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CONVERGENCE OF FORMAL EMBEDDINGS BETWEEN REAL-ANALYTIC HYPERSURFACES IN CODIMENSION ONE

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Abstract

We show that any formal embedding sending a real-analytic strongly pseudoconvex hypersurface $M \subset \mathbb{C}^N$, $N > 1$, into another such hypersurface $M' \subset \mathbb{C}^{N+1}$ is convergent.

1. Introduction and results

By a classical result of Chern-Moser [4], any formal biholomorphic transformation in the complex N-dimensional space, $N > 1$, sending two real-analytic strongly pseudoconvex hypersurfaces into each other is in fact convergent i.e., given by the power series of a local holomorphic map. Several generalizations of this result have recently been established for more general classes of real-analytic hypersurfaces in the equidimensional case (see e.g., [3, 11, 13] and the references therein). On the other hand, it is conjectured that Chern-Moser's result can be extended to formal embeddings sending a real-analytic strongly pseudoconvex hypersurface $M \subset \mathbb{C}^N$ into another such hypersurface $M' \subset \mathbb{C}^{N'}$ with $N' > N$. Here we recall that by a formal embedding $F\colon (\mathbb{C}^N, p) \to (\mathbb{C}^{N'}, p')$ sending M into M', $p \in M$, $p' \in M'$, we mean a formal holomorphic map for which the pullback under F of any local real-analytic defining function for M' near p' vanishes on M (as a formal power series), and for which the induced differential $dF(p)|_{\mathbb{C}T_pM}\colon \mathbb{C}T_pM \to \mathbb{C}T_{p'}M'$ is injective. The main difficulty that

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one has to face in such a setting, and that was pointed out by a number of authors (see e.g., [6, 8, 13]), lies in the codimension $N' - N > 0$ of the embedding and indeed, up to now, only partial results on the above question have been obtained under various additional assumptions on the mapping or manifolds (see e.g., [9, 10]). For instance, it follows from the recent results of [10] that the above conjecture holds if the target hypersurface is moreover assumed to be real-algebraic. In this paper, we make a step towards the understanding of the above problem by giving a complete solution in the one-codimensional case. Indeed, we have:

Theorem 1.1. *Any formal embedding sending a real-analytic strongly pseudoconvex hypersurface $M \subset \mathbb{C}^N$, $N > 1$, into another such hypersurface $M' \subset \mathbb{C}^{N+1}$ is convergent.*

Theorem 1.1 gives also the first positive answer to a long standing open problem which consists in providing a regularity result for embeddings of positive codimension between real-analytic strongly pseudoconvex hypersurfaces that holds at *all* points of the source manifold (see e.g., [6, 8]). As a byproduct of the proof of Theorem 1.1, we are able to treat the convergence problem for formal embeddings between merely Levi-nondegenerate real-analytic hypersurfaces. In what follows, we consider formal maps $F\colon (\mathbb{C}^N, p) \to (\mathbb{C}^{N'}, p')$ sending M into M' that are CR *transversal* i.e., for which $dF(p)(\mathbb{C}T_pM) \not\subset T^{1,0}_{p'}M' \oplus T^{0,1}_{p'}M'$ where $T^{1,0}_{p'}M' \subset \mathbb{C}T_{p'}M'$ (resp. $T^{0,1}_{p'}M' \subset \mathbb{C}T_{p'}M'$) denotes the $(1,0)$ (resp. $(0,1)$) tangent space of M' at p', and we shall prove the following result specific to the one-codimensional situation.

Theorem 1.2. *Any formal CR transversal map sending a real-analytic Levi-nondegenerate hypersurface $M \subset \mathbb{C}^N$, $N > 1$, into another such hypersurface $M' \subset \mathbb{C}^{N+1}$ is convergent.*

Theorem 1.2 is indeed sharp in the sense that the analogous statement does not hold in codimension higher or equal to two (see e.g., [9]). Moreover it turns out that the condition of CR transversality is automatically satisfied by all formal embeddings between real-analytic strongly pseudoconvex hypersurfaces (see e.g., [9, 5]) and therefore, Theorem 1.2 yields Theorem 1.1 as a first (and main) application. It is also noteworthy to mention that Theorem 1.2 provides a convergence result for formal embeddings of positive codimension in other new situations, such as e.g., when the target hypersurface is foliated by complex curves. On the other hand, when there is no complex curve in the target manifold and the embedding is not assumed to be CR transversal, our arguments also give the following.

