PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY, SERIES B
Volume 11, Pages 15–22 (March 5, 2024)
https://doi.org/10.1090/bproc/149

REGULARITY OF SECTIONS OF CR VECTOR BUNDLES

BERNHARD LAMEL AND NORDINE MIR

(Communicated by Harold P. Boas)

Abstract. In this note, we show that every generalized section σ of a CR vector bundle E over a CR manifold M has the property that near most points of its singular support, there exists a proper abstract CR subbundle $F \subset E$ which has the property that every *real* subbundle of E which contains the image of σ also contains F.

1. Introduction

One of the first basic theorems in complex analysis of one variable is the *reflection principle*: If a holomorphic function is defined on a domain in the upper half plane and extends to a real-valued function on the real line (for example, continuously), then the extension is actually real-analytic. In the theory of several complex variables, there are obstructions to automatic analyticity and smoothness of CR maps between real hypersurfaces; in particular, the authors' recent works [7–9] identify the existence of varieties tangent to the image of the boundary values of these maps as one of the main obstructions. The same question naturally arises for CR maps between abstract CR structures and is not yet well understood in this setting, see [2, 5, 10].

In this note, we study the model case of sections of an abstract CR vector bundle E (for definitions, see Section 2); while this case does not exhibit the typical nonlinear constraints of CR mappings between CR manifolds here, the PDEs we encounter do possess a considerably higher complexity than in the embedded case. We shall show that every generalized section σ of a CR vector bundle E over a CR manifold M has the property that near most points of its singular support, there exists an abstract CR subbundle $F \subset E$ which has the property that every *real* subbundle of E which contains the image of σ also contains F. All of the irregularity of a CR section is contained in its "F-component".

In what follows all manifolds and vector bundles are assumed to be $\mathcal{C}^\infty$-smooth and we refer the reader to Section 2 or [3] for the basic notions used here. Our main result is as follows.

Theorem 1. *Let $\sigma\colon M \to E$ be a generalized CR section of a CR vector bundle over the abstract CR manifold M, and assume that σ extends microlocally to a wedge of edge M. Then for every point p in some dense open subset of* $(\operatorname{SingSupp}\sigma)^\circ$, *there exists a neighbourhood U and a nontrivial (smooth) abstract CR vector bundle*

Received by the editors June 18, 2022, and, in revised form, October 29, 2022.
2020 *Mathematics Subject Classification.* Primary 32V05, 32V20.
Key words and phrases. CR vector bundle, CR immersion.
This publication was made possible by NPRP award NPRP-BSRA01-0309-210004 from the Qatar National Research Fund.

$F \subset E|_U$ such that if $R \subset E|_U$ is any (smooth) real subbundle containing the image of $\sigma|_U$, then $F \subset R$. Furthermore, the section

$$\tilde{\sigma}\colon U \to {}^{E|_U}\!\big/_{F}$$

is smooth.

A direct outcome of the construction is the following second main theorem, which gives a nice sufficient condition for automatic regularity. In what follows, given a CR vector bundle E over M, we say that a real subbundle $R \subset E$ is CR nondegenerate (at a point $p \in M$) if the CR derivatives of sections of R span E (at p).

Theorem 2. *Let M and E be as above and let $R \subset E$ be a real subbundle that is CR nondegenerate at a point $p \in M$. Then every generalized CR section σ of R which extends microlocally to a wedge of edge M is smooth near p.*

As an application, we prove regularity of infinitesimal deformations of nondegenerate CR maps between abstract CR manifolds, extending the regularity result for infinitesimal automorphisms of abstract CR structures obtained by Fürdös and the first author in [4]. We refer the reader to Section 5 for the precise definitions.

Theorem 3. *Let M and M' be abstract CR manifolds, and $h\colon M \to M'$ a smooth CR immersion, finitely nondegenerate at p. Then every infinitesimal deformation of h which extends microlocally near p is smooth near p.*

The proofs of these theorems use a construction of “smoothness multiplier ideals” whose properties might be interesting on their own merit. We therefore include a discussion of an algorithm which can be used to construct these ideals.

