Deformations and Embeddings of Compact 3D Strictly Pseudoconvex CR Manifolds II

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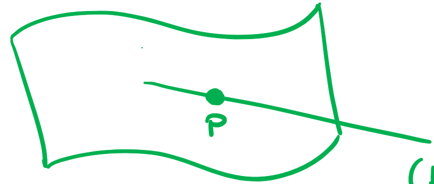
$M^3 \subseteq \mathbb{C}^2$ real hypersurface



(H_p, J_p) max. compl. subspace

$\rightsquigarrow$ CR structure (M, H, J)

- (generically) H <u>contact distribution</u>

  $\hookrightarrow$ $H = \ker \Theta$, Θ 1-form

  $\Theta \wedge d\Theta \neq 0$

- $J : H \to H$, $J^2 = -\mathrm{Id}$.

<u>Notation:</u> $(M, H, J) \longleftrightarrow (M, T^{1,0}) \longleftrightarrow (M, \bar\partial_b)$

$$\mathbb{C}H = \underset{i}{T^{1,0}} \oplus \underset{-i}{T^{0,1}} \quad \text{eigenspaces of } J$$

Often $T^{1,0} = \mathrm{span}\{Z_1\}$ "$\bar\partial_b = Z_{\bar 1} = \overline{Z_1}$"

! $[T^{1,0}, T^{1,0}] \subseteq T^{1,0}$ <u>trivially</u>.

CR Structure of Unit $S^3 \subseteq \mathbb{C}^2$

$$u = 1 - |z|^2 - |w|^2, \qquad \theta = i\partial u|_{TS^3}$$

On S^3

- $\theta = i(z\,d\bar{z} + w\,d\bar{w})|_{TS^3}$ (real)
- $d\theta = i\,(dz \wedge d\bar{z} + dw \wedge d\bar{w})$
- $Z_1 = \bar{w}\,\partial_z - \bar{z}\,\partial_w$
- $Z_{\bar{1}} = w\,\partial_{\bar{z}} - z\,\partial_{\bar{w}}$
- $T = i(z\partial_z + w\partial_w) - i(\bar{z}\partial_{\bar{z}} + \bar{w}\partial_{\bar{w}})$

  (Reeb v.f. of θ)

Abstract Deformations

- Gray's theorem: might as well fix H.
- Eliashberg '89: might as well assume H is standard on S^3.
  ("tight")

$\rightsquigarrow$ Only vary J ($\Leftrightarrow$ keep $\mathrm{span}\{Z_1, Z_{\bar 1}\}$ fixed)

$$\hat{Z}_1 = \frac{1}{\sqrt{1-|\varphi|^2}} \left(Z_1 + \underline{\underline{\varphi_1{}^{\bar 1}}}\, Z_{\bar 1} \right)$$

$|\varphi|^2 < 1$
(pointwise)

$$\hat{\theta}^1 = \frac{1}{\sqrt{1-|\varphi|^2}} \left(\theta^1 - \underbrace{\varphi_{\bar 1}{}^1}_{\text{conj.}}\, \theta^{\bar 1} \right)$$

We get a parametrization of all CR structures on H in this way (up to conjugation).

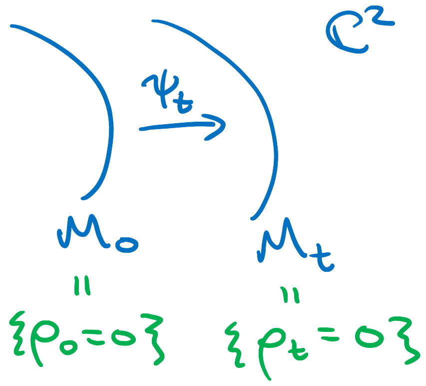
Question: When does $\varphi = \varphi_1^T$ correspond to an *actual* deformation of S^3 inside $\mathbb{C}^2$?

$\longrightarrow$ An embedding problem!

