Algebraic degree of the Bergman kernel

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Roadmap

- Background on the Bergman kernel.
- Main results (in $\mathbb{C}^2$).
  - • Algebraic degree of the Bergman kernel.
  - • Total degree of the Bergman kernel.
- Some generalization and question in higher dimensional case.

Introduction

- Let Ω be a bounded domain in $\mathbb{C}^n$.
- Let $L^2(\Omega)$ denote the Hilbert space with the inner product

$$(f,g) = \int_\Omega f \cdot \bar{g}\, dV_E.$$

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- Let $L^2(\Omega)$ denote the Hilbert space with the inner product

$$(f,g) = \int_\Omega f \cdot \bar{g}\, dV_E.$$

- Let $A^2(\Omega) \subset L^2(\Omega)$ be the subspace of holomorphic functions.
- The *Bergman projection* is the orthogonal projection

$$\Pi : L^2(\Omega) \to A^2(\Omega).$$

The Bergman kernel

- The *Bergman kernel* K_Ω is the distribution kernel of Π:

$$\Pi(f)(x) = \int_\Omega f(y) \cdot K_\Omega(x, y)\, dV_E.$$

- If $\{\varphi_k\}$ is an ONB for $A^2(\Omega)$, then

$$K_\Omega(x, \bar{y}) = \sum_k \varphi_k(x) \cdot \overline{\varphi_k(y)}.$$

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- **Remark.** The Bergman kernel for polarized Kähler manifold* (in the talks by Bayraktar and Coman) is related to but different from the one here.

Some important results

A broad question: Characterize model domains by their Bergman kernels.

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- Cheng-Yau: For any bounded pseudoconvex domain with C^2 boundary, there exists a unique complete KE metric with Ricci curvature -1.
- Yau's question: Classify pseudoconvex domains whose Bergman metrics are KE.
- Cheng's conjecture: Let Ω be a bounded domain in $\mathbb{C}^n$ with smooth and strictly pseudoconvex boundary. Then, the Bergman metric of Ω is KE $\iff$ Ω is biholomorphic to $\mathbb{B}^n$.
- Cheng's conjecture is confirmed by Fu-Wong and Nemirovski-Shafikov for $n = 2$, and by Huang-Xiao for $n \geq 3$.

Algebraic Bergman kernel

Theorem 1 (Ebenfelt, Xiao and $\sim$, 2020)

Let $\Omega \subset \mathbb{C}^2$ be a bounded domain with smooth, strongly pseudoconvex boundary. Then, K_Ω is algebraic (rational) $\iff$ Ω is algebraically (rationally) biholomorphic to $\mathbb{B}^2$.

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Some further questions:

- What if the boundary is pseudoconvex?
- Some characterization on the biholomorphism?
- How about the higher dimensional case $n \geq 3$?

Algebraic degree and the total degree

Suppose the Bergman kernel $K(z, \bar{z})$ of Ω is algebraic. Let

$$P_{\min}(z, \bar{z}, t) = \alpha_d(z, \bar{z})t^d + \ldots + \alpha_0(z, \bar{z}) \in \mathbb{C}[z, \bar{z}, t]$$

be the minimal polynomial of K.

(i) We define the *algebraic degree* of K to be d.

(ii) We define the *total degree* of K to be the degree of $P_{\min}$ in $(z, \bar{z}, t)$.

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Remark.

- K is rational $\iff$ the algebraic degree of K is 1.
- In this case, we can write $K(z,\bar{z}) = \frac{p(z,\bar{z})}{q(z,\bar{z})}$ with $\gcd(p,q) = 1$. Then $qt - p$ is a minimal polynomial of K, and the total degree of K is $\max\{\deg q + 1, \deg p\}$.

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Main result on the algebraic degree

Theorem 2 (Ebenfelt, Xiao and $\sim$, 2021)

Let $\Omega \subset \mathbb{C}^2$ be a smoothly bounded pseudoconvex domain. Assume the Bergman kernel K of Ω is algebraic. Then the boundary $\partial\Omega$ is real algebraic and therefore of finite type. Moreover, if we write d for the algebraic degree of K and $r(\xi)$ for the type of $\partial\Omega$ at $\xi \in \partial\Omega$,

$$\max_{\xi \in \partial\Omega} r(\xi) \leq 2d.$$

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Remark. This inequality is sharp in the following sense.

- Consider the unit ball $\mathbb{B}^2$.
  - $\mathbb{B}^2$ is strongly pseudoconvex $\implies r(\xi) \equiv 2$.
  - $K_{\mathbb{B}^2}(z, \bar{z}) = \frac{2}{\pi^2} \frac{1}{(1-|z|^2)^3}$ is rational $\implies d = 1$.

Remark. (cont.)