Theorem 1.3. *Let $M \subset \mathbb{C}^N$ and $M' \subset \mathbb{C}^{N+1}$ be real-analytic Levi-nondegenerate hypersurfaces through points p and p' respectively, $N > 1$. Assume that M' does not contain any (smooth) complex curve through p'. Then any formal embedding $F\colon (\mathbb{C}^N, p) \to (\mathbb{C}^{N+1}, p')$ sending M into M' is convergent.*

Our approach for the proof of Theorems 1.2 and 1.3, that will be given in §3, goes back to the work of Webster [14] and uses the invariant family of Segre varieties attached to any real-analytic hypersurface in complex space. We also make use of the CR vector field techniques later developed in the works of Huang [7] and Baouendi, Ebenfelt and Rothschild [2]. As a preliminary step for the proof, we need to establish in §2 a useful criterion for the convergence of a formal power series that satisfies a certain type of identity (see Proposition 2.2). The proof of such a statement requires the use of some arguments from our previous works [11, 12].

2. A criterion for the convergence of a formal power series

Throughout the paper, we denote by $\mathbb{C}[[x]]$ (resp. $\mathbb{C}\{x\}$), $x = (x_1, \ldots, x_k)$, the ring of formal (resp. convergent) power series in k indeterminates with complex coefficients. If $x^0 \in \mathbb{C}^k$, $\mathbb{C}[[x - x^0]]$ and $\mathbb{C}\{x - x^0\}$ will denote the corresponding ring of series centered at x^0. Given a formal power series $g(x) \in \mathbb{C}[[x]]$, we also denote by $\overline{g}(x)$ the formal power series obtained from $g(x)$ by taking complex conjugates of its coefficients. Given moreover a (germ at the origin of a) complex submanifold $S \subset \mathbb{C}^k$, we write $g(x) \equiv 0$ for $x \in S$ to mean that $g \circ \nu \equiv 0$ for any parametrization ν of S. We start by stating the following well-known lemma (see e.g., [11, Propositions 4.2 and 6.2]).

Lemma 2.1. *Let $R(x, y) \in \mathbb{C}\{x, y\}$, $x = (x_1, \ldots, x_k)$, $y \in \mathbb{C}$, $h(x) \in \mathbb{C}[[x]]$ with $h(0) = 0$ and $v \in (\mathbb{C}\{t\})^k$, $t = (t_1, \ldots, t_q)$. Then the following holds:*

- (i) *If $R(x, y) \not\equiv 0$ and $R(x, h(x)) \equiv 0$ then $h(x)$ is convergent.*
- (ii) *If $(h \circ v)(t)$ is convergent and v is of generic rank k, then $h(x)$ is itself convergent.*

Let $M \subset \mathbb{C}^N$ be a (germ of a) real-analytic hypersurface through the origin, and $\rho(Z, \overline{Z})$ be a real-analytic defining function for M defined in a connected neighborhood U of 0 in $\mathbb{C}^N$, with non-vanishing gradient on U. Recall that the complexification $\mathcal{M}$ of M is the complex submanifold of $\mathbb{C}^{2N}$ defined as follows:

$$\mathcal{M} := \{(Z,\zeta) \in U \times U^* : \rho(Z,\zeta) = 0\}, \tag{2.1}$$

where for any subset $V \subset \mathbb{C}^N$, we have denoted $V^* := \{\overline{w} : w \in V\}$. Recall also that M is said to be of *finite type* at the origin if there is no complex hypersurface of $\mathbb{C}^N$ contained in M through 0 (see [2]). (Note that when $N = 1$, M cannot have any point of finite type.) We may now formulate one of the main tools in the proof of Theorems 1.2 and 1.3.