2. Preliminaries

Most of the results in this paper will be local, and we therefore only define the necessary notions in the local setting. This means that we will assume that $M \subset \mathbb{R}^n_x \times \mathbb{R}^n_y \times \mathbb{R}^d_s$ is an open neighbourhood of 0, and that the CR structure $\mathcal{V}$ is spanned by the n linearly independent vector fields $\bar{L}_1, \ldots, \bar{L}_n$; we can assume that they are chosen so that they commute (see, e.g., [11, Prop. I.5.1]). We are going to assume that the coordinates (x, y, s) are a *standard coordinate patch* for M; this simply means that for every $(x_0, y_0) \in \pi_1(M)$ the submanifold $N_{(x_0,y_0)} := (\pi_1|_M)^{-1}((x_0, y_0)) = \{(x_0, y_0, s) \in M\}$ of M is totally real and transverse to the complex tangent directions $T^c M = \operatorname{Re} \mathcal{V}$ in M. It follows that $T^0_p M \cong T^*_p N_{\pi_1(p)}$ (induced by the restriction of evaluation of the forms) and we require that this yields a well-defined identification $T^0 M \cong \pi_1(M) \times \mathbb{R}^d$ (for details as to why such standard coordinates exist, see [10, Section 4.1]).

We are considering the vector bundle $E = M \times \mathbb{C}^r$. A (generalized) section u of E is therefore given by an r-tuple of smooth functions (distributions)

$$u = \begin{pmatrix} u_1 \\ \vdots \\ u_r \end{pmatrix}$$

on M. In our basis of CR vector fields $\bar{L}_1, \ldots, \bar{L}_n$, a partial connection D is given by

$$D_j u = \bar{L}_j u + A_j u,$$

where $A_1, \ldots, A_n$ are smooth matrix-valued functions on M. A section u is said to be *CR* if $D_j u = 0$ for $j = 1, \ldots, n$, or equivalently, if

$$\bar{L}_j u_k = -\sum_{\ell} A_{jk}^{\ell} u_\ell, \quad j = 1, \ldots, n, \quad k = 1, \ldots, r.$$

We say that this partial connection is *CR* provided that $\bar{L}_j A_k - \bar{L}_k A_j = A_j A_k - A_k A_j$; this is a *formal integrability condition* automatically satisfied if E has a basis of CR sections, and when thinking about A as a matrix valued form it can be compactly written as $dA = A \wedge A$ when acting on CR vectors.

We will realize E^* as row vectors, and consider the standard dual pairing by matrix multiplication. The dual connection D^* is then defined for row vectors $v = (v_1, \ldots, v_r)$ by

$$D_j^* v = \bar{L}_j v - v A_j,$$

which is the choice that ensures

$$(1) \qquad \begin{aligned} (D_j^* v)u + v(D_j u) &= (\bar{L}_j v - v A_j)u + v(\bar{L}_j u + A_j u) \\ &= (\bar{L}_j v)u + v(\bar{L}_j u) \\ &= \bar{L}_j (vu). \end{aligned}$$

We will write $\mathcal{D}'(M, E)$ for the $\mathcal{C}^\infty(M)$-module of generalized sections of E, consisting in the local representation above of vectors whose components are distributions. The wavefront set of a generalized section $\sigma = (\sigma_1, \ldots, \sigma_r)$ is defined by

$$\mathrm{WF}(\sigma) = \bigcup_{j=1}^{r} \mathrm{WF}(\sigma_j),$$

where the wavefront set of a distribution is defined in the usual manner (see [1,11]). Recall that $T'M = \mathcal{V}^\perp$ and that the *characteristic bundle* of M, denoted T^0M, is the set of all (real) forms annihilating $\mathcal{V}$ and $\bar{\mathcal{V}}$. The elliptic regularity theorem implies that if a section $\sigma \colon M \to E$ is CR (i.e. $D_{\bar{j}}\sigma = 0$ for $j = 1, \ldots, n$), then $\mathrm{WF}(\sigma) \subset T^0M$ (see [1,11]). Following [4], we say that σ extends microlocally to a wedge of edge M if there exists a set $\Gamma \subset T^0M$ such that for every $p \in M$, the fiber Γ_p is a closed non-empty convex cone of $T_p^0M \setminus \{0\}$ satisfying $\mathrm{WF}(\sigma) \subset \Gamma$. For σ as above, we recall that $\mathrm{SingSupp}\,\sigma$ is the closed subset of M consisting of those points $p \in M$ where σ is not $\mathcal{C}^\infty$-smooth in any neighbourhood of p.

