

# Left-invariant CR structures on 3-dimensional Lie groups

Gil Bor and Howard Jacobowitz

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A *CR structure* on a 3-dimensional manifold  $M$  is a rank 2 subbundle  $D \subset TM$  together with an almost complex structure  $J$  on  $D$

$$J : D \rightarrow D, \quad J^2 = -Id .$$

Equivalently, a CR structure is a complex line subbundle  $V \subset TM \otimes \mathbf{C}$  such that  $V \cap \bar{V} = \{0\}$ .

$$D = \{\Re V, \Im V\}$$

Notation:  $(M, V)$  or  $(M, L)$ .

Usually assume  $L, \bar{L}, [L, \bar{L}]$  linearly independent (spc).

Note:  $[L, \bar{L}] \equiv 0$  means that  $M$  is foliated by complex curves.

$f : M \rightarrow \mathbf{C}$  is a CR function if  $Lf = 0$  for each section of  $V$ .

$G$  a 3 dimensional Lie group

$\mathfrak{G}$  its Lie algebra

$$\mathfrak{G} = T_e G = \{L : g_* L_e = L_g\}$$

$(G, L)$  is a CR manifold provided

$$L \cap \bar{L} = \{0\}.$$

Conjugation

$$\Phi : G \rightarrow \text{Aut}(G)$$

$$\Phi(g)h = ghg^{-1}$$

induces a map

$$\text{Ad} : G \rightarrow \text{Aut}(\mathfrak{G})$$

with derivative at the identity

$$\text{ad} : \mathfrak{G} \rightarrow \mathfrak{G}$$

given by

$$\text{ad}(X)Y = [X, Y].$$

## Immersions into $\mathbf{CP}^2$

A complex line sub-bundle  $V \subset T(G) \otimes \mathbf{C}$  determines a point in  $P(\mathfrak{G} \otimes \mathbf{C}) = \mathbf{CP}^2$ ,  $L \rightarrow [L] \in P(\mathfrak{G} \otimes \mathbf{C})$ , and the orbit of  $L$

$$\{X : \exists g \in G, \text{Ad}_g L = X\}$$

gives a map of  $G$  into  $\mathbf{CP}^2$

$$\phi : G \rightarrow \mathbf{CP}^2$$

$$g \rightarrow [\text{Ad}_g L]$$

with differential at the identity

$$\phi_* X = [[X, L]].$$

$\phi$  is always a CR map

$$G \rightarrow \mathbf{CP}^2.$$

$\phi$  is an immersion if

$$[X, L] = 0 \text{ for } X \text{ real} \Rightarrow X = 0.$$

# Spherical and Aspherical

$(M, V)$  is spherical near a point if it is locally CR equivalent to the standard CR structure on  $S^3$ .

## Theorem (Cartan 1932)

*The local CR equivalence group of a 3 dimensional aspherical CR manifold has dimension at most 3.*

Consider two left-invariant aspherical CR structures  $(G_i, V_i)$  on two connected 3-dimensional Lie groups  $G_i$

## Corollary

*If the two CR structures are locally equivalent, then there exists a group isomorphism  $G_1 \rightarrow G_2$  which extends this local CR equivalence.*

## Proof.

- ① Right-invariant vector fields generate left-translations. So  $\mathcal{R}_{G_i} \subset \text{Aut}_{CR}(G_i, V_i)$ .
- ② Since the dimension are equal these groups coincide near the identity.
- ③ The local CR equivalence  $f$  maps  $\mathcal{R}_{G_1} \rightarrow \mathcal{R}_{G_2}$ .
- ④  $f$  preserves the group structure.



## The most interesting examples: $SL(2, \mathcal{R})$ and $SU(2)$

Basis for  $sl(2, \mathcal{R})$

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

Basis for  $su(2)$

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$$

Note

$$sl(2, \mathbf{C}) = su(2) \otimes \mathbf{C}.$$

and

$$P(\mathfrak{g} \otimes \mathbf{C}) = P(sl(2, \mathbf{C})) = \mathbf{CP}^2.$$



## Geometry of $P(\mathfrak{sl}(2, \mathbf{C}))$

Let  $L \in \mathfrak{sl}(2, \mathbf{C})$  be given as

$$\begin{bmatrix} a & b \\ c & -a \end{bmatrix}.$$

The Killing form is

$$K(L, L) = a^2 + bc = -\det L$$

and

$$K(L, L) = 0$$

defines a conic in  $\mathcal{C} = \mathbf{CP}^1 \subset P(\mathfrak{sl}(2, \mathbf{C}))$ . The polar line with respect to  $\mathcal{C}$  of a point  $[L] \in \mathbf{CP}^2$  determines two points in  $\mathcal{C}$ .