Proposition 2.2. *Let $M \subset \mathbb{C}^N$ be a real-analytic hypersurface of finite type through the origin, $N > 1$, and $\mathcal{M} \subset \mathbb{C}^N_Z \times \mathbb{C}^N_\zeta$ its complexification as given by (2.1). Let $H(Z)$ be a formal power series, with $H(0) = 0$, satisfying at least one of the following conditions:*

(i) *There exists $G(\zeta) := (G_1(\zeta), \ldots, G_m(\zeta))$ a vector-valued formal power series and $A(Z,\zeta,X,T) \in \mathbb{C}\{Z,\zeta,X - G(0),T\}$, $X = (X_1, \ldots, X_m)$, $T \in \mathbb{C}$, such that $A(Z,\zeta,G(\zeta),T) \not\equiv 0$ for $(Z,\zeta) \in \mathcal{M}$ and such that $A(Z,\zeta,G(\zeta),H(Z)) \equiv 0$ for $(Z,\zeta) \in \mathcal{M}$.*

(ii) *There exists $B(Z,\zeta,\widetilde{T},T) \in \mathbb{C}\{Z,\zeta,\widetilde{T},T\}$, $T,\widetilde{T} \in \mathbb{C}$, such that $B(Z,\zeta,\widetilde{T},T) \not\equiv 0$ for $(Z,\zeta) \in \mathcal{M}$ and such that $B(Z,\zeta,\overline{H}(\zeta), H(Z)) \equiv 0$ for $(Z,\zeta) \in \mathcal{M}$.*

Then the formal power series $H(Z)$ is necessarily convergent.

Remark 2.3. In the case where the power series A, B in Proposition 2.2 are polynomials in T and $\widetilde{T}$, the above conclusion follows from [12, 10].

Proof. We choose local holomorphic coordinates $Z = (z,w) \in \mathbb{C}^{N-1} \times \mathbb{C}$ so that M is given near the origin by an equation of the form

$$w = Q(z,\overline{z},\overline{w}), \tag{2.2}$$

for some holomorphic function $Q(z,\chi,\tau)$ defined in a neighborhood of $0 \in \mathbb{C}^{N-1} \times \mathbb{C}^{N-1} \times \mathbb{C}$. We may also assume (see e.g., [2]) that Q satisfies

$$Q(z,\chi,\overline{Q}(\chi,z,w)) \equiv w,\ Q(z,0,w) = Q(0,\overline{z},w) \equiv 0. \tag{2.3}$$

Consider the parametrizations v^1, v^2, v^3 of the Segre sets up to order 3 attached to M at the origin (see [2]) given by

$$v^1(t^1) := (t^1, 0), \quad v^2(t^1, t^2) := (t^2, Q(t^2, \overline{v^1}(t^1))), \\ v^3(t^1, t^2, t^3) := (t^3, Q(t^3, \overline{v^2}(t^1, t^2))), \tag{2.4}$$

where each $t^j \in \mathbb{C}^{N-1}$ is sufficiently close to the origin. Recall that since M is of finite type at 0, the holomorphic map

$$\eta\colon (\mathbb{C}^{3N-3}, 0) \ni (t^1, t^2, t^3) \mapsto (v^2(t^1, t^2), \overline{v^3}(t^1, t^2, t^3)) \in (\mathcal{M}, 0) \tag{2.5}$$

is of generic maximal rank $2N-1$ (see e.g., [2]).

Let A, G, H be as in (i). Then after composing the given identities with the map $(Z, \zeta) = \eta(t^1, t^2, t^3)$, we have

$$A(v^2(t^1, t^2), \overline{v^3}(t^1, t^2, t^3), (G \circ \overline{v^3})(t^1, t^2, t^3), (H \circ v^2)(t^1, t^2)) \equiv 0, \tag{2.6}$$

and

$$\Delta(t^1, t^2, t^3, T) := A(v^2(t^1, t^2), \overline{v^3}(t^1, t^2, t^3), (G \circ \overline{v^3})(t^1, t^2, t^3), T) \not\equiv 0, \tag{2.7}$$

in view of the generic rank of η. From (2.7) and (2.6), we may choose a multiindex $\beta_0 \in \mathbb{N}^{N-1}$ such that

$$\widetilde{\Delta}(t^1, t^2, T) := \left[\frac{\partial^{|\beta_0|} \Delta}{\partial t_3^{\beta_0}}(t^1, t^2, t^3, T) \right]_{\big| t^3 = t^1} \not\equiv 0 \qquad \text{and} \\ \widetilde{\Delta}(t^1, t^2, (H \circ v^2)(t^1, t^2)) \equiv 0. \tag{2.8}$$

