The inverse spectral problem for the Leray transform in two settings in $\mathbb{C}^2$

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The Leray transform

Let D be a $\mathbb{C}$-linearly convex domain in $\mathbb{C}^n$ (i.e. its complement is a union of hyperplanes) with a defining function ρ and a sufficiently smooth boundary S (certainly C^1). The Leray transform is defined for functions on S for which the following integral converges:

$$\mathbb{L}(f)(z) = \int_{\zeta \in S} f(\zeta) L(\zeta, z),$$

where the kernel is given by

$$L(\zeta, z) = \frac{1}{(2\pi i)^n} \frac{j^*(\partial\rho \wedge (\bar{\partial}\partial\rho)^{n-1})(\zeta)}{< \partial\rho, \zeta - z >^n},$$

$$< \partial\rho, \zeta - z > := \sum_{k=1}^{n} \frac{\partial\rho}{\partial z_k}(\zeta_k - z_k),$$

and j^* is the pullback of the inclusion $j : S \to \mathbb{C}^n$ acting on $(2n-1)$-forms.

Regularity

We can consider $\mathbb{L}$ as a self-map of $L^p(S, d\sigma)$, at least for some domains. L. Lanzani and E. Stein proved that the Leray transform is bounded for all $1 < p < \infty$ and all bounded, strongly $\mathbb{C}$-linearly convex domains (i.e. for all $z \in D$ and $w \in S$ the distance between z and the complex tangent hyperplane at w is at least some constant times $|z - w|$) with $C^{1,1}$-smooth boundaries (the partial derivatives of ρ are Lipschitz functions). These are sufficient conditions for L to be bounded, but they aren't necessary as demonstrated by the class of l_p balls for $n = 2$ and $p > 1$ given by

$$\rho(z_1, z_2) = |z_1|^p + |z_2|^p - 1.$$

More generally than l_p balls, D. Barrett and L. Lanzani proved that boundedness is achieved for a certain class of Reinhardt domains in $\mathbb{C}^2$ denoted by $\mathcal{R}$. The following setup is a survey of some of their work.

Setup

Put simply, each domain in $\mathcal{R}$ has an osculating l_p ball at every point, as ensured by C^2 smoothness away from the axes and a milder condition at points on the axes. In more detail, let's first parametrize the boundary curve $\gamma := S \cap \mathbb{R}^2_{\geq 0}$ by $\gamma(s) = (r_1(s), r_2(s))$ for $s \in [0, 1]$, where

$$s = -\frac{r_1 \phi'(r_1)}{\phi(r_1) - r_1 \phi'(r_1)}, \quad \rho(z_1, z_2) = |z_2| - \phi(|z_1|),$$

with suitable conditions on ϕ to ensure convexity and enough smoothness (C^2 and concavity away from the axes, C^1 and somewhat more otherwise).

Setup (continued)

It can be worked out for which conditions an osculating l_p ball (generally dilated in each variable separately) exists even at points on the axes, and how the exponent p depends on s. For our purposes, it suffices to note that we can generate all domains in $\mathcal{R}$ via

$$r_1(s) = b_1 \exp\left(-\int_s^1 \frac{dt}{tp(t)}\right), \quad r_2(s) = b_2 \exp\left(-\int_0^s \frac{dt}{(1-t)p(t)}\right), \tag{1}$$

where $b_1, b_2 > 0$ are constants and $p : [0,1] \to (1, \infty)$ is continuous and satisfies integrability on $[0,1]$ for both $\frac{1/p(s) - 1/p(0)}{s}$ and $\frac{1/p(s) - 1/p(1)}{1-s}$. We should keep this in mind for later, but for now the key point is the aforementioned theorem about the Leray transform being bounded for domains in $\mathcal{R}$. But for which measure on S?

Spectrum

Parametrizing S by $(r_1(s)e^{i\theta_1}, r_2(s)e^{i\theta_2})$, it turns out that the most natural rotation-invariant measure is given by

$$d\mu_0 = \frac{1}{4\pi^2} ds \wedge d\theta_1 \wedge d\theta_2.$$

It is not equivalent to the surface measure, but it has the advantage of offering boundedness for as big a class of domains as possible.

Using Fourier decomposition (separately for each variable), we can write $\mathbb{L} = \bigoplus_{n,m\in\mathbb{Z}} \mathbb{L}_{n,m}$, where each $\mathbb{L}_{n,m}$ acts on functions of the form $g(s)e^{i(n\theta_1+m\theta_2)}$. It can be shown that

$$(\mathbb{L}_{n,m})^*_{\mu_0}\mathbb{L}_{n,m}f = < f, \tau_{n,m} > \kappa_{n,m},$$

where $\kappa_{n,m} = \frac{(n+m+1)!}{n!m!}\left(\frac{s}{r_1(s)}\right)^n \left(\frac{1-s}{r_2(s)}\right)^m e^{i(n\theta_1+m\theta_2)}$.

Eigenvalues

The non-zero eigenvalues of $\mathbb{L}^*_{\mu_0}\mathbb{L}$ (which correspond to $\kappa_{n,m}$) are given by

$$\lambda_{n,m} = \left(\frac{(n+m+1)!}{n!m!}\right)^2 \int_0^1 r_1^{2n}(s) r_2^{2m}(s)\, ds \times \int_0^1 \left(\frac{s}{r_1(s)}\right)^{2n} \left(\frac{1-s}{r_2(s)}\right)^{2m} ds \quad (2)$$

for $n, m \in \mathbb{Z}_{\geq 0}$. Note that we can get a basis for $L^2(S, \mu_0)$ by adjoining $\{\kappa_{n,m}\}_{\mathbb{Z}^2}$ to some basis for the (infinite-dimensional) kernel.

