A natural invariant measure for polynomial semigroups, and its properties

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$_1$Preliminaries

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- The Fatou set of $f :=$ the set of normality (or equicontinuity) of $\{f^n : n \in \mathbb{Z}_+\}$.
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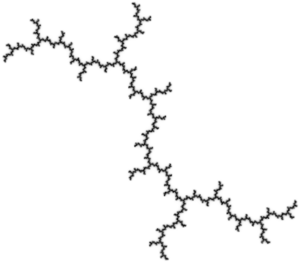
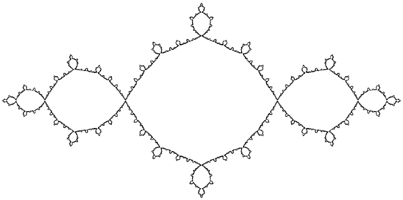
The dynamics on Fatou sets is tame and the structure of Fatou sets is well understood. On the other hand, the dynamics on Julia sets is chaotic and, in the generic case, Julia sets are fractals.

$_2$Some pictures



$z^2 + i$



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- There is considerable success in constructing analogues of the above-mentioned measures, even for multi-valued maps.

Mission statement of this talk: To study these measures from **potential-theoretic** points of view for the case of polynomial semigroups.

What conceptual framework do we have?

We study measures that describe the limiting distribution of the iterated pre-images of **any** point excluding, perhaps, a small set of exceptional points. In short, such a measure is the weak* limit of the sequence $\{\mu_n\}$ (if limit exists), where:

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What is an example of such a measure?

Brolin's theorem

Result (Brolin, 1965)

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- Roughly speaking, an equilibrium measure gives the distribution of a unit charge, in the absence of any external field, on a conductor that minimizes energy.
- (Lyubich, 1983) *Let g be a rational map of degree $d \geq 2$ and a be any point in $\widehat{\mathbb{C}}$ with (perhaps) two exceptions. Then $\{\mu_n\}$ (as defined above) converges to a Borel probability measure μ_g with support $\mathbf{J}(g)$* **[here μ_g has no potential-theoretic interpretation in general]**.

6 Some terminology

A *rational semigroup* S is a semigroup consisting of non-constant rational maps on the Riemann sphere $\widehat{\mathbb{C}}$ with the function-composition as the semigroup operation.

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Consider a generating set $\mathcal{G}$ of S. For $g \in S$, the expression $l(g) = n$ is the shorthand for the following implication:

$$l(g) = n \implies \exists \; g_{i_1}, \ldots, g_{i_n} \in \mathcal{G} \text{ such that } g = g_{i_n} \circ \cdots \circ g_{i_1}.$$

Measures associated with semigroups

Result (Boyd, 1999)

Let S be a finitely generated rational semigroup. Assume that every element of S is of degree at least 2. Let $\mathcal{G} = \{g_1, \ldots, g_N\}$ be a generating set and $D := \sum_{i=1}^{N} \deg(g_i)$.

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$$\mu_n := \frac{1}{D^n} \sum_{\substack{g(z)=a \\ l(g)=n}} \delta_z \xrightarrow{\text{weak*}} \mu_{\mathcal{G}} \quad \text{as } n \to \infty.$$

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We use a result by Dinh–Sibony, stated for correspondences, for a version of the last result that allows degree 1 elements.

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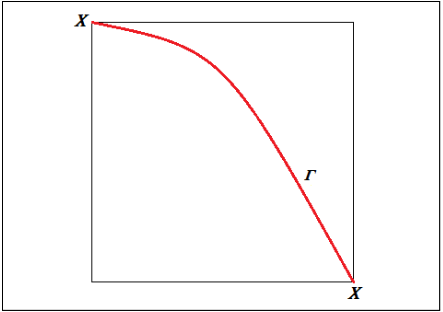

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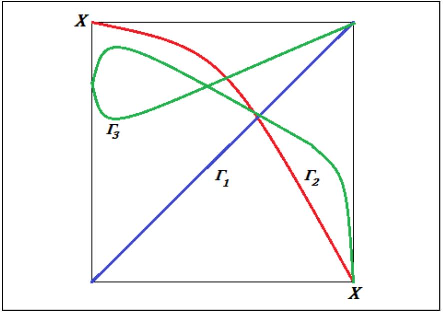

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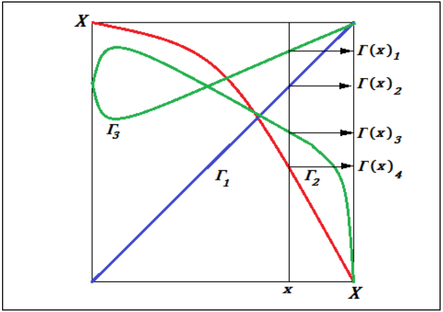

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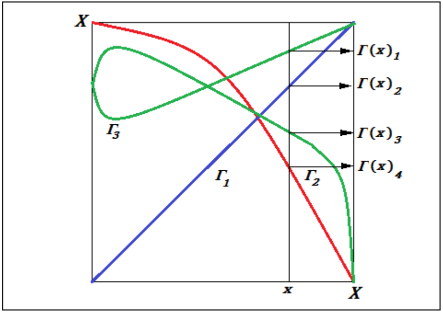
- $F(x) = \cup_i \, \pi_2(\pi_1^{-1}\{x\} \cap \Gamma_i)$.



