Talks » 2021

Polyhedral Approximations of Strongly $\mathbb{C}$-Convex Domains

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Partially based on (ongoing) joint work with
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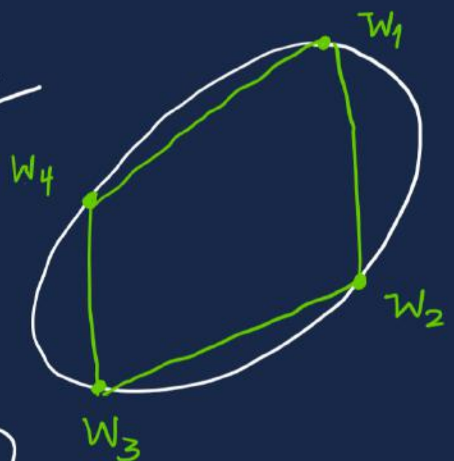
$\mathbb{R}^d$: approximation schemes (based on boundary data)

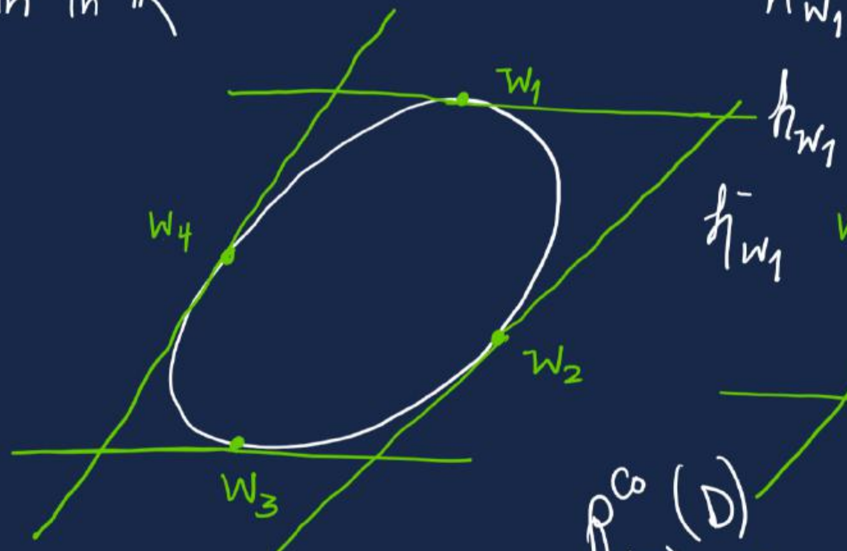
- D: strongly convex domain in $\mathbb{R}^d$

Case I



$\mathcal{P}_n^i(D)$ $\begin{cases} P = \operatorname{cvx}\{w_1, \dots, w_n\} \\ \#\operatorname{vert}(P) \leq n \end{cases}$

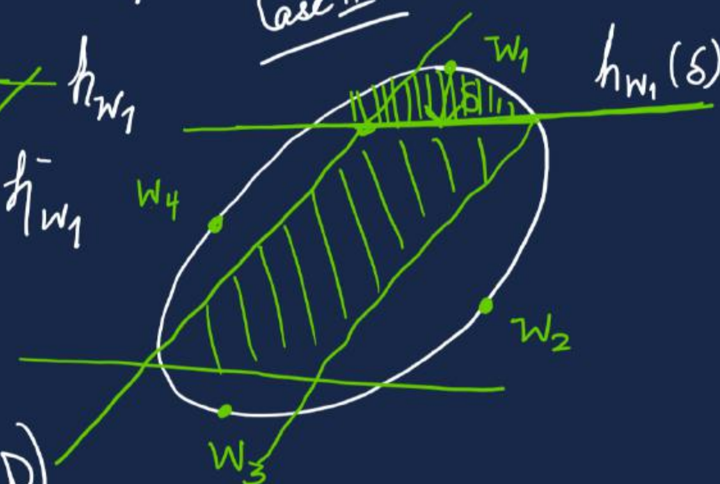
$\delta_V(D,P) = \operatorname{vol}(D \setminus P)$



$P = \bigcap_{1 \leq j \leq n} h^-_{w_j}$

$\#\text{facets}(P) \leq n.$

Case II



$\mathcal{P}^{co}_{(n)}(D)$ $\begin{cases} P = \bigcap_{1 \leq j \leq n} h^-_{w_j}(\delta_j) \\ \#\text{facets} \leq n \end{cases}$

$\delta_A(D,P) = \max_{1 \leq j \leq n} \operatorname{vol}\left(D \cap h^+_{w_j}(\delta_j)\right).$

$\delta(D,P)$ = "distance" b/w D & P.