Limit values

There are three types of limit values:

1. Limit values that correspond to $\min\{n, m\} \to \infty$ with $\frac{n}{m} \to x \in [0, \infty]$. These are given by the function

$$\phi_D : [0, \infty] \to [1, \infty), \quad \phi_D(x) = \frac{\sqrt{p(\frac{x}{1+x})p^*(\frac{x}{1+x})}}{2},$$

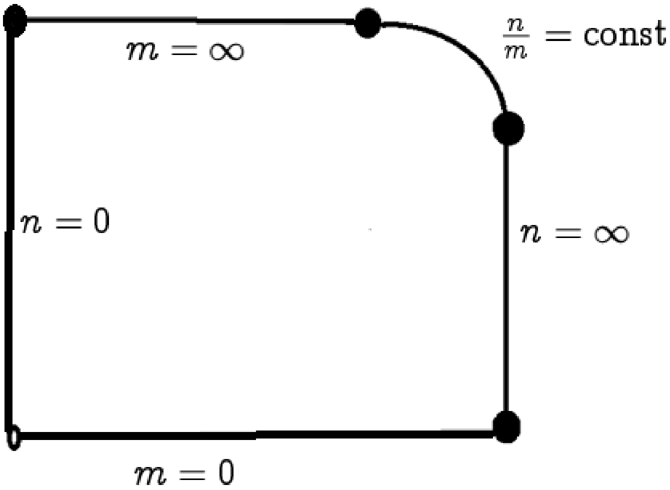
where $p^* = \frac{p}{p-1}$ is the Hölder conjugate of p.

2. Limit values that correspond to horizontal lines $m = m_0$ with $n \to \infty$. These are given by

$$\frac{\Gamma(\frac{2m_0}{p_2}+1)\Gamma(\frac{2m_0}{p_2^*}+1)}{\Gamma^2(m_0+1)(\frac{2}{p_2})^{\frac{2m_0}{p_2}+1}(\frac{2}{p_2^*})^{\frac{2m_0}{p_2^*}+1}}, \quad \text{where} \quad p_2 = p(1).$$

3. Same as above for vertical lines. Just swap m_0 with n_0 and p_2 with $p_1 = p(0)$.

Visualization



Leray spectrum for rigid Hartogs domains

We move onto the second setting explored in this talk.

Definition

We define a class of domains in $\mathbb{C}^2$ and a useful subclass.

- A *rigid Hartogs domain* in $\mathbb{C}^2$ is given by

$$D = \{(z_1, z_2) \in \mathbb{C}^2 : \mathrm{Im}(z_2) > f(|z_1|)\},$$

  where $f : [0, \infty) \to [0, \infty)$. Such a domain is rotationally invariant in z_1 and translation invariant (by real scalars) in z_2.

- We denote by $\widetilde{\mathcal{H}}$ the subclass of rigid Hartogs domains where $f \in C^1[0, \infty) \cap C^2(0, \infty)$ satisfies

$$f'(0) = 0, \quad \forall x > 0 \quad f''(x) > 0.$$

Let D be a rigid Hartogs domain D corresponding to $f(r)$. We consider a family of measures that are invariant under the natural automorphisms (i.e. rotations in z_1 and real translations in z_2), given by (using $\zeta = (r_\zeta e^{i\theta_\zeta}, s_\zeta + if(r_\zeta))$

$$d\sigma_{d,g}(\zeta) = ds_\zeta \wedge d\widetilde{\sigma}_{d,g}(r_\zeta) \wedge d\theta_\zeta,$$

where $d \in \mathbb{R}$, $g : [0, \infty] \to (0, \infty)$ is continuous and

$$d\widetilde{\sigma}_{d,g}(r_\zeta) = ((r_\zeta f'(r_\zeta)')^d g(r_\zeta) dr_\zeta.$$

The special case $\sigma := \sigma_{1,1}$ is in fact the Leray-Levi measure, which we will call the *pairing measure*. We will highlight it for the inverse spectral problem.

Theorem

For a domain $D \in \widetilde{\mathcal{H}}$ with boundary S endowed with $\sigma_{d,g}$ as above, we define the adjoint operator $\mathbb{L}^*_{\sigma_{d,g}}$ relative to the subspace $L^2(S, \sigma_{d,g})$. Then there exists a decomposition $\mathbb{L} = \bigoplus_{k=-\infty}^{\infty} \mathbb{L}_k$ such that the $\mathbb{L}^*_{k,\sigma_{d,g}} \mathbb{L}_k$ are unitarily equivalent to rank 1 projections $P_{k,d,g}$ on $L^2((-\infty, 0) \times (0, \infty))$ corresponding to some functions $v_k(\xi, r)$. More precisely

$$(P_{k,d,g} w)(\xi, r) = \widetilde{C}_{d,g}(\xi, k) < w(\xi, \cdot), \kappa_k(\xi, \cdot) >_\sigma v_k(\xi, r),$$

where

$$\widetilde{C}_{d,g}(k, \xi) = \eta_k^2(\xi) \left\|\tau_k(\xi, \cdot)\right\|^2_{\widetilde{\sigma}_{d,g}},$$
$$\eta_k(\xi) = \frac{(-2\pi\xi)^{k+1}}{k!},$$
$$\tau_k(\xi, r) = r^k e^{2\pi\xi f(r)},$$
$$\kappa_k(\xi, r) = (f'(r))^k e^{2\pi\xi(rf'(r) - f(r))}$$

for $\xi < 0$ and integer $k \geq 0$.

