Chapter 1

What are differential equations?

Many of the equations governing the natural world involve rates at which things happen. These equations are differential equations (a more precise definition is coming up later).

Example 1

An object is falling in the atmosphere near sea level and experiences air resistance. Let's say we want a formula for \( v(t) \), the velocity of the object at time \( t \). It's not obvious how to immediately write down a formula for \( v(t) \). It is easier and more natural to first write down a differential equation describing the object's motion.

To model the problem with a DE, identify the independent and dependent variables, assign letters & units to them, articulate basic principles/laws that govern the problem, as equations.

Newton's 2nd law:

\[
F = ma
\]

\[
F = m \frac{dv}{dt}
\]

\( v \) is velocity in m/s, measured downwards (\( v \) is a function of \( t \)).

\( t \) is time in s

\( m \) is mass in kg

\( F \) is force in Kg, \( m/s^2 \) or N.
What forces act on the object?

(a) Force due to gravity (weight), which equals \( g \cdot m = 9.8 \, \text{m} \)

\[ g = 9.8 \, \text{m/s}^2 \]

(b) Force due to air resistance (drag).

Reasonable assumption: The faster the object falls, the larger the drag is.

The simplest equation which expresses this assumption is that of proportionality:

\[ \text{drag} = -k \cdot V \]

where \( k \) is a constant (the drag coefficient, which depends on the shape of the object. It has units equal to

\[ \frac{\text{units of force}}{\text{units of velocity}} = \frac{\text{kg} \cdot \text{m/s}^2}{\text{m/s}} = \text{kg/s} \]

So finally,

\[ m \frac{dV}{dt} = F \]

\[ m \frac{dv}{dt} = 9.8m - kV \]

Now we can try to "solve" this DE to get a formula for \( V \) in terms of \( t \).

For the sake of simplicity, let's assign some numbers to the constants. Say \( m = 1 \), \( k = \frac{1}{5} \).

\[ \frac{dv}{dt} = 9.8 - \frac{1}{5}V \]
What do we mean by solving this DE? A function \( V \) is a solution of the DE if it satisfies the DE. For example \( V(t) = t^2 \) is not a solution because it does not satisfy the DE:

\[
\frac{dv}{dt} = 9.8 - \frac{1}{5} v \\
2t = 9.8 - \frac{1}{5} t^2 
\]

This does not hold for all values of time \( t \).

Qualitative solution

Without obtaining a formula for \( V \), we can plot possible solutions \( V \).

Step 1: Draw \( V \) and \( t \) axes.
Step 2: For every point on the plane, draw the slope of \( V \).

For example,

- For \( v = 40 \) and any value of \( t \), we get \( \frac{dv}{dt} = 9.8 - \frac{40}{5} = 1.8 \)
- For \( v = 60 \) and any value of \( t \), we get \( \frac{dv}{dt} = 9.8 - \frac{60}{5} = -2.2 \)
- For \( v = 49 \) and any value of \( t \), we get \( \frac{dv}{dt} = 9.8 - \frac{49}{5} = 0 \)

![Graph](Illustration of slope 1.8 for all points with \( v = 40 \))

Illustration of slope 2.2 for all points with \( v = 50 \)

Illustration of slope 0 for all points with \( v = 49 \)
Fill in the rest or use a computer:

This is called a slope field or direction field.

Step 3: Any solution \( v \) of the DE must "follow" the slope field of the DE (shown in blue above).

Observations:

(a) There are many possible solutions \( v \) to the DE, corresponding to the many possible "initial values" of \( v(0) \).

(b) \( v = 49 \) is a constant or equilibrium solution.

(c) If \( v(0) < 49 \), the object's velocity \( v \) increases and approaches 49 m/s as \( t \to \infty \).

(d) If \( v(0) > 49 \), the object's velocity \( v \) decreases and approaches 49 m/s as \( t \to \infty \).

Next, can we find formulas for the many possible solutions \( v \) we have visualized on the slope field?
Analytic solution

\[ \frac{dv}{dt} = 9.8 - \frac{v}{5} \]

\[ \frac{dv}{dt} = -\frac{1}{5}(v - 49) \]

Equilibrium solution: \( v = 49 \) for all \( t \) is a solution as it satisfies the DE:

\[ \frac{dv}{dt} = -\frac{1}{5}(v - 49) \]

\[ 0 = -\frac{1}{5}(49 - 49) \] \( \checkmark \)

Other solutions: Suppose \( v \neq 49 \). Then we can divide by \( v - 49 \) to get

\[ \frac{1}{v-49} \cdot \frac{dv}{dt} = -\frac{1}{5} \]

Integrate both sides with respect to \( t \):

\[ \int \frac{1}{v-49} \cdot \frac{dv}{dt} \, dt = \int -\frac{1}{5} \, dt \]

\[ \int \frac{1}{v-49} \, dv = \int -\frac{1}{5} \, dt \]

\[ \ln |v-49| = -\frac{1}{5} t + C \]

where \( C \) is an arbitrary constant.