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$\mathbb{R}^d$: Some questions

Optimal approximation

1. Asymptotic expansion of $v_n := \inf_{\mathrm{Ord}(P) \le n} \delta(D, P)$ as $n \to \infty$.

2. Description of close-to-optimal polyhedra.

Random approximation

3. Asymptotic behavior of $V_n = \delta(D, P_n)$ as $n \to \infty$, where P_n is a random polyhedron.

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$\mathbb{R}^d$: sample results — case I

Gruber (1993)

$$v_n := \inf\{\mathrm{vol}(D \setminus P) : P \in \mathcal{P}_n^i(D)\}.$$

$$v_n \sim \alpha_d \cdot \sigma_{Bla}(D)^{\frac{d+1}{d-1}} \frac{1}{n^{2/d-1}} \qquad n \to \infty.$$

Schütt-Werner (2003)

$f : bD \longrightarrow (0,\infty)$ continuous pdf,
$\omega_1, \dots, \omega_n \sim f$ i.i.d. points on bD,
$V_n := \mathrm{vol}(D \setminus \mathrm{cvx}\{\omega_1, \dots, \omega_n\})$.

$$\mathbb{E}(V_n) \sim a_d(D,f) \frac{1}{n^{2/d-1}} \qquad n \to \infty$$

Best constant : $a_d(D,f) = \tilde{\alpha}_d \, \sigma_{Bla}(D)^{\frac{d-1}{d+1}}$

$\mathbb{R}^d$: geometry of the approximation

$A : \mathbb{R}^d \to \mathbb{R}^d$ is an affine transformation.

1. A preserves strong convexity.
2. $A\left[\mathcal{P}_n^i(D)\right] = \mathcal{P}_n^i(AD)$.
3. If $|\det A| = 1$, $\delta_V(D,P) = \delta_V(AD, AP)$.

Case II

The problem is invariant under equiaffine transformations. — geometry of the approximation

An equiaffine invariant measure on bD (Blaschke)

$$\sigma_{Bla}(x) = \kappa_D^{\frac{1}{d+1}}(x)\, \sigma_{Euc}(x), \quad x \in bD$$

$$A^* \sigma_{Bla}^{A(D)} = |\det A|^{\frac{d-1}{d+1}} \sigma_{Bla}^{D}.$$

κ_D = Gaussian curvature on bD

σ_{Euc} : surface area measure.

$\mathbb{R}^d$: sample results — case II

$\boxed{\text{Schneider (1986)}}$

$$v_n := \inf \{ \delta_M(D,P) : P \in \mathcal{P}^{o}_{(n)}(D) \}.$$

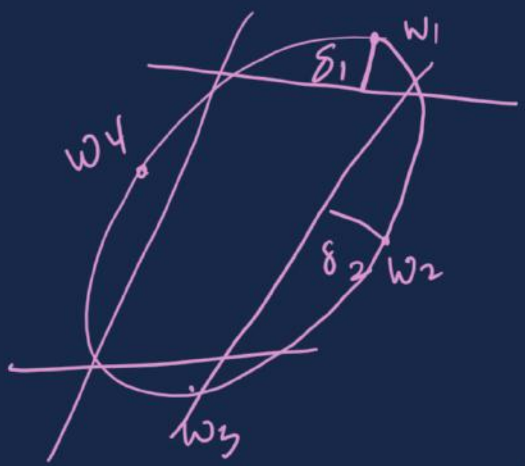
$$V_n \sim \beta_d \, \sigma_{Bla}(D)^{\frac{d+1}{d-1}} \underbrace{\frac{1}{n^{\frac{d+1}{d-1}}}} \qquad n \to \infty.$$

$\boxed{\text{Glasauer - Schneider (1996)}}$

$f : bD \longrightarrow (0,\infty)$ continuous pdf,
$\{w_j\}_{j \in \mathbb{N}} \sim f$ i.i.d. points on bD,
$\delta_1, \ldots, \delta_n$ = function $\{w_1, \ldots, w_n, f\}$: $P \subseteq D$

$$V_n := \delta_M(D,P).$$

$$V_n \xrightarrow{\ P\ } b_d(f,D) \underbrace{\left(\frac{\log n}{n}\right)^{\frac{d+1}{d-1}}} \qquad n \to \infty$$