We say that a $\mathcal{C}^\infty(M)$-submodule Ω of $\mathcal{D}'(M, E)$ is *CR closed* if it has the property that $D_j\sigma \in \Omega$ for every $j = 1, \ldots, n$ and for every $\sigma \in \Omega$. In particular any submodule Ω generated by CR sections is CR closed (see [3]). For a given $\mathcal{C}^\infty(M)$-submodule $\Omega \subset \mathcal{D}'(M, E)$ we denote by $\hat{\Omega}$ its CR closure, the smallest $\mathcal{C}^\infty(M)$-submodule containing Ω which is CR closed. Note that

$$\hat{\Omega} = \langle D_{j_1} \ldots D_{j_r}\sigma \colon r \in \mathbb{N}, \sigma \in \Omega \rangle_{\mathcal{C}^\infty},$$

where, here and in what follows, $\langle A \rangle_{\mathcal{C}^\infty}$ denotes the submodule generated by A over $\mathcal{C}^\infty$. We also set

$$\mathrm{SingSupp}\,\Omega = \overline{\cup_{\sigma \in \Omega} \mathrm{SingSupp}\,\sigma}.$$

From the definition, it is clear that $M \setminus \mathrm{SingSupp}\,\Omega$ is the largest open subset of M where all $\sigma \in \Omega$ are simultaneously smooth.

A *CR subbundle* of a CR vector bundle E is a subbundle F such that all of the CR derivatives of sections of F are again sections of F, i.e. if $\Gamma(M, F)$ is CR closed.

We note that $F \subset E$ is a CR subbundle if and only if $F^\perp \subset E^*$ is a CR subbundle with respect to the natural dual connection.

3. Smoothness multipliers

The goal of this section is to introduce a simple yet powerful tool to study smoothness of generalized CR sections.

Definition 1. For a subset $\Omega \subset \mathcal{D}'(M,E)$ we define its module of (vector) smoothness multipliers by

$$\mathcal{S}(\Omega) = \{\lambda \in \Gamma(M,E^*) \colon \lambda(\sigma) \in \mathcal{C}^\infty(M),\ \forall \sigma \in \Omega\}\,.$$

Note that $\mathcal{S}(\Omega)$ only depends on the $\mathcal{C}^\infty(M)$-module generated by Ω, so we will from now on only talk about modules of smoothness multipliers of $\mathcal{C}^\infty(M)$-modules. The usefulness of smoothness multipliers comes from the fact that they are closed under CR derivatives and characterize smoothness; the reader can easily check Lemmas 1 and 2.

Lemma 1. *If $\Omega \subset \mathcal{D}'(M,E)$ is a CR closed submodule, then $\mathcal{S}(\Omega)$ is CR closed.*

Lemma 2. *A $\mathcal{C}^\infty(M)$-submodule $\Omega \subset \mathcal{D}'(M,E)$ satisfies $\Omega \subset \mathcal{C}^\infty(M,E)$ if and only if $\mathcal{S}(\Omega) = \Gamma(M,E^*)$.*

In a similar way, one can introduce scalar smoothness multipliers:

Definition 2. For a subset $\Omega \subset \mathcal{D}'(M,E)$ we define its module of (scalar) smoothness multipliers by

$$\tilde{\mathcal{S}}(\Omega) = \{\lambda \in \mathcal{C}^\infty(M) \colon \lambda\sigma \in \mathcal{C}^\infty(M,E)\}\,.$$

By Cramer's rule, we have the inclusion $\Lambda^r \mathcal{S}(\Omega) \subset \tilde{\mathcal{S}}(\Omega)$. On the other hand, every scalar multiplier is a vector multiplier in a natural way.

We will use the following simple fact which is a consequence of the smooth version of the edge of the wedge theorem.