Note that by using the identity $v^3(t^1, t^2, t^1) = v^1(t^1)$ (which follows from (2.3)), we may rewrite

$$\widetilde{\Delta}(t^1, t^2, T) = \widehat{\Delta}\left(t^1, t^2, (((\partial^\beta G) \circ \overline{v^1})(t^1))_{|\beta| \le |\beta_0|}, T\right)$$

for some holomorphic function $\widehat{\Delta}$ defined in a neighborhood of

$$\left(0, 0, ((\partial^\beta G)(0))_{|\beta| \le |\beta_0|}, 0\right).$$

Now as in [11, 12], by differentiating the second identity in (2.8) with respect to t^2, setting $t^2 = 0$, using the identity $v^2(t^1, 0) = 0$, and applying Artin's approximation theorem [1], we may find for any positive integer k a convergent power series mapping $Y^k(t^1)$ which agrees up to order k with $(((\partial^\beta G)\circ \overline{v^1})(t^1))_{|\beta|\leq|\beta_0|}$ at the origin and which satisfies the identity

$$\widehat{\Delta}(t^1,t^2,Y^k(t^1),(H\circ v^2)(t^1,t^2))\equiv 0. \tag{2.9}$$

Moreover, by choosing k large enough, we may achieve the condition $\widehat{\Delta}(t^1,t^2,Y^k(t^1),T)\not\equiv 0$ in view of (2.8). We may therefore apply Lemma 2.1 (i) to conclude that $H\circ v^2$ is convergent. Since M is of finite type at 0, it is easy to see that the generic rank of v^2 is N, and hence, from Lemma 2.1 (ii) it follows that H is convergent. The proof of Proposition 2.2 (i) is complete.

Let B,H as in (ii). Expand B as a Taylor series as follows $B(Z,\zeta,\widetilde{T},T)=\sum_{j=0}^{\infty} b_j(Z,\zeta,\widetilde{T})T^j$. There are two cases to consider.

First case. There exists j_0 such that $b_{j_0}(Z,\zeta,\overline{H}(\zeta))\not\equiv 0$ for $(Z,\zeta)\in\mathcal{M}$. Then $B(Z,\zeta,\overline{H}(\zeta),T)$ is nontrivial for $(Z,\zeta)\in\mathcal{M}$ and the convergence of H follows from Proposition 2.2 (i) proved above.

Second case. For all j, $b_j(Z,\zeta,\overline{H}(\zeta))\equiv 0$ for $(Z,\zeta)\in\mathcal{M}$. Since $B(Z,\zeta,\widetilde{T},T)\not\equiv 0$ for $(Z,\zeta)\in\mathcal{M}$, there exists j_1 such that $B_{j_1}(Z,\zeta,\widetilde{T})\not\equiv 0$ for $(Z,\zeta)\in\mathcal{M}$. The convergence of the series H then follows from Lemma 2.1 (i) by noticing that $\overline{B_{j_1}}(\zeta,Z,H(Z))\equiv 0$ for $(Z,\zeta)\in\mathcal{M}$. This completes the proof of Proposition 2.2 (ii). q.e.d.

3. Proofs of Theorems 1.3 and 1.2

Proof of Theorem 1.3. Without loss of generality, we may assume that $p=p'=0$. We choose local holomorphic coordinates $Z'=(z',w')\in\mathbb{C}^N\times\mathbb{C}$ near the origin so that the Levi-nondegenerate real-analytic hypersurface M' is given by the equation

$$\overline{w'}=\Theta(\overline{z'},z',w'), \tag{3.1}$$

where $\Theta=\Theta(\chi',Z')$ is a holomorphic function near the origin in $\mathbb{C}^N\times\mathbb{C}^{N+1}$. We set $\rho'(Z',\overline{Z'}):=\overline{w'}-\Theta(\overline{z'},z',w')$ and may assume (see [4]) that $\Theta(\chi',z',0)$ vanishes at the origin up to order one and also satisfies

$$\Theta_{\chi'_j z'_k}(0)=0,\quad \text{for } j\neq k,\quad \Theta_{\chi'_j z'_j}(0)=\pm 1,\quad j,k=1,\dots,N. \tag{3.2}$$