$$\mathcal{S} = \{ \{ \zeta_1, \zeta_2 \} : \zeta_i \in \mathbf{CP}^1 \}.$$

$$L \rightarrow [L] \in P(\mathfrak{sl}(2, \mathbf{C})) \rightarrow \mathcal{S}$$

is  $SL(2, \mathbf{C})$  equivariant.

$$\begin{array}{ccc} P(\mathfrak{sl}(2, \mathbf{C})) & \xrightarrow{Ad_g} & P(\mathfrak{sl}(2, \mathbf{C})) \\ \downarrow & & \downarrow \\ \mathcal{S} & \xrightarrow{g} & \mathcal{S} \end{array}$$

Classifying the orbits of  $[L]$  under  $SL(2, \mathcal{R})$  or  $SU(2)$  uses that

- $SL(2, \mathcal{R})$  preserves the hyperbolic distance in the upper half-plane.
- $SU(2)$  preserves the Euclidean distance in  $S^2$ .

So the following are equivalent:

- 1  $[L]$  and  $[\tilde{L}]$  are in the same  $Aut_G$  orbit.
- 2  $d(\zeta_1, \zeta_2) = d(\tilde{\zeta}_1, \tilde{\zeta}_2)$ .
- 3 The left-invariant CR structures  $(G, L)$  and  $(G, \tilde{L})$  are CR equivalent under a group automorphism.

## Theorem

Let  $V_t \subset T_{\mathbf{C}}SL(2, \mathcal{R})$ ,  $t \in (-1, 1]$  be the left-invariant complex line bundle spanned at  $e$  by

$$L_t = \begin{pmatrix} i\frac{1+t}{2} & t \\ 1 & -i\frac{1+t}{2} \end{pmatrix} \in \mathfrak{sl}_2(\mathbf{C}).$$

Then

- 1  $V_t$  is a left-invariant CR structure for all  $t \neq 0$ .
- 2  $V_0$  is a foliation by complex curves.
- 3  $V_t$  is spherical if  $t = 1$  or  $-3 \pm 2\sqrt{2}$  and aspherical otherwise.
- 4 Every left-invariant CR structure on  $SL_2(\mathcal{R})$  is CR equivalent to  $V_t$  for a unique  $t$ .
- 5 The aspherical left-invariant CR structures  $V_t$ ,  $t \in (-1, 1) \setminus \{0, -3 + 2\sqrt{2}\}$ , are pairwise non-equivalent, even locally.

The immersion results rely on a geometric observation. Recall the null cone of the Killing form of  $sl(2, \mathcal{R})$

$$\mathcal{C} = \{L : a^2 + bc = 0\}.$$

A left-invariant CR structure on  $SL(2, \mathcal{R})$  is called hyperbolic if the real 2-plane spanned by  $\{\Re L, \Im L\}$  intersects  $\mathcal{C}$  in two lines. It is called elliptic if the intersection is the origin.

Note: One spherical structure is elliptic, the other is hyperbolic.

## Theorem

- ① *The elliptic left-invariant spherical CR structure on  $SL(2, \mathcal{R})$  ( $t = 1$ ) is realizable by the map*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \rightarrow \begin{pmatrix} b + ia \\ d + ic \end{pmatrix}.$$

*It is also realizable as the complement of  $S^3 \cap \mathbf{C}^1$ .*

- ② *The hyperbolic spherical structure on  $SL(2, \mathcal{R})$  ( $t = -3 + 2\sqrt{2}$ ) is realizable as the complement of  $S^3 \cap \mathcal{R}^2$ .*
- ③ *The rest of the left-invariant CR structures on  $SL(2, \mathcal{R})$  are either 4 : 1 covers, in the aspherical elliptic case  $0 < t < 1$ , or 2 : 1 covers, in the aspherical hyperbolic case  $-1 < t < 0$ , of the orbits of  $SL(2, \mathcal{R})$  in  $\mathbf{P}(sl(2, \mathbf{C}))$ .*