The symbol function

Corollary

The image of the function

$$C_{d,g}(\xi,k) \coloneqq \eta_k^2(\xi)\,\|\tau_k(\xi,\cdot)\|^2_{\widetilde{\sigma}_{d,g}}\,\|\kappa_k(\xi,\cdot)\|^2_{\widetilde{\sigma}_{2-d,1/g}}$$
$$=\eta_k^2(\xi)\int_0^\infty r^{2k} e^{4\pi\xi f(r)}((rf'(r))')^d g(r)\,dr$$
$$\times\int_0^\infty (f'(r))^{2k} e^{4\pi\xi(rf'(r)-f(r))}((rf'(r))')^{2-d}\frac{1}{g(r)}\,dr$$

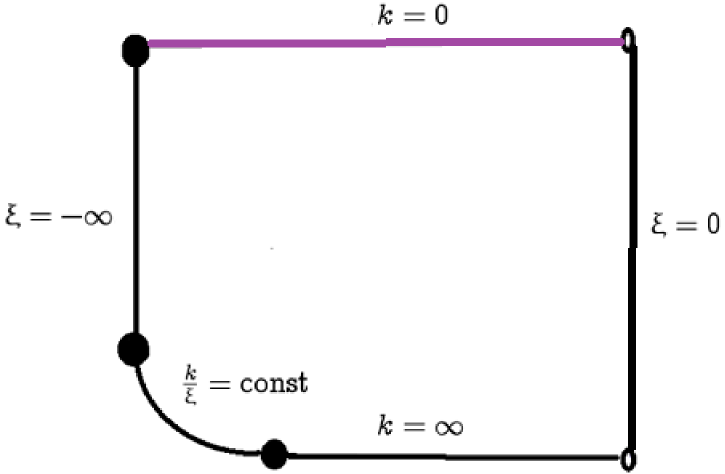
is dense in the spectrum of $\mathbb{L}^*_{\sigma_{d,g}}\mathbb{L}$. *The multiplier* $C_{d,g}(\xi,k)$ *is called the symbol function associated with* D *and* $\sigma_{d,g}$.

Boundedness of $\mathbb{L}$

We want to compute all limit values of the symbol function $C_{d,g}(\xi, k)$. Consider a sequence $((\xi_j, k_j))_{j=0}^{\infty}$, where $\xi_j < 0, k_j \in \mathbb{Z}_{\geq 0}$ for all $j \in \mathbb{Z}_{\geq 0}$. Much like the Reinhardt setting, by potentially passing to a sub-sequence (without changing the subscript) we have four options other than (uninteresting due to continuity) limits in $(-\infty, 0) \times \mathbb{Z}_{\geq 0}$:

1. $\lim\limits_{j \to \infty} \frac{k_j}{\xi_j} \in [-\infty, 0], \quad k_j, |\xi_j| \to \infty.$
2. $\xi_j \to -\infty, \quad k_j = const.$
3. $\xi_j \to 0, \quad k_j = const.$
4. $\xi_j \to \widetilde{\xi} \in [0, \infty), k_j \to \infty.$

Visualization



More setup

We'd like to have an analogue of the function $p(s)$, which recovers convex Reinhardt domains up to dilations. The most intuitive approach is to use translated and dilated versions of M_γ (corresponding to $f(r) = a + br^\gamma$) as model domains, osculating to the second order. This will give rise to a function $\gamma : (0, \infty) \to (1, \infty)$, which we call an osculation function for a domain $D \in \widetilde{\mathcal{H}}$, and is given by

$$\gamma(r) = 1 + \frac{rf''(r)}{f'(r)}$$

Definition

It is sometimes useful to work with the parameter $u = rf'(r)$. Since $r \mapsto rf'(r)$ is increasing, we can write

$$r = r(u) = |z_1|, \quad y(u) = f(r(u)) = \mathrm{Im}(z_2)$$

for the parametrization of the domain (not forgetting $\theta, s = \mathrm{Re}(z_2)$ as the other coordinates).

Theorem

Assume $d \in \mathbb{R}$, $g : (0, \infty) \to (0, \infty)$ *is continuous and* $\lim_{j \to \infty} \frac{k_j}{\xi_j} = -2\pi\widetilde{u}$, *where* $\widetilde{u} \in (0, \infty)$, $|\xi_j|, k_j \xrightarrow[j \to \infty]{} \infty$. *Then for any* $D \in \widetilde{\mathcal{H}}$, *we have*

$$\lim_{j \to \infty} C_{d,g}(\xi_j, k_j) = \Phi(\widetilde{u}) := \frac{\sqrt{\gamma(r(\widetilde{u}))\gamma^*(r(\widetilde{u}))}}{2} = \frac{\gamma(r(\widetilde{u}))}{2\sqrt{\gamma(r(\widetilde{u})) - 1}}.$$

A variant of the Laplace method

Theorem

Let $(a_k(x))_{k\in\mathbb{N}}$ and $\widetilde{h}(x)$ be measurable real-valued functions on $\mathbb{R}$ such that for all $k \in \mathbb{N}$ the function $a_k(x)$ has a unique maximum point $x_k \in \mathbb{R}$ and appropriate regularity and growth conditions are satisfied, then

$$\lim_{k\to\infty} \sqrt{k}\, e^{-k a_k(\widetilde{x})} \int_{\mathbb{R}} e^{k a_k(x)} \widetilde{h}(x)\, dx = \sqrt{\frac{2\pi}{|A|}}\, \widetilde{h}(\widetilde{x}),$$

where $\widetilde{x} = \lim_{k\to\infty} x_k$ and $A = \lim_{k\to\infty} a_k''(\widetilde{x})$.