In these coordinates, we split the formal map F as follows $F=(f,g)\in\mathbb{C}^N\times\mathbb{C}$, where $f=(f_1,\dots,f_N)$. At the source $\mathbb{C}^N$ space, we denote our coordinates by $Z = (z, w) \in \mathbb{C}^{N-1} \times \mathbb{C}$ which we may assume to be normal coordinates as in (2.2) and (2.3). Since F sends M into M', we have the formal identity

$$\overline{g}(\zeta) = \Theta(\overline{f}(\zeta), F(Z)), \text{ for } (Z, \zeta) \in \mathcal{M}, \tag{3.3}$$

where $\mathcal{M}$ is the complexification of M as given by (2.1). Let $\mathcal{L}_1, \ldots, \mathcal{L}_{N-1}$ be a basis of holomorphic vector fields (in a neighborhood of 0 in $\mathbb{C}^N \times \mathbb{C}^N$, with holomorphic coefficients in (Z, ζ)) tangent to (a neighborhood of 0 in) $\mathcal{M}$ that annihilate the projection $\mathbb{C}^N \times \mathbb{C}^N \ni (Z, \zeta) \mapsto Z \in \mathbb{C}^N$. Applying each $\mathcal{L}_j$ to (3.3), we obtain

$$\mathcal{L}_j \overline{g}(\zeta) = \sum_{k=1}^{N} \Theta_{\chi'_k}(\overline{f}(\zeta), F(Z)) \mathcal{L}_j \overline{f}_k(\zeta). \tag{3.4}$$

By (3.4) and our choice of Θ, we have $(\mathcal{L}_j \overline{g})(0) = 0$ for all j and therefore, since F is an embedding, the rank of the matrix $\left(\mathcal{L}_j \overline{f}_k(\zeta)\right)_{\substack{1 \le j \le N-1 \\ 1 \le k \le N}}$ is equal to $N-1$ at the origin. Then after interchanging the components of f if necessary, we may assume that the rank of the matrix $\left(\mathcal{L}_j \overline{f}_k(\zeta)\right)_{\substack{1 \le j \le N-1 \\ 1 \le k \le N-1}}$ equals $N-1$ at the origin. Therefore, by using Cramer's rule to (3.4), taking the complex conjugate of (3.3), we obtain the system of formal equations

$$\begin{cases} g(Z) = \overline{\Theta}(f(Z), \overline{F}(\zeta)) \\ \Theta_{\chi'_k}(\overline{f}(\zeta), F(Z)) = \Theta_{\chi'_N}(\overline{f}(\zeta), F(Z))\, P_k((\mathcal{L}_j \overline{F}(\zeta))_{1 \le j \le N-1}) \\ \qquad\qquad + S_k((\mathcal{L}_j \overline{F}(\zeta))_{1 \le j \le N-1}), \end{cases} \tag{3.5}$$

for $k = 1, \ldots, N-1$ and $(Z, \zeta) \in \mathcal{M}$. Here each P_k, S_k is a convergent power series centered at $((\mathcal{L}_j \overline{F})(0))_{1 \le j \le N-1}$ (that is even rational). In view of (3.2), we may solve the system (3.5) by making use of the implicit function theorem to obtain the vectorial formal identity

$$\begin{aligned} &(f_1(Z), \ldots, f_{N-1}(Z), g(Z)) \\ &\quad = \Psi(Z, \zeta, ((\partial^\alpha \overline{F})(\zeta))_{|\alpha| \le 1}, f_N(Z)), \text{ for } (Z, \zeta) \in \mathcal{M}, \end{aligned} \tag{3.6}$$

where

$$\begin{aligned} \Psi &= (\Psi_1, \ldots, \Psi_{N-1}, \Psi_{N+1}) \\ &= \Psi(Z, \zeta, \Lambda, T) \in \left(\mathbb{C}\{Z, \zeta, \Lambda - ((\partial^\alpha \overline{F})(0))_{|\alpha| \le 1}, T\}\right)^N. \end{aligned}$$