(because remember \( \int \frac{1}{x} \, dx = \ln |x| + C \))
"Exponentiate" both sides:

\[ |v - 4q| = e^{-\frac{1}{5}t + c} \]

\[ |v - 4q| = e^c e^{-\frac{1}{5}t} \]

\[ v - 4q = \pm e^c e^{-\frac{1}{5}t} \]

Since \( c \) is an arbitrary constant, \( \pm e^c \) can be any constant except zero.

So let's write \( v - 4q = Be^{-\frac{1}{5}t} \)

where \( B \) is an arbitrary, non-zero constant.

So the list of all solutions to the DE is

(i) \( v = 4q \)

(ii) \( v = 4q + Be^{-\frac{1}{5}t} \), for any non-zero \( B \)

We can package (i) and (ii) neatly into one formula:

\[ v = 4q + Be^{-\frac{1}{5}t} \text{ for any constant } B \]

So what we did was observe that if we put \( B=0 \) in (ii), then we get (i).

\[ v = 4q + Be^{-\frac{1}{5}t} \text{ is called the general solution.} \]

There are infinitely many choices for \( B \), and these correspond to the infinitely many possible solutions \( v \) which we saw on the slope field.

If the initial value \( v(0) \) is specified then this
information fixes B. For example, suppose the object is dropped from rest at time 0.

So $v(0) = 0$.

\[ v = gt + Be^{-\frac{g}{5}t} \]

0 = gt + Be^{-\frac{g}{5} \cdot 0}

0 = wt + B

B = -gt

So $v = gt - gt e^{-\frac{g}{5}t}$. This is called a particular solution.

Notice that as $t \to \infty$, $e^{-\frac{g}{5}t} \to 0$, so $v \to gt$.

This confirms what we saw on the slope field.
Example 2. In ideal conditions, the population of mice in a certain area increases at a rate proportional to the current population (reasonable).

Suppose that the conditions are not so ideal and owls kill 450 mice per month.

Let \( p(t) \) be the population of mice at time \( t \) months.

Model:

\[
\frac{dp}{dt} = rp - 450
\]

\( \text{mice/month} = \frac{1}{\text{month}} \cdot \text{mice/\text{month}} \)

\( r \) is a constant (with units months\(^{-1}\)).

For simplicity, let's assign a number to \( r \). Say \( r = 0.5 \).

\[
\frac{dp}{dt} = 0.5p - 450
\]

Slope Field: If \( p = p_0 \) is an equilibrium solution, for some constant \( p_0 \), then

\[
\frac{dp}{dt} = 0.5p - 450
\]

\[ 0 = 0.5p_0 - 450 \Rightarrow p_0 = 900. \]

If \( p > 900 \), \( \frac{dp}{dt} = 0.5p - 450 > 0 \), and \( \frac{dp}{dt} \) gets larger as \( p \) gets larger.

If \( p < 900 \), \( \frac{dp}{dt} = 0.5p - 450 < 0 \), and \( \frac{dp}{dt} \) becomes more negative as \( p \) gets smaller.
Observations:

(a) If \( p(0) = 900 \), population remains constant at 900

(b) If \( p(0) > 900 \), population grows to infinity

(c) If \( p(0) < 900 \), population decreases to infinity

(d) Limitations of this model:
   - A population wouldn't really have ideal conditions.
   - As the population grows there would be overcrowding, competition, etc and we wouldn't see such rapid growth
   - A population cannot be negative.

Analytic Solution

\( p = 900 \) is a solution.

Others: If \( p \neq 900 \),

\[
\frac{dp}{dt} = 0.5(p - 900)
\]

\[
\frac{1}{p - 900} \frac{dp}{dt} = 0.5
\]

\[
\int \frac{1}{p - 900} \frac{dp}{dt} \, dt = \int 0.5 \, dt
\]

\[
\int \frac{1}{p - 900} \, dp = \int 0.5 \, dt
\]

\[
\ln |p - 900| = 0.5t + C
\]

where \( C \) is any constant
\[ |p-900| = e^{c \cdot e^{0.5t}} \]
\[ p-900 = \pm e^{c \cdot e^{0.5t}} \]
\[ p = 900 + Be^{0.5t} \]

where \( B \) is any non-zero constant.

Allow \( B = 0 \) because that gives \( p = 900 \), which is the equilibrium solution.

General solution: \[ p = 900 + Be^{0.5t} \]

for an arbitrary constant \( B \).

Now, suppose the initial population is 850.

Then \[ 850 = 900 + Be^{0} \]
\[ \therefore B = -50 \]

Particular solution: \[ p = 900 - 50e^{0.5t} \]

Notice that as \( t \to \infty \), \( p \to -\infty \), just as we saw on the slope field.

**Example 3**  
A pond contains 10^6 gallons of water contaminated with a chemical. Water containing 0.01 grams/gallon of this chemical flows into the pond at a rate of 300 gallons/hour. The mixture flows out at the same rate (so the pond always has 10^6 gallons
of water). Write a DE for the amount of chemical in the pond at time $t$ hours.

Let $a(t)$ be the amount in grams of the chemical at time $t$ hours.