Best constant : $b_d(f,D) \overset{\sim}{=} \beta_d \, \sigma_{Bla} \ldots$

$\mathbb{R}^d$: Riemannian metrics on bD

Caps of P on bD $\longleftrightarrow$ balls of a Riemannian metric $\mathcal{G}$ on bD

optimal / random approximant $\longleftrightarrow$ minimal / random coverings by $\mathcal{G}$-balls

Error $\longleftrightarrow$ dispersion (radius)

$\boxed{\text{Schneider (1981)}}$ r_n = min. radius for covering bD by n equiradial $\mathcal{G}$-balls.

$$\lim_{n\to\infty} r_n n^{1/d} \in (0,\infty).$$

$\boxed{\text{Janson (1985)}}$ $(X_n)_{\mathbb{N}}$ i.i.d. $R_n = \min\{R>0 : bD = \bigcup B_{\mathcal{G}}(x_i; R)\}$.

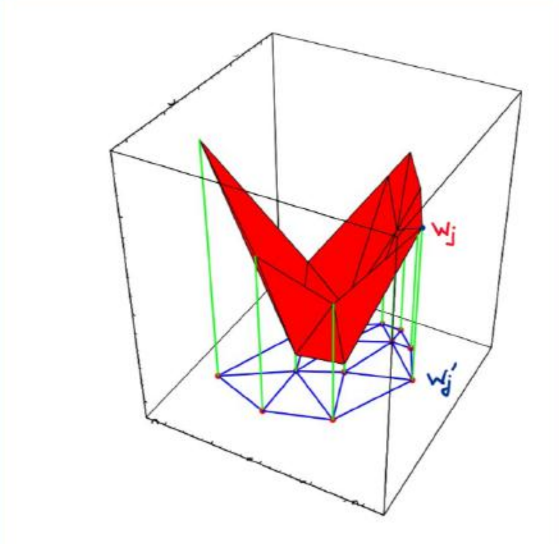
$$\mathbb{P}\text{-}\lim_{n\to\infty} \left(\frac{n}{\log n}\right)^{1/d} R_n \in (0,\infty).$$

$\mathbb{R}^d$: on the dimensional constants.

Model: $M = \{ x_d > x_1^2 + \cdots + x_{d-1}^2 \} \xrightarrow{\text{projection}}$ geom. combinatorics on $(\mathbb{R}^{d-1}, \|\cdot\|)$.

Case I.

Lower facets of $\mathrm{cvx}\{w_1, \dots, w_n\}$ $\longleftrightarrow$ Delaunay triangulation of $w_1', \dots, w_n'$.



A Delaunay triangulation and its lifting to a paraboloid.

Case II. involves minimum covering density of $\mathbb{R}^{d-1}$ by unit Euc. balls.

$\mathbb{C}^d$: strongly $\mathbb{C}$-convex domains

$\Omega = \{\rho < 0\} \subseteq \mathbb{C}^d$: $\mathcal{C}^2$-domain.

Str. $\mathbb{C}$-cvx : $\quad \mathrm{Hess}_{\mathbb{R}}\, \rho(w)\big|_{H_w b\Omega} > 0 \quad \forall w \in b\Omega.$

$\rightarrow$ Same as str. $\mathbb{C}$-lin. cvx.

$\rightarrow$ $\Omega \overset{\text{homeo}}{\simeq} \mathbb{B}^d$.

$\rightarrow$ $w + H_w b\Omega \cap \overline{\Omega} = \{w\}$.

$\rightarrow$ str. $\mathbb{C}$-cvx. invariant under LFTs.

$\rightarrow$ locally str. convexifiable via LFTs.

$(a_{jk}) \in GL(d+1; \mathbb{C})$

$$M: z = (z^1, \dots, z^d) \mapsto \left(\frac{f_1(z)}{f_0(z)}, \dots, \frac{f_d(z)}{f_0(z)} \right)$$

$$f_k(z) = a_{k0} + \sum_{j=1}^{d} a_{kj} z^j.$$

$\mathbb{C}^d$: analytic polyhedra (too general!)

