Section 6.1

Suppose \( a_0, a_1, a_2, a_3, \ldots \) are some real numbers. Then we make the following definition, using the Greek letter sigma:

\[
\sum_{i=m}^{n} a_i = a_m + a_{m+1} + a_{m+2} + \ldots + a_{n-1} + a_n
\]

**Examples:**

1. \( \sum_{i=1}^{4} i^2 = 1^2 + 2^2 + 3^2 + 4^2 = 30 \)

2. \( \sum_{k=3}^{6} \frac{1}{k} = \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} = \frac{19}{20} \)

3. \( \sum_{i=1}^{3} 2 = 2 + 2 + 2 = 6 \)

**Properties:**

1. For any constant \( c \), \( \sum_{i=m}^{n} c a_i = c \sum_{i=m}^{n} a_i \)

**Proof:**

\[
\sum_{i=m}^{n} c a_i = c a_m + c a_{m+1} + \ldots + c a_n = c (a_m + a_{m+1} + \ldots + a_n) = c \sum_{i=m}^{n} a_i
\]

2. \( \sum_{i=m}^{n} a_i + b_i = \sum_{i=m}^{n} a_i + \sum_{i=m}^{n} b_i \)

**Proof:**

\[
\sum_{i=m}^{n} a_i + b_i = (a_m + b_m) + (a_{m+1} + b_{m+1}) + \ldots + (a_n + b_n)
\]

\[
= (a_m + a_{m+1} + \ldots + a_n) + (b_m + b_{m+1} + \ldots + b_n)
\]

\[
= \sum_{i=m}^{n} a_i + \sum_{i=m}^{n} b_i
\]
\[ \sum_{i=m}^{n} a_i - b_i = \sum_{i=m}^{n} a_i - \sum_{i=m}^{n} b_i \]

Some useful summation formulae

1. \[ \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \]
2. \[ \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \]
3. \[ \sum_{i=1}^{n} i^3 = \left( \frac{n(n+1)}{2} \right)^2 \]

Proof of first formula:

\[ \sum_{i=1}^{n} i = 1 + 2 + 3 + 4 + \ldots + n-3 + n-2 + n-1 + n \]
\[ + \sum_{i=1}^{n} i = n + n-1 + n-2 + n-3 + \ldots + 4 + 3 + 2 + 1 \]
\[ = \sum_{i=1}^{n} i = n(n+1) \]
\[ \therefore \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \]

Examples:

1. Find \[ \sum_{i=3}^{n} i (4i^2 - 3) \].

\[ \sum_{i=3}^{n} i (4i^2 - 3) = 4 \sum_{i=3}^{n} i^3 - 3 \sum_{i=3}^{n} i \]
\[ = 4 \left( \frac{n^2 (n+1)^2}{2} - 3 \right) - 3 \left( \frac{n(n+1)}{2} \right) \]
\[ = 4 \left( \frac{n(n+1)(3)}{2} - 9 \right) - 3 \left( \frac{n(n+1)}{2} - 3 \right) \]
2. Find \( \lim_{n \to \infty} \sum_{i=1}^{n} \frac{3}{n} \left[ \left( \frac{i}{n} \right)^2 + 1 \right] \)

\[
\lim_{n \to \infty} \sum_{i=1}^{n} \frac{3}{n} \left[ \left( \frac{i}{n} \right)^2 + 1 \right] = \lim_{n \to \infty} \frac{3}{n^3} \sum_{i=1}^{n} i^2 + \frac{3}{n} \sum_{i=1}^{n} 1
\]

\[
= \lim_{n \to \infty} \frac{3}{n^3} \frac{n(n+1)(2n+1)}{6} + \frac{3}{n} \cdot n
\]

\[
= \lim_{n \to \infty} \frac{3}{n^3} \frac{2n^3 + 3n^2 + n}{6} + 3
\]

\[
= 4
\]
Section 6.2

Example: How do we find the area under the graph of $f(x) = x^2$ from $x = 0$ to $x = 1$?

Let's start by approximating the area.

Right rectangles

We can approximate the area using 4 "right rectangles":

- The upper right corners of rectangles lie on the graph.
- When the rectangles have equal width, the width of each rectangle is $1 - 0 = \frac{1}{4}$ (length of interval divided by the number of rectangles).