(Here we have used the fact that the vector fields $\mathcal{L}_j$, $j = 1, \ldots, N-1$, have holomorphic coefficients in (Z, ζ).) For $k \in \{1, \ldots, N+1\}$, $k \neq N$, we write the Taylor expansion

$$\Psi_k(Z,\zeta,\Lambda,T) = \sum_{i=0}^{\infty} \varphi_{k,i}(Z,\zeta,\Lambda)T^i. \tag{3.7}$$

Applying again each vector field $\mathcal{L}_j$ to (3.6), we obtain for all k as above

$$\begin{aligned} 0 &= \mathcal{L}_j\left(\Psi_k(Z,\zeta,((\partial^\alpha \overline{F})(\zeta))_{|\alpha|\leq 1}, f_N(Z))\right) \\ &= \sum_{i=0}^{\infty} \mathcal{L}_j\left(\varphi_{k,i}(Z,\zeta,((\partial^\alpha \overline{F})(\zeta))_{|\alpha|\leq 1})\right)(f_N(Z))^i, \ (Z,\zeta) \in \mathcal{M}. \end{aligned} \tag{3.8}$$

Note that there is a convergent power series $\widetilde{\Psi}_{k,j}(Z,\zeta,\widehat{\Lambda},T) \in \mathbb{C}\{Z,\zeta,\widehat{\Lambda}-((\partial^\alpha \overline{F})(0))_{|\alpha|\leq 2}, T\}$ such that

$$\begin{aligned} &\widetilde{\Psi}_{k,j}(Z,\zeta,((\partial^\alpha \overline{F})(\zeta))_{|\alpha|\leq 2}, f_N(Z)) \\ &\quad = \mathcal{L}_j\left(\Psi_k(Z,\zeta,((\partial^\alpha \overline{F})(\zeta))_{|\alpha|\leq 1}, f_N(Z))\right). \end{aligned} \tag{3.9}$$

Now, as in [7], we come to a dichotomy which will give the convergence of the map F (in any case).

First case. There exist indices k_0, j_0 and i_0 such that

$$\mathcal{L}_{j_0}\left(\varphi_{k_0,i_0}(Z,\zeta,((\partial^\alpha \overline{F})(\zeta))_{|\alpha|\leq 1}\right) \not\equiv 0 \ \text{ for } \ (Z,\zeta) \in \mathcal{M}.$$

Then the formal power series $\widetilde{\Psi}_{k_0,j_0}(Z,\zeta,((\partial^\alpha \overline{F})(\zeta))_{|\alpha|\leq 2}, T)$ is nontrivial for $(Z,\zeta) \in \mathcal{M}$ and satisfies $\widetilde{\Psi}_{k_0,j_0}(Z,\zeta,((\partial^\alpha \overline{F})(\zeta))_{|\alpha|\leq 2}, f_N(Z)) \equiv 0$ on $\mathcal{M}$. Since M is Levi-nondegenerate and therefore of finite type at 0, the convergence of f_N then follows from Proposition 2.2 (i). Using (3.6), the now established convergence of f_N and Proposition 2.2 (i), we obtain the convergence of all other components of the map F.

Second case. For all indices k, j and i we have

$$\mathcal{L}_j\left(\varphi_{k,i}(Z,\zeta,((\partial^\alpha \overline{F})(\zeta))_{|\alpha|\leq 1}\right) \equiv 0 \ \text{ for } \ (Z,\zeta) \in \mathcal{M}.$$

This means that for any $j \in \{1, \ldots, N-1\}$,

$$\mathcal{L}_j\left(\Psi(Z,\zeta,((\partial^\alpha \overline{F})(\zeta))_{|\alpha|\leq 1}, T)\right) \equiv 0 \ \text{ for } \ (Z,\zeta) \in \mathcal{M}.$$

From this, it is then easy to see that there exists a $\mathbb{C}^N$-valued formal power series mapping $\Phi(Z,T)$ such that

$$\Phi(Z,T) = \Psi(Z,\zeta,((\partial^\alpha \overline{F})(\zeta))_{|\alpha|\le 1},T),\ (Z,\zeta)\in\mathcal{M}. \tag{3.10}$$