An analytic polyhedron in Ω of order $\leq n$:

finite union of connected comp of $\{z \in \Omega : |f_j(z)| < 1,\ 1 \leq j \leq n\}$, that are rel. cpt. in Ω. $f_j \in \mathcal{O}(\Omega)$.

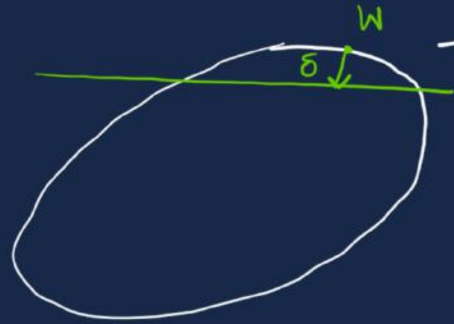
facets $\leq n$.

$\boxed{\text{Bishop (1961)}}$ Given $K \subset^{\mathrm{cpt}} \Omega$, $\exists$ polyhedron P of order $\leq d$ s.t.

$$K \subset P \subset\subset \Omega.$$

$\mathbb{C}^d$: Cauchy-Leray polyhedra

$D=\{r<0\}\subseteq\mathbb{R}^d$



$$\boxed{\frac{\langle \nabla r(w), w-x\rangle}{|\nabla r(w)|}=\delta}$$

$P\in\mathcal{P}^{co}_{(n)}$

$\Downarrow$

$$\underbrace{P=\bigcap_j\{x:\langle\nabla r(w_j),w_j-x\rangle>\delta_j\}}$$

$\Omega=\{\rho<0\}\subseteq\mathbb{C}^d$

$$L(w,z)=\frac{2}{|\nabla\rho(w)|}\sum\frac{\partial\rho}{\partial z^j}(w)\,(w^j-z^j),\quad w\in b\Omega,\ z\in\mathbb{C}^d$$

Let $\mathcal{W}=\{w_1,\dots,w_n\}\subseteq b\Omega$ and $\Delta=\{\delta_1,\dots,\delta_n\}\subseteq(0,\infty)$

$P_{\mathcal{W},\Delta}=$ union of all conn. comp. $\bigcap\{z\in\mathbb{C}^d:|L(w_j,z)|>\delta_j\ \forall j\}$ that meet Ω.

$\mathcal{P}^{co}_{(n)}(\Omega)=\{\mathcal{P}_{\mathcal{W},\Delta}:\mathcal{P}_{\mathcal{W},\Delta}\subseteq\Omega\}$.

invariant under LFTs.

$\mathbb{C}^d$: an optimal approximation result.

Theorem (G.) Let $\Omega \subseteq \mathbb{C}^d$ be a C^3-smooth str. $\mathbb{C}$-cvx. domain. Then,

$$\nu_n = \inf\{\mathrm{vol}(\Omega \setminus P) : P \in \mathcal{P}^{co}_{(n)}\} \sim \underbrace{k_d} \cdot \underbrace{\Big(\underbrace{\int_{b\Omega} \nu_\Omega^{\frac{1}{d+1}} d\sigma_{Euc}}_{\sigma_\Omega}\Big)^{\frac{d+1}{d}}} \underbrace{\frac{1}{n^{1/d}}} \quad \text{as } n \to \infty,$$

where $k_d > 0$ is independent of Ω & $\nu_\Omega : b\Omega \to (0,\infty)$ is continuous.

$\nu_\Omega :=$ analogue of K_D $\qquad$ det (Shape operator $|_{H_w b\Omega}$)

$$\sigma_\Omega := \nu_\Omega^{\frac{1}{d+1}} \sigma_{Euc}.$$

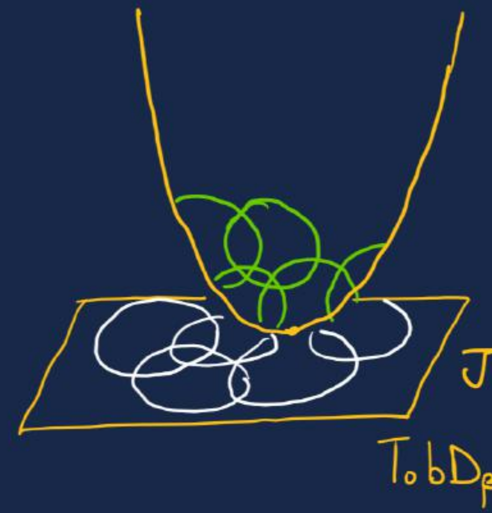
$$M^* \sigma_{M(\Omega)} = |\det J_{\mathbb{C}} M|^{\frac{2d}{d+1}} \sigma_\Omega.$$