Proposition 1. *Let M and E be as above and $\Omega \subset \mathcal{D}'(M,E)$ such that every $\sigma \in \mathcal{D}'(M,E)$ extends microlocally to a wedge of edge M. If there exists a smooth $\lambda \in \Gamma(M,E^*)$ such that $\operatorname{Re}\lambda(\sigma)$ is smooth for all $\sigma \in \Omega$, then $\lambda \in \mathcal{S}(\Omega)$.*

Proof. Let $\sigma \in \Omega$. Then $\lambda(\sigma) + \overline{\lambda(\sigma)} = f \in \mathcal{C}^\infty(M,E)$. Pick $p \in M$ and $\Gamma_p \subset T^0_p M \setminus \{0\}$, a closed non-empty convex subcone, such that $\mathrm{WF}(\sigma)|_p \subset \Gamma_p$. Then $\mathrm{WF}(\bar\sigma)|_p \subset (-\Gamma_p)$ and therefore $\mathrm{WF}(\lambda(\sigma))|_p \subset \Gamma_p \cap (-\Gamma_p) = \emptyset$, and so $\lambda(\sigma)$ is smooth near every $p \in M$ (see [10]). Hence $\lambda \in \mathcal{S}(\Omega)$. □

4. The algorithm

In this section, we describe the algorithm to produce smoothness multipliers, and use it to deduce regularity of generalized sections. For this, we start out with a CR closed module $\Omega \subset \mathcal{D}'(M,E)$ such that every $\sigma \in \mathcal{D}'(M,E)$ extends microlocally to a wedge of edge M. We then define

$$\mathcal{S}_0(\Omega) := \langle \{\lambda \in \Gamma(M,E^*) \colon \operatorname{Re}\lambda(\sigma) = 0,\ \forall \sigma \in \Omega\} \rangle_{\mathcal{C}^\infty(M)}\,.$$

By Proposition 1, we have that $\mathcal{S}_0(\Omega) \subset \mathcal{S}(\Omega)$. Now, we define an increasing sequence of $\mathcal{C}^\infty(M)$-submodules of $\Gamma(M,E^*)$ by setting, for each $k \geq 0$,

$$\mathcal{S}_k(\Omega) = \langle \mathcal{S}_{k-1}(\Omega), D_j \mathcal{S}_{k-1}(\Omega) \colon j = 1,\dots,n \rangle_{\mathcal{C}^\infty(M)}\,.$$

By Lemma 1, we have $\mathcal{S}_k(\Omega) \subset \mathcal{S}(\Omega)$ for all $k \geq 0$, and therefore also $\mathcal{S}_\infty(\Omega) := \bigcup_{k\geq 0} \mathcal{S}_k(\Omega) \subset \mathcal{S}(\Omega)$. The main observation about these is that we can effectively compute the modules $\mathcal{S}_k(\Omega)$, and thus, if $\mathcal{S}_k(\Omega) = \Gamma(M, E^*)$ for some k, we have that $\Omega \subset \mathcal{C}^\infty(M, E)$.

We next look at $\mathcal{S}_\infty(\Omega)$ and the upper semicontinuous integer valued function $d\colon M \to \{0, \dots, r\}$, $d(p) = r - \dim \mathcal{S}_\infty(\Omega)(p)$. For $s \in \{0, \dots, r\}$, consider the open sets

$$(d^{-1}(\{s\}))^\circ = M^s_\Omega$$

and note that the open subset of M

$$M_\Omega = \bigcup_{j=0}^{r} M^j_\Omega =: M^0_\Omega \cup N_\Omega$$

is dense in M. We now have the following characterization of the behaviour of Ω on M_Ω (see [10] for a somewhat analogous approach in a different context).