Remarks:

- **Lempert Stability:** Small deformations of S^3 (or $M^3 \subseteq \mathbb{C}^2$) that are embeddable in $\mathbb{C}^N$ are embeddable in $\mathbb{C}^2$ so we always consider embeddability in $\mathbb{C}^2$.
- In higher dimensions every compact (integrable) CR manifold is embeddable (Boutet de Monvel '75).
- Integrability condition: $\qquad \bar{\partial}_b \bar{\varphi} + \bar{\varphi} \wedge \bar{\varphi} = 0.$

Solution at the linearized level:



Pull back by $\psi_t : M_0 \to M_t$ (contact diffeo^m)

to get a family of CR structures

on M_0.

$$\rightsquigarrow \quad Z_1^t = \frac{1}{\sqrt{1-|\varphi(t)|^2}}\left(Z_1 + \underline{\varphi_1{}^{\bar 1}(t)}\, Z_{\bar 1}\right)$$

$$\rightsquigarrow \quad \dot\varphi_{11} = \frac{d}{dt}\Big|_{t=0} \varphi_{11}(t)$$

Geometric Calculation (well known): $\dot\varphi_{11}$ is a linearized embeddable deformation iff

$$\dot\varphi_{11} = (\nabla_1\nabla_1 + i A_{11}) f \quad \text{for some } f. \qquad (*)$$

(complex)

$\operatorname{Re} f = \dot\rho|_{M_0}$ (normal velocity) $\qquad \frac{d}{dt}\Big|_{t=0}(e^{\gamma_t}\rho_t)\Big|_{M_0} = e^{\gamma_0}\dot\rho|_{M_0}$

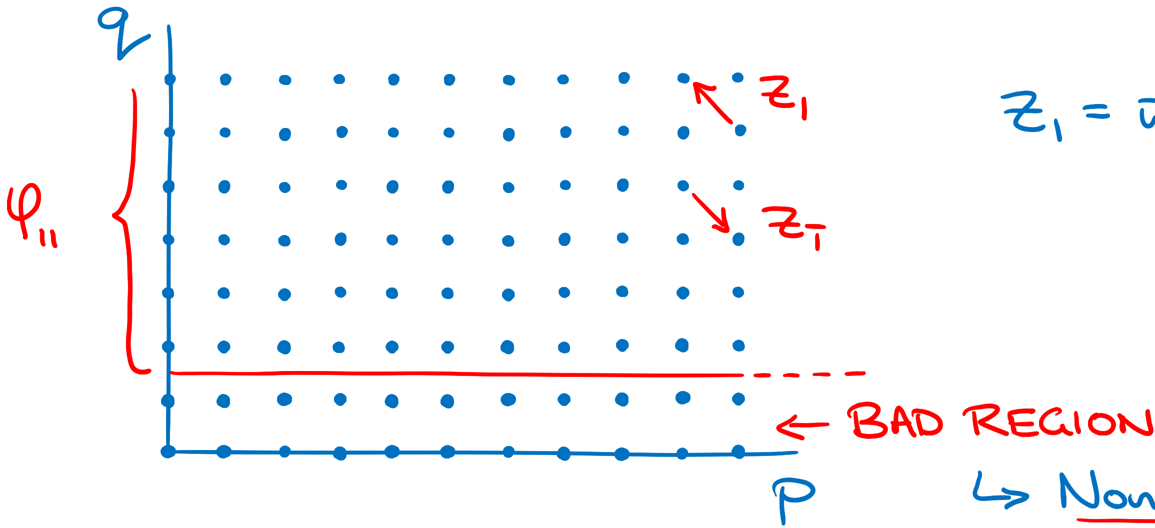
On S^3 with $(Z_1, Z_{\bar{1}}, T)$ standard:

$$A_{11} = 0, \quad \omega_1{}^1{}_1 = 0$$

$$\rightsquigarrow \quad \dot{\varphi} = (\nabla_1\nabla_1 + iA_{11}) f = \underline{\underline{Z_1 Z_1 f}}.$$

We understand this using spherical harmonics: $H_{p,q} \subseteq P_{p,q}$

(e.g., $(z-w)^p(\bar{z}+\bar{w})^q \in H_{p,q}$)