- Consider the egg domains $E_d = \{|z|^2 + |w|^{2d} \le 1\}$ for any $d \ge 2$.
  - E_d has type $2d$ for points with $w = 0$.
  - **D'Angelo's formula.**

$$K\big((z,w),\overline{(z,w)}\big) = \sum_{k=0}^{2} c_k \frac{(1-|z|^2)^{-2+\frac{k}{d}}}{\big((1-|z|^2)^{\frac{1}{d}} - |w|^2\big)^{1+k}},$$

    with $c_0 = 0, c_1 = \frac{1}{\pi^2} \cdot \frac{d-1}{d}$, and $c_2 = \frac{1}{\pi^2} \cdot \frac{2}{d}$.
  - K is of algebraic degree d.

A corollary

Corollary 1

Let $\Omega \subset \mathbb{C}^2$ be a smoothly bounded pseudoconvex domain. If the Bergman kernel K_Ω is rational, then $r(\xi) = 2$ for all $\xi \in \partial\Omega$, i.e., $\partial\Omega$ is strongly pseudoconvex. In this case, there is a rational biholomorphism from Ω to the unit ball $\mathbb{B}^2$.

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Remark.

- The condition "smoothly" (i.e., smooth boundary) cannot be dropped, because the bidisk $D(0,1) \times D(0,1)$ also has rational Bergman kernel. (More examples like generalized Hartogs triangles, certain class of elementary Reinhardt domains by the work of Chakrabarti, Edholm, Huo, Zeytuncu...)

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- The condition "rational" cannot be relaxed to "algebraic", because the egg domain E_d has algebraic Bergman kernel.

Main ingredients for the improvement

- **The Fefferman/Boute de Monvel-Sjöstrand Asymptotics.**
If $\Omega = \{\rho > 0\} \Subset \mathbb{C}^n$ has smooth, strongly pseudoconvex boundary, then $\exists\, \phi, \psi \in C^\infty(\overline{\Omega})$ such that

$$K_\Omega = \frac{\phi}{\rho^{n+1}} + \psi \log \rho.$$

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- **Hsiao and Savale's generalization to pseudoconvex domains of finite type in $\mathbb{C}^2$.**
  Given $\xi \in \partial\Omega$ of type r, the Bergman kernel $K(z, \bar{z})$ has the following asymptotic expansion when $z \to \xi$ along a transversal direction:

$$K(z, \bar{z}) = \rho^{-2-\frac{2}{r}} \left(\sum_{j=0}^{N} c_j \rho^{\frac{j}{r}} + O(\rho^{\frac{N+1}{r}}) \right) + \psi \log \rho.$$

Sketch of the proof of Theorem 2

- **Algebraicity.**

$$\alpha_d(z,\bar{z})K^d + \cdots + \alpha_0(z,\bar{z}) \equiv 0, \quad (\alpha_d \not\equiv 0).$$

$$\Longrightarrow^{*} a_d(z,\bar{z}) = 0 \text{ on } \partial\Omega \implies \partial\Omega \text{ is algebraic} \implies \text{finite type.}$$

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$\Longrightarrow^* a_d(z,\bar{z}) = 0$ on $\partial\Omega \Longrightarrow \partial\Omega$ is algebraic $\Longrightarrow$ finite type.

- **Hsiao and Savale's asymptotics.**
  Take $N = 0$. On a transversal line $L(t)$,

$$K|_L = \rho^{-2-\frac{2}{r}}\left(c_0 + O(\rho^{\frac{1}{r}})\right) + \psi \log \rho = \rho^{-2-\frac{2}{r}}\left(c_0 + O(\rho^{\frac{1}{r}})\right).$$

$\Longrightarrow$

$$\alpha_d(t)\left(c_0^d + O(t^{\frac{1}{r}})\right) + \alpha_{d-1}(t)t^2 t^{\frac{2}{r}}\left(c_0^{d-1} + O(t^{\frac{1}{r}})\right) + \cdots + \alpha_0(t)t^{2d}t^{\frac{2d}{r}} = 0.$$

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- **Fu-Wong's type lemma.** If

$$\sum_{j=0}^{r-1} \beta_j(t) t^{\frac{j}{r}} \left(c_0^j + o(1)\right) \equiv 0 \quad \text{on} \quad (0, \varepsilon),$$

then each $\beta_j(t)$ for $0 \le j \le r-1$ vanishes to infinite order at 0.

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  then each $\beta_j(t)$ for $0 \leq j \leq r-1$ vanishes to infinite order at 0.
- **Conclusion.** Assume $2d < r$.
  Then $\alpha_d \equiv 0$ and this is a contradiction.

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Main result on the total degree

Theorem 3 (Ebenfelt, Xiao and $\sim$, 2021)

Let $\Omega \subset \mathbb{C}^2$ be a smoothly bounded pseudoconvex domain. Let K be the Bergman kernel of Ω. If K is algebraic, then

(a) *The total degree of $K \geq 7$.*

(b) *The total degree of $K = 7$ if and only if Ω is the unit ball up to a complex linear transformation. In this case, K is rational.*

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E.g.