The subclass $\mathcal{H}$

Definition

Let $\mathcal{H}$ denote the subclass of domains $D \in \widetilde{\mathcal{H}}$ for which $\gamma(r)$ has a continuous extension to $[0, \infty]$ and the integrals

$$\int_0^1 \frac{\gamma(t) - \gamma_0}{t}\,dt, \quad \int_1^\infty \frac{\gamma(t) - \gamma_\infty}{t}\,dt$$

converge.

Part of this definition is natural in light of the previous theorem ($\gamma(r)$ must be bounded away from $1, \infty$ for $\mathbb{L}$ to be bounded). The existence of γ_0, γ_∞ and the integral conditions are convenient for the computation of the other limit values.

Theorem

Define the function

$$J_{\gamma,d}(k) = \frac{\gamma^{\frac{2k}{\gamma}+\frac{d-1}{\gamma^*}+1}(\gamma^*)^{\frac{2k+1-d}{\gamma^*}+1}}{k!^2 2^{2k+2}}\Gamma(\frac{2k}{\gamma}+\frac{d-1}{\gamma^*}+1)\Gamma(\frac{2k+1-d}{\gamma^*}+1).$$

Then for $D \in \mathcal{H}$ *and the measure* $\sigma_{d,g}$*, where* $d \in \mathbb{R}$ *and* $g : (0,\infty) \to (0,\infty)$ *has a positive continuous extension to* $[0,\infty]$*, we get the following asymptotic results:*

1. $C_{d,g}(\xi_j, k_j) \to J_{\gamma_0,v}(k) \quad$ *as* $\quad \xi_j \to -\infty, \quad k_j = const.$
2. $C_{d,g}(\xi_j, k_j) \to J_{\gamma_\infty,v}(k) \quad$ *as* $\quad \xi_j \to 0, \quad k_j = const.$
3. $C_{d,g}(\xi, k_j) \to \frac{\gamma_\infty}{2\sqrt{\gamma_\infty - 1}}(\gamma_\infty - 1)^{\frac{d-1}{\gamma_\infty^*}} \quad$ *as* $\xi_j \to \widetilde{\xi} \in [0,\infty), \quad k_j \to \infty.$

Corollary

Let $D \in \mathcal{H}$ have boundary S. Then for any continuous $g : [0, \infty] \to (0, \infty)$, $\mathbb{L}$ is bounded with respect to $L^2(S, \sigma_{d,g})$ if and only if

$$1 - \min\{\gamma_0^*, \gamma_\infty^*\} < v < 1 + \min\{\gamma_0^*, \gamma_\infty^*\}.$$

This holds automatically for any $v \in [0, 2]$.

Comparing the two settings

	Reinhardt	Rigid Hartogs
Models	l_p balls	M_γ domains
Osculation	$p(s)$	$\gamma_c(u)$
S Limits	$\frac{\sqrt{p(s)p^*(s)}}{2}$	$\frac{\sqrt{\gamma_c(u)\gamma_c^*(u)}}{2}$
Recovery	$r_1(s) = b_1 \exp\left(\int_0^s \frac{dt}{tp(t)}\right)$	$r(u) = c_1 \int_0^u \frac{dt}{t\gamma(t)}$
	$r_2(s) = b_2 \exp\left(\int_s^1 \frac{dt}{(1-t)p(t)}\right)$	$y(u) = \int_0^u \frac{dt}{\gamma(t)} + c_2$
Duality	$p^*(s)$	$\gamma_c^*(u)$
Measure	μ_0	σ

Here $\gamma_c(u) := \gamma(r(u))$.

Can you hear the shape of a sufficiently smooth convex Reinhardt domain?

For the Laplacian operator with Dirichlet boundary conditions, the spectrum is not generally known to classify the domain unless we restrict the problem to a rather small class (for example, see Zelditch's paper cited in the references).

In our case, the group on $\widetilde{\mathcal{R}}$ for which the Leray kernel transforms nicely, is generated by all dilations and the reflection $R(z_1, z_2) = (z_2, z_1)$. Also, passing to the polar (dual) domain preserves the spectrum when the measure is μ_0, which is easy to check as the integrals in (2) are swapped under duality.

Unlike the Laplacian, the eigenvalues are not ordered, but rather given as function values on a lattice. Sticking to this marking means remembering the toroidal frequencies, and gives us a better chance of recovering the domain from its marked spectrum.

The slope approach

As mentioned before, by considering sequences of lattice points with a convergent slope $u \in [0, \infty]$, we arrive at the function

$$\phi_D : [0, \infty] \to [1, \infty), \quad \phi_D(x) = \frac{\sqrt{p(\frac{x}{1+x})p^*(\frac{x}{1+x})}}{2}.$$

Note that $p \mapsto pp^* = \dfrac{p^2}{p-1}$ is a covering map of degree 2 with $p \mapsto p^*$ as the deck transformation. Thus, it is clear that using ϕ_D, the marked spectrum recovers $p(s)$ up to $p \mapsto p^*$ for all $s \in [0, 1]$. Does this mean that this rather simple recovery method works up to dilations and the duality map (reflection entails reflecting the marking, giving $p(1-s)$), as one would hope based on the generating formula for (1)?