By (3.10) and Proposition 2.2 (i) applied to the real-analytic hypersurface $M\times\mathbb{C}\subset\mathbb{C}^{N+1}$ (that is of finite type), we obtain that $\Phi(Z,T)$ defines a holomorphic map near $0\in\mathbb{C}^{N+1}$. We may therefore rewrite (3.6) as follows

$$(f_1(Z),\ldots,f_{N-1}(Z),g(Z)) = \Phi(Z,f_N(Z)). \tag{3.11}$$

For $(Z,T)\in\mathbb{C}^N\times\mathbb{C}$ sufficiently close to the origin, we set

$$h(Z,T) := (\Phi_1(Z,T),\ldots,\Phi_{N-1}(Z,T),T,\Phi_{N+1}(Z,T))\in\mathbb{C}^{N+1}. \tag{3.12}$$

Consider the holomorphic function defined near the origin in $\mathbb{C}^N\times\mathbb{C}^N\times\mathbb{C}\times\mathbb{C}$ by

$$R(Z,\zeta,T,\widetilde{T}) := \rho'(h(Z,T),\overline{h}(\zeta,\widetilde{T})). \tag{3.13}$$

Note that necessarily $R(Z,\zeta,T,\widetilde{T})$ does not vanish identically for $(Z,\zeta,T,\widetilde{T})\in(\mathcal{M}\times\mathbb{C}^2,0)$ since otherwise M' would contain a complex curve through the origin which is impossible by assumption. In view of (3.12) and (3.11) we have $h(Z,f_N(Z)) = F(Z)$ and therefore since F sends M into M', we have

$$R(Z,\zeta,f_N(Z),\overline{f_N}(\zeta))\equiv 0,\ \text{for } (Z,\zeta)\in\mathcal{M}. \tag{3.14}$$

Then by applying Proposition 2.2 (ii) to (3.14), we obtain that f_N is convergent and hence F too in view of (3.11). The proof of Theorem 1.3 is complete. q.e.d.

Proof of Theorem 1.2. Let $F\colon(\mathbb{C}^N,p)\to(\mathbb{C}^{N+1},p')$ be a given formal CR transversal map sending M into M'. As in the proof of Theorem 1.3, we may assume that $p=p'=0$. Since M and M' are Levi-nondegenerate and F is CR transversal, F is a formal embedding (see e.g., [9, 5]). Therefore we may follow the same proof as that of Theorem 1.3 and we notice that the only place (in that proof) where the fact that the target hypersurface does not contain any complex curve through 0 is used is to show that the holomorphic function R given by (3.13) does not vanish identically for $(Z,\zeta,T,\widetilde{T})\in(\mathcal{M}\times\mathbb{C}^2,0)$. We now show that this latter condition holds automatically under the assumptions of Theorem 1.2, which will finish its proof. By contradiction suppose that $R(Z,\zeta,T,\widetilde{T}) \equiv 0$ for $(Z,\zeta,T,\widetilde{T}) \in (\mathcal{M}\times\mathbb{C}^2,0)$. This means in view of (3.13) that $h(Z,T)\in M'$ for all $(Z,T)\in M\times\mathbb{C}$ sufficiently close to the origin. We now claim that h is local biholomorphism of $\mathbb{C}^{N+1}$. Indeed, by using e.g., the arguments of [5, Lemma 5.1], we may assume, after composing the map F with an automorphism of the hyperquadric tangent to M', that the following normalization condition holds for the N-th component of F:

$$\frac{\partial F_N}{\partial z}(0)=0. \tag{3.15}$$

Since F is CR transversal we have $\partial g/\partial w(0)\neq 0$ (see e.g., [9, 5]) and therefore, since F is an embedding, the map $Z\mapsto (f_1(Z),\ldots,f_{N-1}(Z), g(Z))$ is a formal biholomorphism of $\mathbb{C}^N$. From this fact and the identities (3.11), (3.12) and (3.15), it is easy to see that the Jacobian matrix of h has indeed rank $N+1$ at the origin, which proves the claim. We therefore have a local biholomorphism h of $\mathbb{C}^{N+1}$ satisfying $h(M\times\mathbb{C})=M'$ (as germs through the origin), which is impossible in view of the Levi-nondegeneracy assumption on M'. The proof of Theorem 1.2 is therefore complete. q.e.d.

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