The range of the Killing operator

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The Killing operator

- Riemannian manifold M with metric g_{ab}
- X^a a (locally defined) vector field on M

Lie derivative — Levi-Civita connection

$$X^b \longmapsto \tfrac{1}{2}\mathcal{L}_X g_{ab} = \tfrac{1}{2}\big(\nabla_a X_b + \nabla_b X_a\big) \equiv \nabla_{(a}X_{b)}$$

Killing $= \mathcal{K}$

Euclidean three-space

$$X_b \longmapsto \partial_a X_b = \partial_{(a}X_{b)} + \partial_{[a}X_{b]}$$

symmetric + skew

Killing — curl

$$\{\text{functions}\} \xrightarrow{\text{grad}} \left\{\begin{array}{c}\text{vector}\\ \text{fields}\end{array}\right\} \xrightarrow{\text{curl}} \left\{\begin{array}{c}\text{vector}\\ \text{fields}\end{array}\right\} \xrightarrow{\text{div}} \{\text{functions}\}$$

$\rightsquigarrow\rightsquigarrow\rightsquigarrow$ de Rham complex

The Saint-Venant operator

curl (on $\mathbb{R}^3$) $X_a \longmapsto \epsilon_a{}^{bc}\partial_b X_c$ where ϵ_{abc} = volume form

curlcurl (on $\mathbb{R}^3$) $h_{ab} \longmapsto \epsilon_a{}^{ce}\epsilon_b{}^{df}\partial_c\partial_d h_{ef}$

Theorem (Saint-Venant 1864)

$$\left\{\begin{array}{c}\text{vector}\\ \text{fields}\end{array}\right\} \xrightarrow{\text{Killing}} \left\{\begin{array}{c}\text{symmetric}\\ \text{tensors}\end{array}\right\} \xrightarrow{\text{curlcurl}} \left\{\begin{array}{c}\text{symmetric}\\ \text{tensors}\end{array}\right\}$$

$$\text{div}\downarrow$$

$$\left\{\begin{array}{c}\text{vector}\\ \text{fields}\end{array}\right\}$$

is locally exact

In particular, locally

$$h_{ab} = \partial_{(a}X_{b)} \quad \Leftrightarrow \quad \underbrace{\partial_a\partial_c h_{bd} - \partial_b\partial_c h_{ad} - \partial_a\partial_d h_{bc} + \partial_b\partial_d h_{ac} = 0}_{\text{'integrability conditions'}}$$

Proof?

Killing fields in flat space

$$\partial_{(a}X_{b)} = 0 \quad \Leftrightarrow \quad \begin{array}{rcl} \partial_a X_b &=& \mu_{ab} \\ \partial_a \mu_{bc} &=& 0 \end{array}$$

skew

optional extra

$$\mu_{bc} = \partial_b X_c \Rightarrow \mu_{bc} = \partial_{[b}X_{c]} \Rightarrow \partial_{[a}\mu_{bc]} = 0$$
$$\Leftrightarrow \partial_a \mu_{bc} = \partial_c \mu_{ba} - \partial_b \mu_{ca} = \partial_c \partial_b X_a - \partial_b \partial_c X_a = 0 \ \checkmark$$

Prolongation connection

$$\begin{array}{c} \Lambda^1 \\ \oplus \\ \Lambda^2 \end{array} \ni \left[\begin{array}{c} X_b \\ \mu_{bc} \end{array} \right] \longmapsto \left[\begin{array}{c} \partial_a X_b - \mu_{ab} \\ \partial_a \mu_{bc} \end{array} \right]$$

Flat

$$\therefore \ \dim\{\text{Killing fields on } \mathbb{R}^n\} = \frac{n(n+1)}{2}$$

translations: n

rotations: $n(n-1)/2$

Killing fields in curved space

$$\nabla_{(a}X_{b)} = 0 \quad \Leftrightarrow \quad \begin{array}{rcl} \nabla_a X_b &=& \mu_{ab} \\ \nabla_a \mu_{bc} &=& R_{bc}{}^d{}_a X_d \end{array}$$

skew

optional extra

Prolongation connection

$$\begin{array}{c}\Lambda^1 \\ \oplus \\ \Lambda^2\end{array} \ni \begin{bmatrix} X_b \\ \mu_{bc} \end{bmatrix} \stackrel{D_a}{\longmapsto} \begin{bmatrix} \nabla_a X_b - \mu_{ab} \\ \nabla_a \mu_{bc} - R_{bc}{}^d{}_a X_d \end{bmatrix}$$

Curvature

$$D_{[a}D_{b]} \begin{bmatrix} X_c \\ \mu_{cd} \end{bmatrix} = \begin{bmatrix} 0 \\ R_{ab}{}^e{}_{[c}\mu_{d]e} + R_{cd}{}^e{}_{[a}\mu_{b]e} - \frac{1}{2}(\nabla^e R_{abcd})X_e \end{bmatrix}$$

Flat $\Leftrightarrow$ $R_{abcd} = \text{constant} \times (g_{ac}g_{bd} - g_{bc}g_{ad})$

$\therefore \ \dim\{$Killing fields on n-sphere$\} = n(n+1)/2$ $\qquad \mathfrak{so}(n+1)$ ✓

Proof of Saint-Venant (and more!)