The slope approach (continued)

No, because in general the equation $p(s) = 2$ has $k \geq 0$ solutions in $(0, 1)$, and these solutions partition $[0, 1]$. We can apply $p \mapsto p^*$ on any subinterval independently of the other ones, giving us 2^{k+1} options for p given the same ϕ_D. They all satisfy continuity and the boundary conditions, although if we impose a differentiability condition, that could help (but even if we don't mind the loss of generality, there are still pesky exceptions involving critical points). If $k = 0$, this ambiguity doesn't arise at all. For finite $k > 0$ (and the infinite case under a natural constraint), it turns out that if you basically add another term to the asymptotic expansion, it will reveal the difference between the two domains up to dilations and duality. There's something special about the case $k = 1$ in that a single eigenvalue (of a certain kind) detects the difference.

Some geometry

By the way, whether or not $p(s) > 2$ or $p(s) < 2$ can be described geometrically as follows: Consider the osculating (up to first order) dilated l_2 ball in $\mathbb{C}^2$ of the form $\frac{|z_1|^2}{a^2} + \frac{|z_2|^2}{b^2} = 1$ at the point $(r_1(s), r_2(s))$. If $p(s) > 2$, this dilated l_2 ball is locally inside the Reinhardt domain (globally if the inequality holds for all $s \in [0, 1]$). If the inequality is reversed, then the dilated l_2 ball contains the Reinhardt domain locally (globally). We call the former case Reinhardt convexity and the latter Reinhardt concavity.

Definition

If we have $p(s_0) = 2$, we call s_0 a **Reinhardt vertex**. If $p \equiv 2$ on a subinterval, we only count the endpoints as Reinhardt vertices. For a domain $D \in \widetilde{\mathcal{R}}$, we denote its set of Reinhardt vertices by V_D.

The 1-vertex case

Note that V_D is an invariant of essentially isospectral domains in $\widetilde{\mathcal{R}}$, i.e. domains with the same pp^* function, since the vertices correspond to solutions of $pp^* = 4$.

Theorem

Let $\mathcal{R}_1$ denote the collection of all domains $D \in \widetilde{\mathcal{R}}$ such that card$(V_D \setminus \{0,1\}) = 1$. Then the marked spectrum map $\psi : \mathcal{R}_1 \to l_\infty$ given by $\psi(D) = \{\lambda_{n,m}\}_{\mathbb{Z}^2_{\geq 0}}$ is injective modulo dilations and duality.

Remark

For essentially isospectral domains, we can distinguish between them (modulo dilations and duality) using just a single eigenvalue of the form $\lambda_{n,0}$ or $\lambda_{0,m}$, where $n, m \in \mathbb{N}$. If the unique Reinhardt vertex in $(0,1)$ happens to be rational, then $\lambda_{n,m}$ also suffices for any $n, m \in \mathbb{N}$ such that $\dfrac{n}{n+m} = a$.

Proof (Step 1)

Let $D, \widetilde{D} \in \mathcal{R}_1$ have the same marked spectrum, and let $p(s)$, $\widetilde{p}(s)$ be the respective osculation functions. We know that both domains share the same Reinhardt vertex $a \in (0, 1)$. Thanks to duality, we may assume $\widetilde{p}(s) = p(s) \geq 2$ on $[0, a]$ (we can arrange for $p(s) \geq 2$ and $\widetilde{p}(s) \geq 2$ on $[0, a]$ separately). Then either $\widetilde{p}(s) = p(s)$ on $(a, 1]$ and we are done, or $\widetilde{p}(s) = p^*(s)$ on $(a, 1]$. We assume the latter.

Step 2

We observe that for $s \in [0, a]$ we have (using formulas (1) for $r_1(s), r_2(s)$)

$$\widetilde{r_1}(s) = \alpha r_1(s), \quad \widetilde{r_2}(s) = r_2(s),$$

for the constant $\alpha = \exp(\int_a^1 (\frac{1}{tp(t)} - \frac{1}{tp^*(t)})dt)$. On (a,1] we have

$$\widetilde{r_1}(s) = \frac{s}{r_1(s)}, \quad \widetilde{r_2}(s) = \beta \frac{1-s}{r_2(s)},$$

for the constant $\beta = \exp(\int_0^a (\frac{1}{(1-t)p^*(t)} - \frac{1}{(1-t)p(t)})dt)$.

Step 3

Now we want to show that $\forall n \in \mathbb{N} \quad \widetilde{\lambda}_{n,0} - \lambda_{n,0} \neq 0$. The calculation for $\lambda_{0,m} - \lambda_{0,m}$ is similar, but we can simply consider the reflected domains $R(D), R(\widetilde{D})$ which swap the indices. Using (2), we get that $\frac{1}{(n+1)^2}(\widetilde{\lambda}_{n,0} - \lambda_{n,0})$ is given by

$$\left(\alpha^{2n}\int_0^a r_1^{2n}(s)\,ds + \int_a^1 (\frac{s}{r_1(s)})^{2n}\,ds\right) \times \left(\alpha^{-2n}\int_0^a (\frac{s}{r_1(s)})^{2n}\,ds + \int_a^1 r_1^{2n}(s)\,ds\right) - \left(\int_0^a r_1^{2n}(s)\,ds + \int_a^1 r_1^{2n}(s)\,ds\right) \times \left(\int_0^a (\frac{s}{r_1(s)})^{2n}\,ds + \int_a^1 (\frac{s}{r_1(s)})^{2n}\,ds\right) = \left(\alpha^{2n}\int_0^a r_1^{2n}(s)\,ds - \int_0^a (\frac{s}{r_1(s)})^{2n}\,ds\right) \times \left(\int_a^1 r_1^{2n}(s)\,ds - \alpha^{-2n}\int_a^1 (\frac{s}{r_1(s)})^{2n}\,ds\right) \quad (3)$$