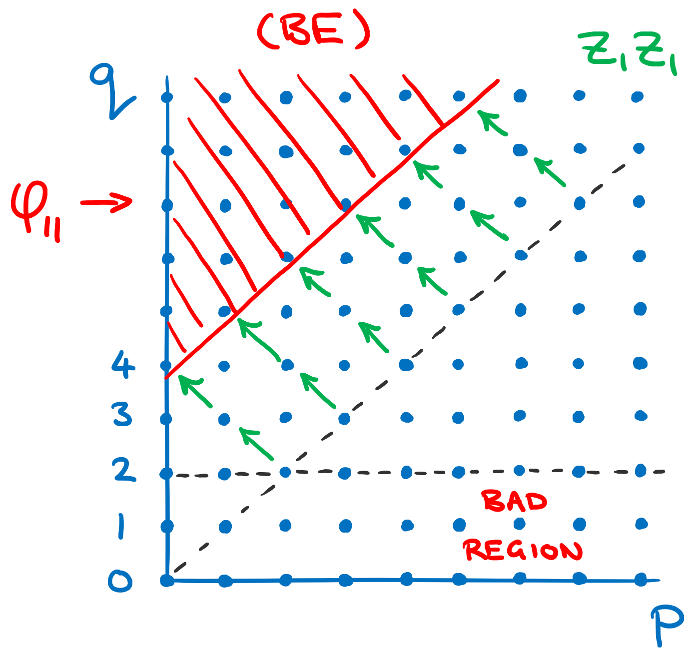
$$Z_1 = \bar{w}\partial_z - \bar{z}\partial_w$$

↳ Nonembeddability is generic!

Interesting/Important Result:

Burns-Epstein, JAMS '90:

φ_{11} is embeddable if $\varphi_{11} \in \overline{\bigoplus_{q \geq p+4} H_{p,q}}$.



Good multiplicative properties

(Using our approach we get analyticity of the embedding in t for $t\varphi_{11}$.)

Why 4? $\mathcal{L}_T Z_1 = -2i Z_1$

$Z_1 + \underline{\underline{\varphi_1}}^{T} Z_{\bar{T}}$

Normalizing the CR Structure Using Contact Diffeo^{m}s

Q: How do contact diffeo^{m}s act on deformations?

Linearized level: contact (Hamiltonian) vector fields

$$X_g = gT + i(\nabla^1 g)Z_1 - i(\nabla^{\bar 1} g)Z_{\bar 1}$$

correspond to $f = ig$ $\quad (\mathrm{Re}\, f = \dot\rho|_M = 0)$.

$\leadsto$ Linearization of natural action

$$\{\text{Contact Diffeo's}\} \times \left\{\begin{matrix}\text{CR structures}\\ \text{on } (S^3,H)\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\text{CR structures}\\ \text{on } (S^3,H)\end{matrix}\right\}$$