de Rham complex coupled with prolongation connection

$$\begin{array}{ccccccccc}
E & \xrightarrow{D} & \Lambda^1 \otimes E & \longrightarrow & \Lambda^2 \otimes E & \longrightarrow & \Lambda^3 \otimes E & \longrightarrow & \cdots \\
\| & & \| & & \| & & \| & & \\
\begin{array}{c}\Lambda^1\\ \oplus \\ \Lambda^2\end{array} & \nearrow & \begin{array}{c}\Lambda^1 \otimes \Lambda^1\\ \oplus \\ \Lambda^1 \otimes \Lambda^2\end{array} & \cong & \begin{array}{c}\Lambda^2 \otimes \Lambda^1\\ \oplus \\ \Lambda^2 \otimes \Lambda^2\end{array} & \twoheadrightarrow & \begin{array}{c}\Lambda^3 \otimes \Lambda^1\\ \oplus \\ \Lambda^3 \otimes \Lambda^2\end{array} & &
\end{array}$$

Diagram chasing $\rightsquigarrow$

Calabi operator

$$\Lambda^1 \xrightarrow{\mathcal{K}} \bigodot^2 \Lambda^1$$

$$h_{ab} \stackrel{\mathcal{C}}{\longmapsto} \nabla_{(a}\nabla_{c)}h_{bd} - \nabla_{(b}\nabla_{c)}h_{ad} - \nabla_{(a}\nabla_{d)}h_{bc} + \nabla_{(b}\nabla_{d)}h_{ac}$$
$$- R_{ab}{}^{e}{}_{[c}h_{d]e} - R_{cd}{}^{e}{}_{[a}h_{b]e}$$

Locally symmetric spaces

Locally symmetric $\Leftrightarrow \nabla_a R_{bcde} = 0$

Example: Fubini-Study metric on complex projective space

F. Costanza, ME, T. Leistner, B. McMillan, arXiv:2112.00841
A Calabi operator for Riemannian locally symmetric spaces

$$\nabla_a R_{bcde} = 0 \Rightarrow \quad \begin{array}{ccc} \Lambda^1 & \xrightarrow{\quad\mathcal{K}\quad} & \Box\!\Box \\ d \big\downarrow & & \big\downarrow \mathcal{C} \\ \Lambda^2 & \xrightarrow{\quad\mathcal{R}\quad} & \boxplus \end{array} \quad \text{commutes}$$

$$\mu_{cd} \longmapsto 2R_{ab}{}^{e}{}_{[c}\mu_{d]e} + 2R_{cd}{}^{e}{}_{[a}\mu_{b]e}$$

a complex of linear differential operators

$$\Box \xrightarrow{\mathcal{K}} \Box\!\Box \xrightarrow{\mathcal{L}} \overline{\boxplus} \equiv \boxplus / \mathcal{R}(\Lambda^2)$$

Integrability conditions

Theorem (CELM) On an irreducible Riemannian locally symmetric space, the complex $\square \xrightarrow{\mathcal{K}} \square\square \xrightarrow{\mathcal{L}} \overline{\boxplus}$ is locally exact

Example the round (unit) n-sphere: due to Calabi 1961

$\mathcal{R} : \wedge^2 \to \boxplus$ vanishes, $\overline{\boxplus} = \boxplus$, and $\mathcal{L} = \mathcal{C}$ is given by

$$\begin{aligned} h_{ab} \mapsto \; & \nabla_{(a}\nabla_{c)}h_{bd} - \nabla_{(b}\nabla_{c)}h_{ad} - \nabla_{(a}\nabla_{d)}h_{bc} + \nabla_{(b}\nabla_{d)}h_{ac} \\ & + g_{ac}h_{bd} \qquad - g_{bc}h_{ad} \qquad - g_{ad}h_{bc} \qquad + g_{bd}h_{ac} \end{aligned}$$

Example the Fubini-Study metric on $\mathbb{CP}_n$: Kähler form J_{ab}

$$R_{abcd} = \underbrace{g_{ac}g_{bd} - g_{bc}g_{ad}}_{\text{cf. sphere}} + \underbrace{J_{ac}J_{bd} - J_{bc}J_{ad} + 2J_{ab}J_{cd}}_{\text{new bits}}$$

Complex projective space cont'd

What about $\mathcal{R} : \Lambda^2 \to \boxplus$?

$\boxplus$ splits into ten irreducible pieces (Tricerri-Vanhecke 1981)

$$\boxplus = \boxplus_\perp \oplus \Lambda^2$$

$\boxplus_\perp$: seven pieces

Λ^2: three pieces

$$\mathcal{R}\uparrow$$

$$\Lambda^2 = \Lambda^{1,1}_{\mathbb{R}} \oplus C$$

$\Lambda^{1,1}_{\mathbb{R}}$: two pieces killed by $\mathcal{R}$

C: one piece injected by $\mathcal{R}$

Therefore (CELM)

$\square \xrightarrow{\mathcal{K}} \square\!\square \xrightarrow{\mathcal{L}} \overline{\boxplus} = \boxplus / C$ is locally exact

In fact, $\square \xrightarrow{\mathcal{K}} \square\!\square \longrightarrow \boxplus_\perp$ is globally exact

E for $\mathbb{CP}_2$
E-Goldschmidt
E-Slovák

Other (pseudo-)Riemannian metrics

Products of irreducible Riemannian locally symmetric spaces?

CELM: $\square \xrightarrow{\mathcal{K}} \square\square \xrightarrow{\mathcal{L}} \overline{\boxplus}$ is locally exact **unless** there is

- at least one flat factor **and**
- at least one Hermitian factor

Examples

$$S^2 \times S^2 \checkmark \quad S^1 \times S^3 \checkmark \quad \mathbb{CP}_1 \times \mathbb{CP}_5 \checkmark \quad S^1 \times S^2 \times$$

Warped products

Example Schwarzschild solution (Khavkine 2019)

Kerr solution
(Aksteiner, Andersson, Bäckdahl, Khavkine, Whiting 2021)

THE END

THANK YOU