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- One can figure out the weight of each irred. component of $|\Gamma| \star |\Gamma'|$ from the **following example**. . .

Composing two holomorphic correspondences, cont'd.

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Example. If

$$\Gamma := \sum_{1\leq i\leq N} \mathsf{graph}(g_i) \quad \text{and} \quad \Gamma' := \sum_{1\leq j\leq M} \mathsf{graph}(f_j),$$

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Note. $d_{top}(\Gamma) = \sum_{i=1}^{N} \deg(g_i)$ and $d_{top}({}^{\dagger}\Gamma) = N$.

Measures associated with correspondences

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- comes from dualising $(\pi_1)_*$,
- is the interpretation of "$(\pi_2^*(\Omega) \wedge [\Gamma])$" in this case.

A question

Let S be a finitely generated rational semigroup with a generating set $\mathcal{G} = \{g_1, \dots, g_N\}$ be a generating set. Consider $\mathit{\Gamma}_{\mathcal{G}} := \sum_{1 \leq i \leq N} \mathsf{graph}(g_i)$.

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Question: If each element of S is a polynomial, then is $\mu_{\mathcal{G}}$ the equilibrium measure of $\mathbf{J}(S)$?

Polynomial semigroups

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We call a rational semigroup S a *polynomial semigroup* if

- every element of S is a polynomial;
- every degree 1 element in S have ∞ as an attracting fixed point.

Logarithmic potentials

Let σ be a Borel probability measure on $\mathbb{C}$ with compact support. Its *logarithmic potential* is the function $U^\sigma : \mathbb{C} \to (-\infty, \infty]$ defined by

$$U^\sigma(z) = \int_{\mathbb{C}} \log \frac{1}{|z-t|} d\sigma(t).$$

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Key Proposition (L., 2020)

Let S be a finitely generated polynomial semigroup with a finite set of generators $\mathcal{G}$. Let $\boldsymbol{C}(\mathcal{G}) := \{c \in \mathbb{C} : g'(c) = 0$ for some $g \in \mathcal{G}\}$.

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- $\mathcal{E}(S) := \{z \in \widehat{\mathbb{C}} : \cup_{g \in S} g^{-1}\{z\}$ is a finite set$\}$.

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- $\mathcal{E}(S)$:=$\{z \in \widehat{\mathbb{C}} : \cup_{g \in S}\, g^{-1}\{z\}$ is a finite set$\}$.
- (Arsove, 1960) *U^σ is finite and continuous at z_0 if σ satisfies*

$$\sigma(D(z,r)) \leq C r^\alpha \quad \forall r \in (0, r_0),$$

where $|z - z_0| < \delta$ and C, α, r_0, δ are positive constants depending only on σ and z_0.

External fields and our first theorem

Let Σ be a compact subset of $\mathbb{C}$ and $Q : \Sigma \to (-\infty, \infty]$ be lower semi-continuous and $Q(z) < \infty$ on a set of positive capacity. The function Q is called an *external field*.

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$$I_Q(\sigma) := \int_{\mathbb{C}} \int_{\mathbb{C}} \log \frac{1}{|z-t|} d\sigma(z) d\sigma(t) + 2 \int_{\mathbb{C}} Q d\sigma.$$

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Remark: Hypothesis can be made "generator independent"; no time to discuss it.

SKETCHES OF A FEW PROOFS . . .

A counting lemma

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$$M := \max\{|g_i'(z)| : z \in \mathbf{J}(S),\ i \in \{1, \ldots, N\}\},$$

$$R := \frac{D}{N} \quad \text{and} \quad \lambda := \frac{\log R}{\log M},$$

where $D := \sum_{i=1}^{N} \deg(g_i)$. Thus $M = R^{\frac{1}{\lambda}}$. Note, $R > 1$ and $M > 1$.