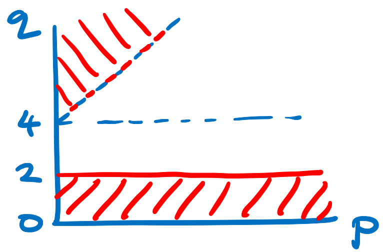
at (id, J_{std}) is

$g \in C^\infty(S^3,\mathbb{R})$

$$(g, \dot\varphi_{11}) \longmapsto \dot\varphi_{11} + i Z_1 Z_1 g.$$

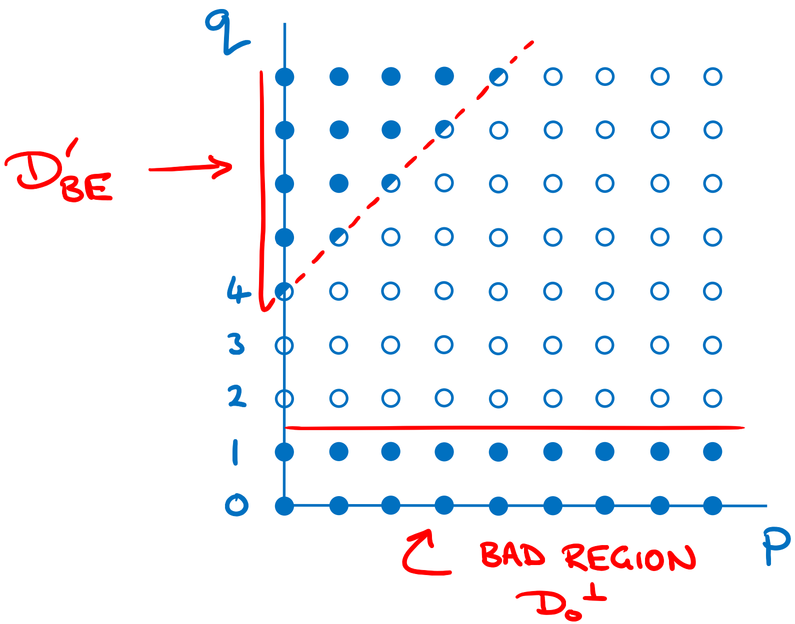
Infinitesimal Slice:

(best choice!)



Modified Cheng-Lee Slice Theorem:

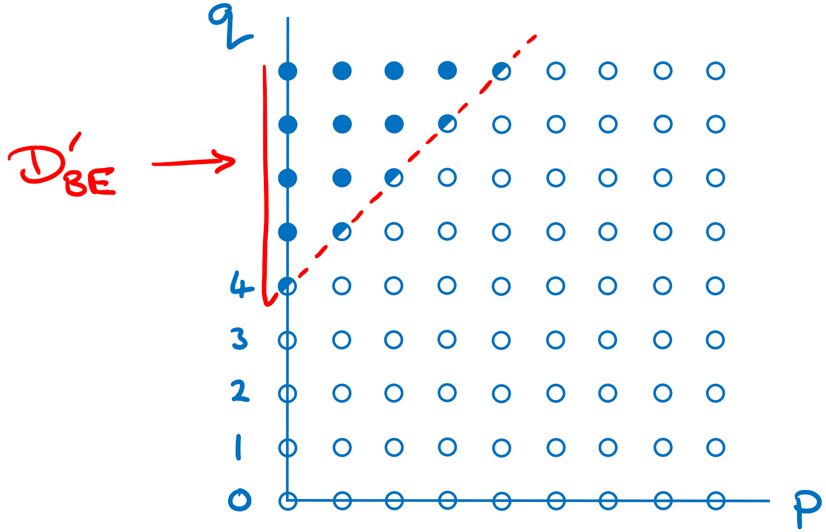
Informal Statement: Any sufficiently small deformation of the standard CR structure on (S^3, H) can be pulled back by a contact diffeo^m to one with deformation tensor φ "in"



- Proof essentially follows Cheng-Lee, AJM '95.
- Advantage is that we connect with embeddability.
- Burns-Epstein showed that the set of nonembeddable $\varphi \in (D'_{BE} \oplus D_0^\perp) - D'_{BE}$ is a G_δ-set.
- Epstein (Ann. Math. '98) showed this G_δ is open.
- We show it is everything (for small φ).

A Slice Theorem For Embeddable Structures:

Slice:



* Need to use "marked" str.s to deal with noncompact symmetry gp. of std CR sphere (as in Cheng-Lee).

$\mathrm{Aut}(S^3) \circlearrowright D'_{BE}$

Consequence: The space of (marked) embeddable CR structures is a smooth tame Frechet submanifold near the standard CR sphere.

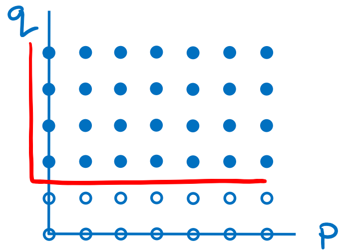
Relies on: Lempert (BSMF '81), Bland-Duchamp (Invent '91).

$\rightsquigarrow$ analytic discs, circular representation

In finite regularity: Bland (Acta '94), cf. B-D (JDG '11, JSG '14).

Q: Can we get around the need to normalize?

In practice, finding the contact diffeo is a problem.