Theorem 4. *Let E be CR vector bundle over the CR manifold M, Ω a CR closed module of $\mathcal{D}'(M, E)$ such that every $\sigma \in \mathcal{D}'(M, E)$ extends microlocally to a wedge of edge M. With the notation introduced above, we have $M^0_\Omega \subset M \setminus \operatorname{SingSupp} \Omega$, and $N_\Omega \cap (\operatorname{SingSupp} \Omega)^\circ$ is dense in $(\operatorname{SingSupp} \Omega)^\circ$. Furthermore, for any $k \geq 1$, there exists a smooth CR closed subbundle $F_k \subset E|_{M^k_\Omega}$ of rank k such that:*

(i) *for any real subbundle $R \subset E$ with $\Omega|_{M_k} \subset \mathcal{D}'(M_k, R)$, we have $F_k \subset R$;*
(ii) *for every $\sigma \in \Omega$, the section $\tilde\sigma\colon M^k_\Omega \to E|_{M^k_\Omega}/F_k$ is smooth.*

Proof. For any $p \in M^0_\Omega$, $\dim \mathcal{S}_\infty(\Omega)(p) = r$, so every $\sigma \in \Omega$ is smooth on M^0_Ω. Hence, $M^0_\Omega \subset M \setminus \operatorname{SingSupp} \Omega$. The density of $N_\Omega \cap (\operatorname{SingSupp} \Omega)^\circ$ in $(\operatorname{SingSupp} \Omega)^\circ$ is then clear from the construction.

On the other hand, for $k = 1, \dots, r$, we have that $\dim \mathcal{S}_\infty(\Omega|_{M^k_\Omega})(p) = r - k$ for any $p \in M^k_\Omega$. We can thus write

$$\mathcal{S}_\infty(\Omega|_{M^k_\Omega}) = \Gamma(M^k_\Omega, F_k^\perp) \tag{2}$$

for a subbundle $F_k \subset E^{**}|_{M^k_\Omega} = E|_{M^k_\Omega}$ of rank k. This bundle F_k is a CR subbundle because $\mathcal{S}_\infty(\Omega)$ is CR closed by construction. In particular, given any $\lambda \in \Gamma(M, E^*)$ such that $\operatorname{Re} \lambda(\sigma) = 0$ for every $\sigma \in \Omega|_{M^k_\Omega}$, then $\lambda \in \mathcal{S}_0(\Omega|_{M^k_\Omega}) \subset \mathcal{S}_\infty(\Omega|_{M^k_\Omega})$, and so $\lambda(\omega) = 0$ for every $\omega \in \mathcal{C}^\infty(M^k_\Omega, F_k)$. This proves (i) in the theorem. Finally, (2) implies that $\mathcal{S}(\Omega|_{M^k_\Omega}) = \Gamma(M^k_\Omega, E/F_k)$ and hence (ii) follows. ☐

5. Proof of the main theorems and applications

We are now in a position to prove the two main theorems.

Proof of Theorem 1. Since σ is CR, the submodule generated by σ is CR closed, and all its elements extend microlocally to wedge of edge M. Now observe that $(\operatorname{SingSupp}\langle\sigma\rangle_{\mathcal{C}^\infty(M)})^\circ = (\operatorname{SingSupp} \sigma)^\circ$, so the conclusion follows directly from Theorem 4. ☐

Proof of Theorem 2. We again apply Theorem 4 to the submodule Ω generated by σ. Under the given assumptions, we have $p \in M^0_\Omega$ which implies that σ is smooth near p. ☐

We now come to the applications of our main theorems. First, we discuss infinitesimal deformations of CR maps between abstract CR manifolds. We work locally, choosing coordinates $x=(x_1,\dots,x_{2n+d})\in\mathbb{R}^{2n+d}$ for M, and $y=(y_1,\dots,y_{2n'+d'})\in\mathbb{R}^{2n'+d'}$ for M', with structure bundles $\mathcal{V}$ and $\mathcal{V}'$, respectively. A $\mathcal{C}^1$-smooth map $h=(h_1,\dots,h_{2n'+d'})\colon M\to M'$ is CR if $h_*\bar L_p\in\mathcal{V}'_{h(p)}$ for every CR vector $\bar L_p\in\mathcal{V}_p$, and for every $p\in M$. Recall that the *holomorphic cotangent bundles* are defined by $T'M=\mathcal{V}^\perp$ and $T'M'=(\mathcal{V}')^\perp$ (see e.g. [11]); in order to avoid using new terminology, we shall (as is commonly done) refer to sections of these as *holomorphic forms*. Then we see that h is CR if and only if

$$\sum_j \omega^j_{h(p)}\bar L_p h_j=0,\qquad \text{for any } \bar L_p\in\mathcal{V}_p, \text{ and } \omega=\sum\omega^j_{h(p)}dy_j\in T'_{h(p)}M'. \tag{3}$$