Step 4

If we can show that

$$\forall s \neq a \quad \alpha r_1(s) \neq \frac{s}{r_1(s)},$$

then it will follow that both factors in (3) are non-zero and thus $\forall n \in \mathbb{N} \quad \widetilde{\lambda}_{n,0} \neq \lambda_{n,0}$. Taking the logarithmic derivative of the quotient of both sides, we get

$$\frac{d}{ds} \log \frac{\alpha r_1(s)}{\frac{s}{r_1(s)}} = \frac{2r_1'(s)}{r_1(s)} - \frac{1}{s} = \frac{1}{s}(\frac{2}{p(s)} - 1).$$

This means that the quotient is decreasing on $[0, a]$ and is either monotone or constant on $[a, 1]$ (since $p(s) \neq 2$ unless $p(s) \equiv 2$ on (a,1) by assumption, and the latter implies $\widetilde{D} = D$). Note that $\alpha r_1(a) = \frac{a}{r_1(a)}$ by continuity of $\widetilde{r_1}(s)$ at $s = a$. □

The general case

Theorem

Let $\mathcal{R}_m$ denote the collection of all domains $D \in \widetilde{\mathcal{R}}$ such that V_D can be represented as a monotone (possibly finite) one-sided sequence $\{v_n\}$. Then the marked spectrum map $\psi : \mathcal{R}_m \to l_\infty$ given by $\psi(D) = \{\lambda_{n,m}\}_{\mathbb{Z}^2_{\geq 0}}$ is injective modulo dilations and duality.

Remark

Note that if we compare a domain from $\mathcal{R}_m$ with a domain from $\widetilde{\mathcal{R}} \setminus \mathcal{R}_m$, they certainly don't have the same marked spectrum since they have different sets of Reinhardt vertices. We need to restrict to $\mathcal{R}_m$ for this particular proof, but that is not to say that the condition is necessary for injectivity (an open question).

Proof (Step 1)

Let $D, \widetilde{D} \in \mathcal{R}_m$ have the same marked spectrum, and let $p(s)$, $\widetilde{p}(s)$ be the respective osculation functions. We know that $pp^* = \widetilde{p}\widetilde{p}^*$ and in particular, both domains share the same Reinhardt vertices. Without loss of generality, there are at least two Reinhardt vertices in $(0, 1)$ (due to the previous theorem), the corresponding sequence is increasing (otherwise, consider the reflected domains) and for small s we have $\widetilde{p}(s) = p(s)$ (otherwise, we can replace D by D^*). Now let $[a, b]$ be the closest interval to 0 such that a, b are Reinhardt vertices (for both domains) and $\widetilde{p}(s) = p^*(s) \neq 2$ on (a, b) (if it doesn't exist, then $\widetilde{D} = D$ up to dilations, and we are done).

Step 2

We recall from Barrett and Lanzani's 2009 paper that we have

$$\int_0^1 r_1^{2n}(s) r_2^{2m}(s)\,ds \sim \sqrt{\frac{\pi nm}{(n+m)^3}}\, p(s_x) r_1^{2n}(s_x) r_2^{2m}(s_x),$$

$$\int_0^1 \left(\frac{s}{r_1(s)}\right)^{2n} \left(\frac{1-s}{r_2(s)}\right)^{2m} ds \sim \sqrt{\frac{\pi nm}{(n+m)^3}}\, p^*(s_x) \left(\frac{s_x}{r_1(s_x)}\right)^{2n} \left(\frac{1-s_x}{r_2(s_x)}\right)^{2m},$$

as $n, m \to \infty$ with $\frac{n}{m} \to x \in (0, \infty)$, where $s_x = \frac{x}{1+x}$ is the limit point of the (unique) maxima for these integrands. The above estimates hold even if we replace the integral bounds $0, 1$ by any c, d such that $\frac{x}{1+x} \in (c, d)$, due to the Laplace-like method involved.

Step 3

We observe that for $s \in [0, a]$ we have by assumption $\widetilde{p}(s) = p(s)$, yielding (using formulas (1) for $r_1(s), r_2(s)$)

$$\widetilde{r_1}(s) = \alpha r_1(s), \quad \widetilde{r_2}(s) = r_2(s),$$

for the constant $\alpha = \exp(\int_a^1 (\frac{1}{tp(t)} - \frac{1}{t\widetilde{p}(t)}) dt)$. On [a,b] we have

$$\widetilde{r_1}(s) = \beta \frac{s}{r_1(s)}, \quad \widetilde{r_2}(s) = \gamma \frac{1-s}{r_2(s)},$$

for similar constants $\beta, \gamma > 0$. We write

$$f_{n,m}(s) = r_1^{2n}(s) r_2^{2m}(s), \quad g_{n,m}(s) = (\frac{s}{r_1(s)})^{2n} (\frac{1-s}{r_2(s)})^{2m}.$$

We define $\widetilde{f}_{n,m}, \widetilde{g}_{n,m}$ similarly.