- If $\Omega = \mathbb{B}^2$, then $K_{\mathbb{B}^2}(z, \bar{z}) = \frac{2}{\pi^2} \frac{1}{(1-|z|^2)^3}$ and its minimal polynomial over $\mathbb{C}[z, \bar{z}]$ is

$$(1 - |z|^2)^3 t - \frac{2}{\pi^2} = 0.$$

So the total degree is 7.

A corollary

Recall that if $K = \frac{p}{q}$ is rational, then the total degree is $\max\{\deg p, \deg q + 1\}$. By this relation, we get

Corollary 2

Let $\Omega \subset \mathbb{C}^2$ be a smoothly bounded pseudoconvex domain. Let K be the Bergman kernel of Ω. If K is rational, by writing $K = \frac{p}{q}$ for some polynomials with $\gcd(p, q) = 1$, *we have*

(a) $\max\{\deg p, \deg q\} \geq 6$.

(b) $\max\{\deg p, \deg q\} = 6$ *holds if and only if Ω is a the unit ball up to a complex linear transformation.*

Sketch of the proof of Theorem 3.

- **The Feffermen expansion nearby strongly pseudoconvex points.** Let p be a strongly pseudoconvex point on $\partial\Omega$. Then we have the Fefferman expansion in a neighborhood U of p.

$$K = \frac{\phi}{\rho^3} + \psi \log \rho \implies \frac{1}{K} = \frac{\rho^3}{\phi + \psi\rho^3 \log \rho} = O(\rho^3).$$

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- **Algebraicity.**

$$a_d = \frac{1}{K}\left(-a_{d-1} - a_{d-2}\frac{1}{K} - \cdots - a_0\frac{1}{K^{d-1}}\right) = O(\rho^3)$$

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- $\mathcal{I} := \{a \in \mathbb{C}[z, \bar{z}] : a \equiv 0 \text{ on } \partial\Omega, \text{ and } \bar{a} = a\} \subset \mathbb{R}[\operatorname{Re} z, \operatorname{Im} z]$. $\mathcal{I}$ is a principal ideal and we take r as a generator. Then $\deg r \geq 2$.

Sketch of the proof of Theorem 3

- **Compare the vanishing order.**

$$a_d = r^3 q(z, \bar{z}).$$

- **Count the degree.**

$$\text{total degree } \geq \deg a_d + d \geq 7.$$

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- **Equality.**
  total degree of $K = 7 \implies \deg r = 2, \deg q = 0$ and $d = 1$.
  Ω is a real ellipsoid by a complex linear transformation.
- By Theorem 1, Ω biholomorphic to $\mathbb{B}^2$.
  Then*, Ω is biholomorphic to $\mathbb{B}^2$ by a complex linear transformation.

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Remark If the total degree ≤ 9, then $\deg r < 3$ and we can still prove $\Omega = \mathbb{B}^2$. So there is a gap from the smallest total degree to the second smallest one.

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Some generalization to higher dimensions

In the higher dimensional case, we can only prove a weaker result.

Theorem 4 (Ebenfelt, Xiao and $\sim$, 2021)

Let $\Omega \subset \mathbb{C}^n (n \geq 2)$ be a smoothly bounded pseudoconvex domain. Let K be the Bergman kernel of Ω. If K is algebraic, then

(a) *The total degree of $K \geq 2n + 3$.*

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We conjecture that part (b) can be improved to

Conjecture.

(b′) The total degree of $K = 2n + 3$ if and only if Ω is $\mathbb{B}^n$ up to a complex linear transformation.

Some generalization to higher dimensions

This conjecture is confirmed if in addition we assume Ω is close to $\mathbb{B}^n$ under the Hausdorff distance.

Theorem 5 (Ebenfelt, Xiao and $\sim$, 2021)

Let $\Omega \subset \mathbb{C}^n (n \geq 2)$ be a smoothly bounded pseudoconvex domain. Suppose the Hausdorff distance $d_H(\Omega, \mathbb{B}^n)$ is sufficiently small. Let K be the Bergman kernel of Ω. If K is algebraic, then the total degree of $K = 2n + 3$ if and only if Ω is the unit ball up to a complex linear transformation in $\mathbb{C}^n$.

Sketch of the proof of Theorem 5

- $\Omega = \Phi(E(A))$ by some complex linear transformation Φ and some ellipsoid $E(A) = \{|z|^2 + \sum A_j(z_j^2 + \overline{z_j}^2) < 0\}$.
- $d_H(\Omega, \mathbb{B}^n)$ is small $\implies A = (A_1, \cdots, A_n)$ is sufficiently small.

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- **Ramadamov conjecture is true for E_A with small A by Hirachi.**
  $\implies A = 0$ and $E_A = \mathbb{B}^n$.

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Remark. The above conjecture is implied by the RC for ellipsoids.

Thank you for your attention!