Now consider a deformation of a CR map $h(x)=h_0(x)$ by CR maps $h_t(x)$, defined for $t\in(-\varepsilon,\varepsilon)$, $\varepsilon>0$. We write $\dot h(x)=\frac{d}{dt}|_{t=0}h_t(x)$ for the infinitesimal deformation associated to $h(x,t)$ (regarding it as a section of h^*TM' in the natural way) and let $\omega=\sum\omega^j(y)dy_j\in\Gamma(M',T'M')$ be a holomorphic form on M', and $\bar L$ a CR vector field on M. Differentiating the equation $\sum_j\omega^j(h(x,t))\bar L h_j(x,t)=0$ with respect to t, and evaluating at $t=0$, one gets

(4)
$$\begin{aligned}0&=\sum_{j,k}\frac{\partial\omega^j}{\partial y_k}(h(x))\dot h_k(x)\bar L h_j(x)+\sum_j\omega^j(h(x))\bar L\dot h_j(x)\\&=d\omega_{h(x)}(\dot h(x),((h_*)_x\bar L_x))+\sum_{j,k}\frac{\partial\omega^j}{\partial y_k}(h(x))\dot h_j(x)\bar L h_k(x)+\sum_j\omega^j(h(x))\bar L\dot h_j(x)\\&=d\omega_{h(x)}(\dot h(x),((h_*)_x\bar L_x))+\bar L_x(\omega_{h(\cdot)}(\dot h(\cdot))).\end{aligned}$$

Provided that h is an immersion, this equation suggests the introduction of a CR vector bundle structure D on $h^*T'M'$ as follows. For a section η of $h^*T'M'$ and $p\in M$, choose a small enough neighbourhood U of p so that $h(U)$ is an embedded submanifold of M', and a section ω of $T'M'$, defined in a neighbourhood of $h(U)$ such that $\eta=h|_U^*\omega$. Extend $h_*\bar L$ to a vector field $\bar K$ on M' and over $h(U)$, we define

$$D_{\bar L}\eta|_U:=D_{\bar K}\omega,$$

where the last D is the natural CR structure on $T'M'$ given by

$$(D_{\bar K}\omega)(X)=d\omega(\bar K,X),\quad \bar K\in\Gamma(M',\mathcal{V}'),\quad \omega\in\Gamma(M',T'M').$$

It is easy to verify that this does not depend on the extension $\bar K$, but only on $\bar L$ (and U). With these definitions, (4) therefore reads

$$D_{\bar L}\eta(\dot h)=\bar L(\eta(\dot h)), \tag{5}$$

and as usual, the dual structure on $(h^*T'M')^*\supset h^*T'M$ is defined by

$$\bar L\eta(X)=D_{\bar L}\eta(X)+\eta(D^*_{\bar L}X).$$

We therefore have:

Lemma 3. *With the natural CR bundle structures just introduced, any deformation (h_t) of a CR immersion h gives rise to a CR section $\dot h$ of $(h^*T'M')^*$. Furthermore (since $\dot h$ is a section of TM'), for any characteristic form θ' on M', we have $\operatorname{Im}\theta'(\dot h)=0$.*

We can now define what an infinitesimal deformation of a CR immersion h is.

Definition 3. We say that a generalized section $X \in \mathcal{D}'(M, h^*T'M')^*$ is an infinitesimal deformation of a CR immersion $h\colon M \to M'$ if, with the natural CR connection defined above, we have

$$D_{\bar L} X = 0, \quad \bar L \in \Gamma(M, \mathcal{V}), \text{ and } \operatorname{Im} \theta(X) = 0, \quad \theta \in \Gamma(M, h^* T^0 M).$$

Let us recall the following definition which basically dates back to [6] (in the smooth context, one can also consult [3]):

Definition 4. We say that an immersive CR map $h\colon M \to M'$ is k-nondegenerate at $p \in M$ if

$$E_k(p) = \operatorname{span}\left\{ (D_{\bar L_1} \cdots D_{\bar L_k} \theta')(p) \colon \bar L_j \in \Gamma(M, \mathcal{V}), \theta' \in \Gamma(M, h^* T^0 M') \right\} = T'_{h(p)} M',$$

with k being the smallest integer with this property. We say that h is finitely nondegenerate at p if it is k-nondegenerate for some integer k.