Step 4

Now we want to improve the estimate for $\widetilde{\lambda}_{n,m}$. Splitting each integral in the product from (2) into three pieces and using the previous slide, we get

$$\left(\frac{n!m!}{(n+m+1)!}\right)^2 \widetilde{\lambda}_{n,m} = \left(\int_0^1 \widetilde{f}_{n,m}\,ds\right)\left(\int_0^1 \widetilde{g}_{n,m}\,ds\right)$$

$$= \left(\int_0^a f_{n,m}\,ds\right)\left(\int_0^a g_{n,m}\,ds\right) + \left(\frac{\alpha}{\beta}\right)^{2n}\gamma^{-2m}\left(\int_0^a f_{n,m}\,ds\right)\left(\int_a^b f_{n,m}\,ds\right)$$

$$+ \left(\frac{\alpha}{\beta}\right)^{-2n}\gamma^{2m}\left(\int_0^a g_{n,m}\,ds\right)\left(\int_a^b g_{n,m}\,ds\right) + \text{error}, \quad (4)$$

where the error terms are products of integrals that are negligible (in the sense of little o notation) compared to those listed above. This is due to Watson's lemma, as follows.

Watson's Lemma

Let f be continuous on $[c, d]$ such that it attains a unique global minimum at c and $f'(0) > 0$ exists. If h is bounded, Lebesgue-measurable on $[c, d]$ and continuous at 0, then

$$\int_c^d e^{-mf(x)} h(x)\,dx \sim \frac{h(c)e^{-mf(c)}}{mf'(c)} \quad \text{as} \quad m \to \infty.$$

Application (Step 5)

Let $x > 0$ be a rational number (to simplify the argument) such that $\dfrac{u}{1+u} \in (0, a)$. We apply the lemma to

$$f_x(s) = -2x \log r_1(s) - 2 \log r_2(s), \quad h \equiv 1, \quad c = a, \quad d = b.$$

Then letting $n = mx, m \to \infty$, we get

$$\int_a^b f_{n,m}(s)\,ds \sim \frac{r_1(a)^{2n} r_2(a)^{2m} a(1-a)p(a)}{2m(a-(1-a)x)}.$$

Step 5 (continued)

In fact, we're considering a subsequence $\{m_k\}_{k\in\mathbb{N}}$ for which $m_k x \in \mathbb{N}$, but we omit the subscript. For the record, $f(s)$ is increasing on $[a,1]$, since we have for all $u \in [0,a]$

$$f'(s) = -2\frac{r_1'(s)}{r_1(s)}(x - \frac{s}{1-s}) > 0.$$

Similarly, using $g_u(s) = -2x\log(\dfrac{s}{r_1(s)}) - 2\log(\dfrac{1-s}{r_2(s)})$ we get

$$\int_a^b g_{n,m}(s)\,ds \sim \frac{(\frac{a}{r_1(a)})^{2n}(\frac{1-a}{r_2(a)})^{2m} a(1-a)p^*(a)}{2m(a-(1-a)x)}.$$

If you replace the bounds a, b by $b, 1$, the new estimates are relatively negligible since $-f, -g$ are decreasing on $[a,1]$.

Step 6

Going back to (4), writing a similar computation for $\lambda_{n,m}$ and then taking the difference, we get

$$\left(\frac{n!m!}{(n+m+1)!}\right)^2\left(\widetilde{\lambda}_{n,m}-\lambda_{n,m}\right)=\left((\frac{\alpha}{\beta})^{2n}\gamma^{-2m}\int_0^a f_{n,m}(s)\,ds\right.$$
$$\left.-\int_0^a g_{n,m}(s)\,ds\right)\times\left(\int_a^b f_{n,m}(s)\,ds-(\frac{\beta}{\alpha})^{2n}\gamma^{2m}\int_a^b g_{n,m}(s)\,ds\right)+error. \tag{5}$$

We want to show that the product is asymptotically larger than the error, which is

$$O(\frac{1}{m^{1.5}}\max\{(\frac{\beta}{\alpha})^{2n}\gamma^{2m}g_{n,m}(s_x)g_{n,m}(b),\quad f_{n,m}(s_x)g_{n,m}(b),$$
$$(\frac{\alpha}{\beta})^{2n}\gamma^{-2m}f_{n,m}(s_x)f_{n,m}(b),\quad g_{n,m}(s_x)f_{n,m}(b)\}). \tag{6}$$

Step 6 (first factor)

Note that

$$(\frac{\alpha}{\beta})^{2n}\gamma^{-2m}\int_0^a f_{n,m}(s)\,ds-\int_0^a g_{n,m}(s)\,ds$$
$$\sim\sqrt{\frac{\pi nm}{(n+m)^3}}\Bigg(\sqrt{p(s_x)}(\frac{\alpha}{\beta}r_1(s_x))^{2n}(\gamma^{-1}r_2(s_x))^{2m}-$$
$$\sqrt{p^*(s_x)}(\frac{s_x}{r_1(s_x)})^{2n}(\frac{1-s_x}{r_2(s_x)})^{2m}\Bigg). \quad (7)$$

Since $p(s)\neq 2$ on $(0,a)$, one of these two exponential terms dominates. If $p(s)>2$, then the latter term dominates (see the next slide). Even without that, there is no cancellation since $p(s_x)\neq 2$.

Step 6 (second factor)

As seen in step 5, if we apply Watson's lemma to each integral separately and take the difference, the estimates cancel out since $p(a) = 2 = p^*(a)$. If we replace the lower bound a by any $t \in (a, b)$, we get an estimate we can work with, but for that we need to know that on (a, b) we have

$$(\frac{\alpha}{\beta})^{2n}\gamma^{-2m} f_{n,m}(s) \neq g_{n,m}(s),$$

to avoid any cancellation. Note that we get equality for $s = a$ and if we take the logarithmic derivative of the quotient, we get

$$2m(\frac{2}{p(s)} - 1)(\frac{x}{s} - \frac{1}{1-s}) \neq 0.$$

This is because $s_x = \dfrac{x}{1+x} \notin (a, b)$ by assumption.