<u>Idea:</u> Given $\dot{\varphi}_{11} \in$  (i.e. $\varphi_{11} = Z_1 Z_1 f_0$)

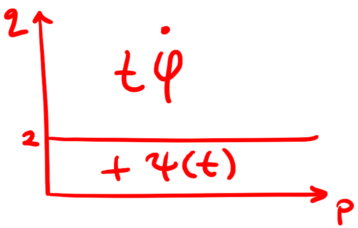
solve

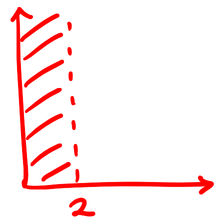
$$\left(\nabla_1^t \nabla_1^t + i A_{11}(t)\right) f_t = \frac{\frac{d}{dt}\varphi_{11}(t)}{1 - |\varphi(t)|^2} \qquad (*_t)$$

for $\varphi(t) = t\dot{\varphi} + O(t^2)$ and $f_t = f_0 + O(t)$.

use this to construct a family of CR embeddings realizing $\varphi(t)$.

To make this well-posed:

$\varphi(t) =$ 

$f_t - \lambda \perp \ker Z_1 Z_1 =$ 

(λ: const. > 0)

From solutions of $(*_t)$ to embeddings:

Theorem (C.-Ebenfelt, in preparation):
Given φ_t, f_t solving $(*_t)$, with $\operatorname{Re} f_t$ having strict sign, $\exists$ a smooth family of embeddings $\Psi_t : S^3 \to \mathbb{C}^2$ realizing φ_t at each time t.

At $t=0$, the correct variational vector field is

$$\operatorname{Re}\left(f_0 \xi^0 + i \nabla' f_0 \cdot Z_1 \right), \qquad \text{where } \xi^0 = \tfrac{1}{2}(JT + iT).$$

But finding the variational vector field (along "$\Psi_t(S^3)$") in terms of f_t is much more difficult.

We refine and reverse engineer a construction in Hirachi-Marugame-Matsumoto (Adv. Math '17) using Fefferman's ambient metric construction & the Graham-Lee connection.

cf. Question in Cheng '14 "Flows and a...

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Transport equation:

$$\begin{cases} \dot{\Psi}_t = \frac{1}{2} J \Psi_{t*} \mathrm{Re}\left(e^{-2\gamma_t} f_t T\right) + \cdots \\ \dot{\gamma}_t = -e^{-2\gamma_t} \mathrm{Re}\left(f_t + \cdots\right) \end{cases}$$

Role of γ_t:

$$(\Psi_t)^* \Theta_t = e^{\gamma_t} \Theta$$

conformal factor

$$M_0 = S^3 \xrightarrow[\text{contact diffeo}]{\Psi_t} M_t$$

Θ

$$\Theta_t = i\partial\rho_t\big|_{M_t}$$

What about solving $(*_t)$?

Theorem (C.-Ebenfelt, in preparation):

Given $\dot\varphi \in \operatorname{image}(Z_1 Z_1)$, sufficiently small,
$\exists\ \varphi(t)$, f_t solving $(*_t)$ such that

(i) $\varphi(t) = t\dot\varphi + \psi(t)$ with $\psi(t) = O(t^2)$
and $\psi(t) \in \ker(Z_{\bar 1} Z_{\bar 1})$,

(ii) $\operatorname{Re} f_t > 0$.

Indication of proof:

$$(*_t) \iff (Z_1 + \varphi_1{}^{\bar 1} Z_{\bar 1})^2 f_t - \varphi_1{}^{\bar 1}{}_{,\bar 1}(Z_1 + \varphi_1{}^{\bar 1} Z_{\bar 1}) f_t - i\lambda \varphi_{11,0} f_t = \varphi_{11}'$$

decompose:

(Nash-Moser)

"sub-elliptic" $\leftarrow Z_1 Z_1 f_t + \cdots$

solve for f_t & plug in

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iT & Δ_b comparable in strip $\rightarrow$ "sub-parabolic" $\leftarrow \psi_{11}' + i\lambda\psi_{11,0} + \ldots$

p

"standard parabolic methods"

Thank You!