## Theorem

Let  $V_t \subset T_{\mathbf{C}}SU(2)$ ,  $t \geq 1$  be the left-invariant complex line bundle spanned at  $e \in SU(2)$  by

$$L_t = \begin{pmatrix} 0 & t-1 \\ t+1 & 0 \end{pmatrix}.$$

- 1  $V_t$  is a left-invariant CR structure on  $SU(2)$  for all  $t \geq 1$ .
- 2  $V_t$  is spherical if and only if  $t = 1$ .
- 3 Every left-invariant CR structure on  $SU(2)$  is CR equivalent to  $V_t$  for a unique  $t$ .
- 4 The aspherical left-invariant CR structures  $V_t$ ,  $t > 1$ , are pairwise non-equivalent, even locally.
- 5  $V_1$  is realized by the usual CR structure on  $S^3 \subset \mathbf{C}^2$ . The aspherical structures are realized as 4 : 1 covers of the adjoint orbits of  $SU(2)$  in  $\mathbf{P}(\mathfrak{sl}(2, \mathbf{C}))$ .

# The Heisenberg Group

$$H = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in \mathcal{R} \right\}$$

and

$$\mathfrak{h} = \left\{ \begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} : a, b, c \in \mathcal{R} \right\}$$

$H$  is isomorphic to the group on the hyperquadric

$$\{(z, w) \in \mathbf{C}^2 : \Im w = |z|^2\}.$$



## Theorem

- ① *Every plane in  $\mathfrak{h}$  containing*

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

*gives a foliation of  $H$  by complex curves.*

- ② *Every other plane is in the orbit of*

$$L = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & i \\ 0 & 0 & 0 \end{pmatrix}$$

*under  $\text{Aut}(H)$ .*

- ③  *$L$  gives a spherical CR structure equivalent to the usual one on the hyperquadric.*

## Observation

Let  $V_t, t \in \mathcal{R}$  be a smooth family of CR structures on some  $M^3$ , such that

$$V_t \equiv V_s \text{ for } s \neq 0, t \neq 0$$

but

$$V_1 \not\equiv V_0.$$

Then  $V_0$  is a foliation of  $M$  by complex curves.

Example

$$L_t = \begin{pmatrix} 0 & i & 1 \\ 0 & 0 & t \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & i & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

$\{L_t, \bar{L}_t\}$  is not involutive for  $t \neq 0$  but  $\{L_0, \bar{L}_0\}$  is involutive.

# The Euclidean Group

$$E = \left\{ \begin{pmatrix} \cos \theta & \sin \theta & u \\ -\sin \theta & \cos \theta & v \\ 0 & 0 & 1 \end{pmatrix} \right\}$$
$$\mathfrak{e} = \left\{ \begin{pmatrix} 0 & x & y \\ -x & 0 & z \\ 0 & 0 & 0 \end{pmatrix} \right\}$$

## Theorem

Let  $V \subset T_{\mathbf{C}}E$  be the left-invariant line bundle whose value at  $e \in E$  is spanned by

$$L = \begin{pmatrix} 0 & -i & 1 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then

- 1 Every spc left-invariant CR structure on  $E$  is CR equivalent to  $V$  by  $\text{Aut}(E)$ .
- 2  $V$  is aspherical.
- 3  $V$  is realized in  $\mathbf{P}(\mathfrak{e} \otimes \mathbf{C}) = \mathbf{C}P^2$  by the adjoint orbit of  $[L]$ . This is CR equivalent to the real hypersurface  $[\Re(z_1)]^2 + [\Re(z_2)]^2 = 1$  in  $\mathbf{C}^2$ .

$$L_t = \begin{pmatrix} 0 & -it & a \\ it & 0 & bi \\ 0 & 0 & 0 \end{pmatrix}$$

is an aspherical CR structure for  $t \neq 0$  but a foliation by complex curves for  $t = 0$ .