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Assume $\sharp\big(\boldsymbol{C}(g_1, \ldots, g_N)\big) > 1$. Then there exist $r_0 > 0$ and $\kappa \in \mathbb{Z}_+$ such that for any $r \in (0, r_0]$ and $y \in \mathbf{J}(S)$, we have

$$\sharp((F^n)^\dagger(y) \cap D(z, r))^\bullet \le \max\left(D^{n - \frac{\nu}{\kappa} + 1} N^{\frac{\nu}{\kappa} - 1}, (D - \tfrac{1}{2})^n\right)$$

for all $n \in \mathbb{N}$ and $z \in \mathbb{C}$, where $\nu \in \mathbb{Z}_+$ is the unique integer such that

$$r \in I(\nu) := \left(r_0 R^{\frac{-2\nu}{\lambda}}, r_0 R^{\frac{-2(\nu-1)}{\lambda}}\right].$$

$_{17}$A counting lemma, continued: Main idea

Goal: $\sharp((F^n)^\dagger(y) \cap D(z,r))^\bullet \leq \max\left(D^{n-\frac{\nu}{\kappa}+1} N^{\frac{\nu}{\kappa}-1}, (D-\frac{1}{2})^n\right) \ \forall n \in \mathbb{N}, \ \forall z \in \mathbb{C}.$

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- Let $\delta_4 > 0$ be the Lebesgue number of the following cover: $\{D(\xi, r(\xi)) : \xi \in \bar{\mathbf{J}}^{\delta_2}\}$, where $r(\xi) > 0$ is such that $g_i|_{D(\xi, r(\xi))}$ is injective for $i = 1, 2, \dots, N$.

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Write:

$$r_0 := \frac{\min\{\delta_2, \delta_3, \delta_4\}}{4} \;\; \text{and} \;\; \kappa = 1.$$

With this choice of r_0 and κ, the inequality follows by induction on n.

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What if $C(g_1, \dots, g_N) \cap \mathbf{J}(S) \neq \emptyset$? Let $\delta_1 > 0$ be so small that:

- $D(c_j, 2\delta_1)$ are pairwise disjoint for $j = 1, 2, \dots, q$,

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A counting lemma, continued: Main idea

Goal: $\sharp((F^n)^\dagger(y) \cap D(z,r))^\bullet \leq \max\left(D^{n-\frac{\nu}{\kappa}+1} N^{\frac{\nu}{\kappa}-1}, (D-\frac{1}{2})^n\right)\ \ \forall n \in \mathbb{N},\ \forall z \in \mathbb{C}.$

What if $C(g_1, \ldots, g_N) \cap \mathbf{J}(S) \neq \emptyset$? Let $\delta_1 > 0$ be so small that:

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This δ_1 and a parameter δ_3 (as in last slide) describe two **partial** open covers of $\bar{\mathbf{J}}^{\delta_2}$. These form an open cover serving the same purpose as in the last slide.

$_{18}$A counting lemma, continued: Main idea

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Value of κ: Let κ be such that
$\sum_{i:g_i'(x)\neq 0} \left(\frac{D}{N}\right)^{1/\kappa} + \sum_{i:g_i'(x)=0} \operatorname{ord}_x(g_i) \leq D - \frac{1}{2} \quad \forall x \in \boldsymbol{C}(g_1, g_2, \dots, g_N).$

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The proof is divided between the following essential cases:

- *Case 1:* $\sharp\big(\boldsymbol{C}(\mathcal{G})\big) > 1$.

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We'll only address Case 1. For n sufficiently large:

$$\mu_n(D(z,r)) = \frac{1}{D^n}\sharp((F^n)^\dagger(a) \cap D(z,r))^\bullet \leq \left(\frac{D}{N}\right)^{1-\frac{\nu}{\kappa}}.$$

19 Sketch of the proof of the Key Proposition

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Since $r > r_0 R^{-2\nu/\lambda}$ and recalling that $R := D/N$, we get

$$\mu_n(D(z,r)) \leq \left(\frac{R}{{r_0}^{\lambda/2\kappa}}\right) r^{\frac{\lambda}{2\kappa}} = C_1 r^{\alpha}.$$

Since $\mu_n \to \mu_{\mathcal{G}}$ in the weak* topology, $\mu_{\mathcal{G}}(D(z,r)) \leq C_1 r^{\alpha}$. ■

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$$G_{\mathcal{G}}(z) := \limsup_{n\to\infty} \frac{1}{D^n} \log\left(\prod_{l(g)=n} |g(z) - a| \right),$$

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- Since $\mu_n \to \mu_{\mathcal{G}}$ in the weak* topology, we get

$$U^{\mu_{\mathcal{G}}}(z) + G_{\mathcal{G}}(z) \leq \frac{\log A}{D - N} \quad \text{for every } z \in \mathbb{C},$$

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where $A = |\mathrm{lead}(g_1) \times \mathrm{lead}(g_2) \times \cdots \times \mathrm{lead}(g_N)|$.

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- As $U^{\mu_{\mathcal{G}}}$ is continuous, it follows that

$$U^{\mu_{\mathcal{G}}}(z) + G^*_{\mathcal{G}}(z) = \frac{\log A}{D - N} \quad \forall z \in \mathbb{C}.$$

Lower bound for capacity of the Julia set

Theorem (L., 2020)

Let $(S, \mathcal{G})$ *be as in our main theorem and let* $Q_{\mathcal{G}}$ *denote the external field associated with* $(S, \mathcal{G})$.

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THANK YOU!