\mathbb{C}^N)$, which is twice more than claimed. To complete the proof of the theorem, we apply again Proposition 10.2 with $k = k_1$, which may be assumed to be even (the other case being analogous) and obtain the existence of a map $\Psi^0_{k_1}\colon \mathbb{C}^{nk_1} \times \mathcal{G}^1_0(\mathbb{C}^N) \times \mathcal{G}^{k_1\kappa_0}_0(\mathbb{C}^N) \to \mathbb{C}^N$ holomorphic in a neighbourhood of $\{0\} \times \mathcal{G}^1_0(\mathbb{C}^n) \times \mathcal{G}^{k_1\kappa_0}_0(\mathbb{C}^N)$ whose components belong to the space $\mathcal{E}^{k_1\kappa_0}_{k_1n}$ such that for every $H \in \mathrm{Aut}(M,0)$ and for every $t^{[k_1]}$ sufficiently close to the origin in $\mathbb{C}^{k_1n}$ the following identity holds:

$$(H \circ v^{k_1})(t^{[k_1]}) = \Psi^0_{k_1}(t^{[k_1]}, j^1_0\overline{H}, j^{k_1\kappa_0}_0 H). \tag{11.1}$$

By mimicing the arguments of [3, §4.2], one may construct by using (11.1) and the above map Ψ^0, another map $\Phi\colon \mathbb{C}^N \times \mathcal{G}^1_0(\mathbb{C}^N) \times \mathcal{G}^{k_1\kappa_0}_0(\mathbb{C}^N) \to \mathbb{C}^N$ holomorphic in a neighbourhood of $\{0\} \times \mathcal{G}^1_0(\mathbb{C}^N) \times \mathcal{G}^{k_1\kappa_0}_0(\mathbb{C}^N)$ whose components belong to the space $\mathcal{E}^{k_1\kappa_0}_N$ such that for every $H \in \mathrm{Aut}(M,0)$ and for every Z sufficiently close to the origin in $\mathbb{C}^N$ the following identity holds:

$$H(Z) = \Phi(Z, j^1_0\overline{H}, j^{k_1\kappa_0}_0 H);$$

the details are left to the reader. Then setting $\Psi(Z, \Lambda^{k_1\kappa_0}) := \Phi(Z, \overline{\Lambda^1}, \Lambda^{k_1\kappa_0})$ yields the required parametrization. The proof of Theorem 7.3 is complete. ☐

Proof of Theorem 7.4. Theorem 7.4 follows directly from Theorem 7.3 and from repeating the arguments of [3, §4.3]. We leave the details to the reader. ☐

Proofs of Theorem 2.1 *and Theorem* 2.2. Firstly, note that to prove Theorem 2.1, it is enough to prove the parametrization property for the stability group of (M,p) for every $p \in M$. Then Theorem 2.1 (resp. Theorem 2.2) follows immediately from Theorem 7.3 (resp. Theorem 7.4), Proposition 7.2 and the upper-semicontinuity of the map $M \ni q \mapsto \kappa_M(q)$ (see §7.1). ☐

Proof of Theorem 2.3. Firstly, when M is generic, Theorem 2.3 follows from the fact that compact real-analytic CR submanifolds of $\mathbb{C}^N$ do not contain any complex-analytic subvariety of positive dimension ([19]) and hence are essentially finite at all points, from the upper-semicontinuity of the map $p \mapsto \ell_p$ in Theorem 2.2 and from the reflection principle proved in [7]. When M is not generic, one proceeds as follows. By [8], there exists $r \in \{1, \dots, N-1\}$ such that for every point $p \in M$ there exists a neighbourhood U_p of p in $\mathbb{C}^N$ and a neighbourhood W_p of 0 in $\mathbb{C}^N$ such that $M \cap U_p$ is biholomorphic to $W_p \cap (\widehat{M_p} \times \{0\})$, where $\widehat{M_p}$ is a real-analytic generic submanifold of $\mathbb{C}^r$ containing the origin, minimal at each of its points and not containing any complex-analytic subvariety of positive dimension. By using Theorem 2.2, for every $p \in M$ we may assume, shrinking $\widehat{M_p}$ near the origin if necessary, that there exists a positive integer k_p so that germs at any point $q \in \widehat{M_p}$ of smooth CR diffeomorphisms mapping $\widehat{M_p}$ into another real-analytic generic submanifold of $\mathbb{C}^r$ of the same CR dimension as that of M are uniquely determined by their k_p-jets at q. Then the theorem easily follows by using a finite covering of M by the neighbourhoods U_p for $p \in M$. The proof of Theorem 2.3 is therefore complete. □

Proofs of Theorems 1.1 *and* 1.2. Since real-analytic hypersurfaces which contain no complex-analytic subvariety of positive dimension are automatically minimal (see e.g. [8]), Theorem 1.1 (resp. Theorem 1.2) follows immediately from Theorem 2.1 (resp. from Theorem 2.2). □

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Universität Wien, Fakultät für Mathematik, Nordbergstrasse 15, A-1090 Wien, Austria

E-mail address: `lamelb@member.ams.org`

Université de Rouen, Laboratoire de Mathématiques Raphaël Salem, UMR 6085 CNRS, Avenue de l'Université, B.P. 12, 76801 Saint Etienne du Rouvray, France

E-mail address: `Nordine.Mir@univ-rouen.fr`