At any given time $t$, the concentration of the chemical in the pond is $\frac{a}{10^6}$.

\[
\frac{da}{dt} = 0.01 \cdot 300 - \frac{a}{10^6} \cdot 300
\]

\[
\frac{\text{grams}}{\text{hour}} = \frac{\text{grams}}{\text{gallon}} \cdot \frac{\text{gallons}}{\text{hour}} - \frac{\text{grams}}{\text{gallon}} \cdot \frac{\text{gallons}}{\text{hour}}
\]

\[
\frac{da}{dt} = 3 - \frac{3}{10^4} \cdot a
\]

Similar to $\frac{dv}{dt} = 9.8 - \frac{v}{5}$, so it will have a similar slope field, with all solutions tending to the equilibrium solution in the long run.
The equilibrium solution of \( \frac{da}{dt} = 3 - \frac{3}{10^4} a \)
is \( a = 10^4 \) grams.

So eventually the pond will contain close to \( 10^4 \) grams of the chemical, no matter how much it started with.

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**Classification of Differential Equations**

**DE's**

- Ordinary differential equations (ODE's)
  - Example: \( \frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = \cos x \)
  - Only "ordinary" derivatives are involved.
  - \( y \) is a function of \( x \).

- Partial differential equations (PDE's)
  - Example: \( 4\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \)
  - Partial derivatives are involved.
  - \( u \) is a multivariable function of \( x \) & \( t \).

In this course, we'll only study ODE's.

So we'll be studying equations of the form

\[
F(x, y, y', \ldots, y^{(n)}) = 0
\]

(an equation involving \( x, y, y', \ldots, y^{(n)} \)).
The highest order of derivative which occurs is called the order of the DE

**Example**
\[
\frac{d^2y}{dx^2} + 2 \frac{dy}{dx} + y = \cos x
\]

is a 2nd order ODE.

We'll mostly study 1st and 2nd order ODE's in this course.

**ODE's**

**Linear**

If \( F \) is a linear function of \( y, y', y'', \ldots, y^{(n)} \).

Remember, linear means a (multivariate) polynomial of degree 1.

So the ODE must be of the form

\[
a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \ldots + a_2(x)y'' + a_1(x)y' + a_0(x)y = b(x)
\]

where \( a_n(x), a_{n-1}(x), \ldots, a_0(x), b(x) \) are any functions of \( x \).

**Examples**

(a) \[
\frac{d^2y}{dx^2} + 2 \frac{dy}{dx} + y = \cos x
\]

is linear

(b) \[
\frac{dy}{dx} + x^3 y = e^x
\]

is linear

Note: \( F \) is allowed to be non-linear in \( x \).

**Non-linear**

If \( F \) is not a linear function of \( y, y', y'', \ldots, y^{(n)} \).

**Examples**

(a) \[
\frac{dy}{dx} + y^2 = x
\]

is non-linear (due to the \( y^2 \) term)

(b) \[
\frac{d^2\theta}{dt^2} + \sin \theta = 0
\]

is non-linear (due to the \( \sin \theta \)-term — a polynomial in \( \theta \) can only involve powers of \( \theta \))

(c) \[
\frac{d^2y}{dx^2} + y \frac{dy}{dx} = 3
\]

is non-linear (due to the \( y \frac{dy}{dx} \) term — the multivariate polynomial \( y'' + y'y = 3 \) is of degree 2).
We'll also study systems of differential equations for two or more functions whose behavior is correlated.

Example: If \( x(t) \) = population of a prey at time \( t \) and \( y(t) \) = " " " predator " " "

then \( x \) and \( y \) should depend on each other in some way. (If \( y \) grows, \( x \) dies out. If \( x \) dies out, \( y \) grows. Etc.)

A model: \[
\frac{dx}{dt} = \alpha x + \beta xy
\]
\[
\frac{dy}{dt} = -\gamma y + \delta xy
\]

This is a system of differential equations. We cannot solve one without the other.

Example: Classify the following ODE. For which values of \( r \) does the ODE have a solution of the form \( y = e^{rt} \)?

\[
y''' - 3y'' + 2y' = 0
\]

It's a 3\(^{rd}\) order linear ODE.

To check if \( y = e^{rt} \) is a solution, just plug it in and see if it satisfies the DE.
\[ y = e^{rt} \]
\[ y' = re^{rt} \]
\[ y'' = r^2 e^{rt} \]
\[ y''' = r^3 e^{rt} \]

\[ y''' - 3y'' + 2y' = 0 \]
\[ r^3 e^{rt} - 3r^2 e^{rt} + 2re^{rt} = 0 \]
\[ e^{rt} (r^3 - 3r^2 + 2r) = 0 \]

For the left hand side to always be zero, the constant \( r^3 - 3r^2 + 2r \) should be zero.

\[ r^3 - 3r^2 + 2r = 0 \]
\[ r(r^2 - 3r + 2) = 0 \]
\[ r(r - 2)(r - 1) = 0 \]

\[ \Rightarrow r = 0, 1, 2. \]