# Cartan's Moving Frames

$$\begin{array}{ccc} H & \longrightarrow & SU(2, 1) \\ & & \downarrow \\ & & Q \end{array}$$

$\dim SU(2, 1) = 8, \quad \dim H = 5.$

Maurer-Cartan form  $\Theta : TSU(2, 1) \rightarrow \mathfrak{su}(2, 1)$

Let  $M$  be a three dimensional (spc) CR structure.

$$\begin{array}{ccc} H^5 & \longrightarrow & B^8 \\ & & \downarrow \\ & & M^3 \end{array}$$

$\Theta : TB \otimes \mathbf{C} \rightarrow \mathfrak{su}(2, 1) \otimes \mathbf{C}.$

## Theorem

With each spc CR 3-manifold  $M$  there is canonically associated a bundle  $B \rightarrow M$  with Cartan connection  $\Theta : TB \rightarrow \mathfrak{su}(2, 1)$ , satisfying

- 1 The eight components of  $\Theta$  are pointwise linearly independent.
- 2 (The CR structure equations) There exist functions  $R, S : B \rightarrow \mathbf{C}$  such that

$$\begin{aligned}d\theta &= i\theta_1 \wedge \bar{\theta}_1 - \theta \wedge (\theta_2 + \bar{\theta}_2), \\d\theta_1 &= -\theta_1 \wedge \theta_2 - \theta \wedge \theta_3, \\d\theta_2 &= 2i\theta_1 \wedge \bar{\theta}_3 + i\bar{\theta}_1 \wedge \theta_3 - \theta \wedge \theta_4, \\d\theta_3 &= -\theta_1 \wedge \theta_4 - \bar{\theta}_2 \wedge \theta_3 - R\theta \wedge \bar{\theta}_1, \\d\theta_4 &= i\theta_3 \wedge \bar{\theta}_3 - (\theta_2 + \bar{\theta}_2)\theta_4 + (S\theta_1 + \bar{S}\bar{\theta}_1) \wedge \theta.\end{aligned}$$

- 3 (Spherical structures)  $M$  is spherical if and only if  $R \equiv 0$ .
- 4 Any local CR diffeomorphism of CR manifolds  $f : M \rightarrow M'$  lifts uniquely to a map  $\tilde{f} : B \rightarrow B'$  with  $\tilde{f}^*\Theta' = \Theta$ .

$\phi, \phi_1$  is an adapted coframe for the CR structure  $V$  on  $M$  if

1

$\phi$  is real.

2

$$\phi(V) = 0, \quad \phi_1(V) = 0.$$

3

$$\phi \wedge \phi_1 \wedge \bar{\phi}_1 \neq 0.$$

4

$$d\phi = i\phi_1 \wedge \bar{\phi}_1.$$

Each adapted coframe has a unique lift to  $\sigma : M \rightarrow B$  such that  $\sigma^*\theta = \phi$  and  $\sigma^*\theta_1 = \phi_1$ .



Pulling back the other components of  $\Theta$  we obtain

$$d\phi = i\phi_1 \wedge \bar{\phi}_1 - \phi \wedge (\phi_2 + \bar{\phi}_2),$$

$$d\phi_1 = -\phi_1 \wedge \phi_2 - \phi \wedge \phi_3,$$

$$d\phi_2 = 2i\phi_1 \wedge \bar{\phi}_3 + i\bar{\phi}_1 \wedge \phi_3 - \phi \wedge \phi_4,$$

$$d\phi_3 = -\phi_1 \wedge \phi_4 - \bar{\phi}_2 \wedge \phi_3 - r\phi \wedge \bar{\phi}_1,$$

$$d\phi_4 = i\phi_3 \wedge \bar{\phi}_3 + (s\phi_1 + \bar{s}\bar{\phi}_1) \wedge \phi.$$

$M$  is spherical if and only if the Cartan relative curvature invariant  $r$  is identically 0.

## Lemma

Let  $M$  be a manifold with a CR structure given by a coframe  $\phi, \phi_1$  satisfying

$$\begin{aligned}d\phi &= i\phi_1 \wedge \bar{\phi}_1, \\d\phi_1 &= a\phi_1 \wedge \bar{\phi}_1 + b\phi \wedge \phi_1 + c\phi \wedge \bar{\phi}_1,\end{aligned}\tag{1}$$

for some complex constants  $a, b, c$ . Then

$$r = ic\left(\frac{|a|^2}{3} + \frac{3ib}{2}\right).$$

If a left-invariant CR structure has a coframe satisfying (1) then  $a, b, c$  are constants and so the structure is spherical if and only if

$$c\left(\frac{|a|^2}{3} + \frac{3ib}{2}\right) = 0.$$