Proof of Theorem 3. Since h is assumed to be finitely nondegenerate, we can apply Theorem 2, with $E = (h^*T'M')^*$ and $R = (h^*T^0M')^*$ (the latter being the real dual, of course). ☐

Actually, we also obtain an interesting statement corresponding to the existence of a non-smooth infinitesimal deformation in the spirit of Theorem 1, which we note for completeness here:

Theorem 5. *If $h\colon M \to M'$ is a CR immersion which has a non-smooth infinitesimal deformation, then for p in a dense, open subset of M, the bundle $h^*T'M'$ contains a nontrivial CR subbundle near p.*

References

[1] Shiferaw Berhanu, Paulo D. Cordaro, and Jorge Hounie, *An introduction to involutive structures*, New Mathematical Monographs, vol. 6, Cambridge University Press, Cambridge, 2008, DOI 10.1017/CBO9780511543067. MR2397326

[2] S. Berhanu and Ming Xiao, *On the C^∞ version of the reflection principle for mappings between CR manifolds*, Amer. J. Math. **137** (2015), no. 5, 1365–1400, DOI 10.1353/ajm.2015.0034. MR3405870

[3] Peter Ebenfelt and Bernhard Lamel, *Finite jet determination of CR embeddings*, J. Geom. Anal. **14** (2004), no. 2, 241–265, DOI 10.1007/BF02922071. MR2051686

[4] Stefan Fürdös and Bernhard Lamel, *Regularity of infinitesimal CR automorphisms*, Internat. J. Math. **27** (2016), no. 14, 1650112, 25, DOI 10.1142/S0129167X16501123. MR3593674

[5] Sung-Yeon Kim and Dmitri Zaitsev, *Equivalence and embedding problems for CR-structures of any codimension*, Topology **44** (2005), no. 3, 557–584, DOI 10.1016/j.top.2004.11.004. MR2122216

[6] Bernhard Lamel, *A reflection principle for real-analytic submanifolds of complex spaces*, J. Geom. Anal. **11** (2001), no. 4, 627–633, DOI 10.1007/BF02930759. MR1861300

[7] Bernhard Lamel and Nordine Mir, *Convergence of formal CR mappings into strongly pseudoconvex Cauchy-Riemann manifolds*, Invent. Math. **210** (2017), no. 3, 963–985, DOI 10.1007/s00222-017-0743-3. MR3735633

[8] Bernhard Lamel and Nordine Mir, *Convergence and divergence of formal CR mappings*, Acta Math. **220** (2018), no. 2, 367–406, DOI 10.4310/ACTA.2018.v220.n2.a5. MR3849288

[9] Bernhard Lamel and Nordine Mir, *On the C^∞ regularity of CR mappings of positive codimension*, Adv. Math. **335** (2018), 696–734, DOI 10.1016/j.aim.2018.07.004. MR3836676

[10] Bernhard Lamel and Nordine Mir, *Regularity of CR mappings of abstract CR structures*, Internat. J. Math. **31** (2020), no. 1, 2050009, 38, DOI 10.1142/S0129167X20500093. MR4060572

[11] François Trèves, *Hypo-analytic structures*, Princeton Mathematical Series, vol. 40, Princeton University Press, Princeton, NJ, 1992. Local theory. MR1200459

Texas A&M University at Qatar, Division of Arts and Sciences, PO Box 23874, Education City, Doha, Qatar and Faculty of Mathematics, University of Vienna, Oskar-Morgenstern-Platz 1, 1090 Vienna, Austria
Email address: bernhard.lamel@qatar.tamu.edu, bernhard.lamel@univie.ac.at

Texas A&M University at Qatar, Division of Arts and Sciences, PO Box 23874, Education City, Doha, Qatar
Email address: nordine.mir@qatar.tamu.edu