Exponents

	Haus. dim. of metric on bB^d	exp. of measure	exp. of $1/n$
$\mathbb{R}^d$	$d-1$	$d+1/d-1$	$2/d-1$
$\mathbb{C}^d$	$2d$	$d+1/d$	$1/d$.
	$\}\ \eta$	$\}\ \frac{\eta+2}{\eta}$	$\}\ \frac{2}{\eta}$

$\mathbb{C}^2$: model domains

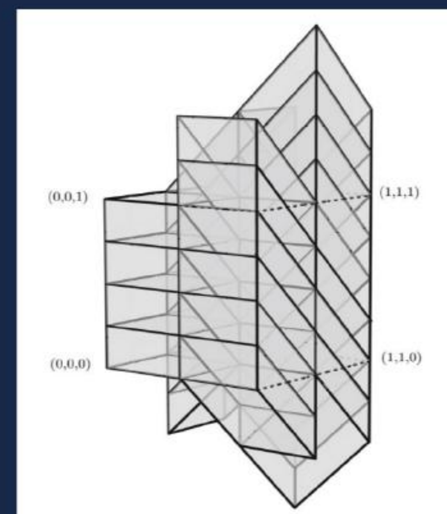
$$D_{\alpha,\beta} := \{ \operatorname{Im} z^2 > \alpha |z^1|^2 + \beta \operatorname{Re}(z^1)^2 \}, \quad 0 \le \beta < \alpha.$$



$\rightarrow$ $\tilde{\nu}_{D_{\alpha,\beta}} \equiv \dfrac{\alpha^2 - \beta^2}{16}$

$\rightarrow$ $\{ L_{\alpha,\beta}(w,z) < \delta \} \cap bD_{\alpha,\beta} \xrightarrow{\text{project}}$ balls in a quasimetric $d_{\alpha,\beta}$.

$\rightarrow$ $d_{\alpha,\beta}$ is left-invariant under a nonabelian group str.



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$\mathbb{C}^d$: random approximations — the model

Let $f : b\Omega \to (0,\infty)$ be a continuous p.d.f.

$\mathcal{X} = \{W_j\}_{j\in\mathbb{N}}$ i.i.d. points on $b\Omega$

$P_n(\delta) := P_{\mathcal{X}_n,\delta}$, where $\mathcal{X}_n = \{W_1,\dots,W_n\}$ & $\delta = \{\delta,\dots,\delta\}$.

$$V_n(\delta) := \mathrm{vol}(\Omega \setminus P_n(\delta))\,\mathbb{1}_{[P_n(\delta)\subseteq\Omega]} + \mathrm{vol}(\Omega)\,\mathbb{1}_{[P_n(\delta)\subseteq\Omega]^c}.$$

Theorem (Athreya–G.–Yogeshwaran) Suppose $\delta_n \to 0$ such that $\{\mathbb{P}[P_n(\delta_n)\subseteq\Omega] \to 1$ as $n\to\infty$.$\}$

Then,

$$\frac{V_n(\delta_n)}{\delta_n} \xrightarrow{P} \sigma_{Euc}(b\Omega) \quad \text{as } n\to\infty.$$

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$\mathbb{C}^d$: random approximations

Recall: $V_n(\delta_n) \overset{P}{\sim} \delta_n$ if $\mathbb{P}[P_n(\delta_n) \subseteq \Omega] \to 1$.

Lemma. $\exists\, c_D, C_D > 0$ s.t. if

a) $\delta_n \underbrace{\left(\frac{\log(n)}{n}\right)^{-\frac{1}{d}}}_{} \geq C_D$

$$\mathbb{P}[P_n(\delta_n) \subseteq \Omega] \to 1 \quad \text{as} \quad n \to \infty,$$

b) $\delta_n \left(\frac{\log(n)}{n}\right)^{-1/d} \leq c_D$

$$\mathbb{P}[P_n(\delta_n) \subseteq \Omega] \to 0 \quad \text{as} \quad n \to \infty.$$

optimal $V_n \sim \frac{1}{n^{1/d}}$ v/s random $V_n \sim \left(\frac{\log n}{n}\right)^{1/d}$