Wrapping up

Plugging (7) into (5) and applying Watson's lemma to the second factor (for $[t, b]$), we get for $n = mx, \quad m \to \infty$

$$\begin{aligned} &\left(\frac{n!m!}{(n+m+1)!}\right)^2 |\widetilde{\lambda}_{n,m} - \lambda_{n,m}| \\ \gtrsim \frac{1}{m^{1.5}} \max\{ &(\frac{\beta}{\alpha})^{2n}\gamma^{2m} g_{n,m}(s_x)g_{n,m}(t), \quad f_{n,m}(s_x)g_{n,m}(t), \\ &(\frac{\alpha}{\beta})^{2n}\gamma^{-2m} f_{n,m}(s_x)f_{n,m}(t), \quad g_{n,m}(s_x)f_{n,m}(t)\}. \end{aligned}$$

The error resulting from (6) is negligible as the functions

$$-f_x = \frac{1}{m}\log(f_{n,m}), \quad -g_x = \frac{1}{m}\log(g_{n,m})$$

are both decreasing on $[s_x, 1]$. It follows that the eigenvalues differ for sufficiently large m (possibly depending on x). □

Corollary

If two domains $D, \widetilde{D} \in \mathcal{R}_m$ have the same marked spectrum outside a finite set, i.e. $\exists N \in \mathbb{N} \quad \forall n, m > N \quad \widetilde{\lambda}_{n,m} = \lambda_{n,m}$, then they are the same up to dilations and duality. This implies that the image of $\psi : \mathcal{R}_m \to l_\infty$ is not invariant under any finite permutation but the identity, and in fact for any such permutation P and any $D \in \mathcal{R}_m$ we have $P(\psi(D)) \notin \mathrm{Im}(\psi)$.

Proof.

The previous proof still applies as we only need asymptotics for it. If a finite permutation of some spectrum (for a domain in $\mathcal{R}_m$) corresponds to another domain in $\mathcal{R}_m$, we immediately see that this permutation is trivial. $\square$

The rigid Hartogs setting

We consider rigid Hartogs domain in $\widetilde{\mathcal{H}}$. The osculation function γ_c plays a pivotal role here.

Definition

If we have $\gamma_c(u_0) = 2$, we call u_0 a **Hartogs vertex**. If $\gamma_c \equiv 2$ on a subinterval, we only count the endpoints as Hartogs vertices. For a domain $D \in \mathcal{H}$, we denote its set of Hartogs vertices by V_D (not to be confused with the Reinhardt notation, based on the context).

Note that V_D is an invariant of essentially isospectral domains in $\widetilde{\mathcal{H}}$, i.e. domains with the same Φ_D function, since the vertices correspond to solutions of $\gamma\gamma^* = 4$.

The 1-vertex case

Theorem

Let $\mathcal{H}_1$ denote the collection of all domains $D \in \widetilde{\mathcal{H}}$ such that card$(V_D \setminus \{0, \infty\}) = 1$. Let $\mathcal{S}$ denote the space of sequences (indexed by $k \in \mathbb{Z}_{\geq 0}$) of continuous functions on $(-\infty, 0)$. Then the semi-marked spectrum map $\Omega : \mathcal{H}_1 \to \mathcal{S}$ given by $\Omega(D) = C(\xi, k)$ is injective modulo dilations in z_1, translations in z_2 and duality.

The general case

Theorem

Let $\mathcal{H}_m$ denote the collection of all domains $D \in \widetilde{\mathcal{H}}$ such that V_D can be represented as a monotone (possibly finite) one-sided sequence (v_n). As before, let $\mathcal{S}$ denote the space of sequences (indexed by $k \in \mathbb{Z}_{\geq 0}$) of continuous functions on $(-\infty, 0)$. Then the semi-marked spectrum map $\Omega : \mathcal{H}_m \to \mathcal{S}$ given by $\Omega(D) = C(\xi, k)$ is injective modulo dilations in z_1, translations in z_2 and duality.

Corollary

If two domains $D, \widetilde{D} \in \mathcal{H}_m$ *have symbol functions that coincide outside a compact set, i.e.* $\exists K \in \mathbb{N} \quad \forall k > K, \xi < -K \quad C(\xi, k) = \widetilde{C}(\xi, k)$, *then they are the same up to dilations in* z_1, *translations in* z_2 *and duality.*

Proof.

The proof of the above theorem only relies on asymptotics as $\xi, k \to \infty$. □

Corollary

Let $D \in H_m$. If the symbol function $C(\xi, k)$ is independent of ξ, then D is an M_γ domain up to coordinate dilations and translations in z_2. Otherwise, the continuous spectrum of $\mathbb{L}_\sigma^* \mathbb{L}$ is non-empty.

Proof.

For such a domain, the symbol function must coincide with the symbol function of M_{γ_0} (or equivalently M_{γ_∞}), since the boundary values of a symbol function are determined by γ_0 (at ∞) and γ_∞ (at 0). The the conclusion follows from the previous theorem. □

Open questions (the main ones)

1. In the Reinhardt setting, what happens when we remove the marking? Are there, at least for some domains, transformations that permute the eigenvalues other than reflection?
2. In the rigid Hartogs setting, what happens when two domains have different symbol functions $C(\xi, k)$, $\widetilde{C}(\xi, k)$ such that $C(\cdot, k)$, $\widetilde{C}(\cdot, k)$ have the same range for all $k \in \mathbb{Z}_{\geq 0}$? Does it happen outside of dilations in z_2?
3. In both settings and in any context, what happens in higher dimensions?

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