Sum of areas of 4 right rectangles:

\[
\frac{1}{4} \cdot f(\frac{1}{4}) + \frac{1}{4} \cdot f(\frac{3}{4}) + \frac{1}{4} \cdot f(\frac{3}{4}) + \frac{1}{4} \cdot f(\frac{3}{4})
\]

\[
= \frac{1}{4} \left( f(\frac{1}{4}) + f(\frac{3}{4}) + f(\frac{3}{4}) + f(\frac{3}{4}) \right)
\]
The more rectangles we use, the better the approximation gets.

If we use \( n \) right rectangles of equal width, each has width \( \frac{1}{n} \). Very thin when \( n \) is big.

Sum of areas of \( n \) right rectangles

\[
S_n = \frac{1}{n} \left( f\left(\frac{1}{n}\right) + f\left(\frac{2}{n}\right) + f\left(\frac{3}{n}\right) + \ldots + f\left(\frac{n}{n}\right) \right)
\]

\[
= \frac{1}{n} \left( \frac{1^2}{n^2} + \frac{2^2}{n^2} + \frac{3^2}{n^2} + \ldots + \frac{n^2}{n^2} \right)
\]

\[
= \frac{1}{n} \cdot \frac{1}{n^2} \left( 1^2 + 2^2 + 3^2 + \ldots + n^2 \right)
\]

\[
= \frac{1}{n^3} \sum_{i=1}^{n} i^2 = \frac{1}{n^3} \frac{n(n+1)(2n+1)}{6}
\]

\[
= \frac{2n^2 + 3n + 1}{6n^2}
\]

To get the exact area \( A \) under the graph, let \( n \to \infty \).

\[
A = \lim_{n \to \infty} S_n = \lim_{n \to \infty} \frac{2n^2 + 3n + 1}{6n^2} = \frac{2}{6} = \frac{1}{3}
\]
Using 4 "left rectangles" of equal width ...

Sum of areas of 4 left rectangles

\[ \frac{1}{4} \left( f\left( \frac{1}{4} \right) + f\left( \frac{1}{2} \right) + f\left( \frac{3}{4} \right) + f\left( 1 \right) \right) \]

Sum of areas of \( n \) left rectangles

\[ S_n = \frac{1}{n} \left( f\left( \frac{0}{n} \right) + f\left( \frac{1}{n} \right) + f\left( \frac{2}{n} \right) + \ldots + f\left( \frac{n-1}{n} \right) \right) \]

\[ = \frac{1}{n} \left( \frac{0^2}{n^2} + \frac{1^2}{n^2} + \frac{2^2}{n^2} + \ldots + \frac{(n-1)^2}{n^2} \right) \]

\[ = \frac{1}{n} \cdot \frac{1}{n^2} \left( 1^2 + 2^2 + \ldots + (n-1)^2 \right) \]

\[ = \frac{1}{n^3} \sum_{i=1}^{n-1} i^2 = \frac{1}{n^3} \frac{(n-1)(n-1+1)(2(n-1)+1)}{6} \]

\[ = \frac{2n^2-3n+1}{6n^2} \]

Exact area \( A = \lim_{n \to \infty} S_n = \lim_{n \to \infty} \frac{2n^2-3n+1}{6n^2} = \frac{2}{6} = \frac{1}{3} \).

Same answer!
Midpoint rectangles

Using 4 "midpoint rectangles" of equal width,

\[ y = f(x) = x^2 \]

\[ \text{Sum of areas of 4 midpoint rectangles} = \frac{1}{4} \left( f\left( \frac{1}{2} \right) + f\left( \frac{3}{8} \right) + f\left( \frac{5}{8} \right) + f\left( \frac{7}{8} \right) \right) \]

\[ \text{Sum of areas of } n \text{ midpoint rectangles} = S_n = \frac{1}{n} \left( f\left( \frac{1}{2n} \right) + f\left( \frac{3}{2n} \right) + f\left( \frac{5}{2n} \right) + \ldots + f\left( \frac{2n-1}{2n} \right) \right) \]

\[ = \frac{1}{n} \left( \frac{1^2}{4n^2} + \frac{3^2}{4n^2} + \frac{5^2}{4n^2} + \ldots + \frac{(2n-1)^2}{4n^2} \right) \]

It's also true that \( A = \lim_{n \to \infty} S_n = \frac{1}{3} \) with midpoint rectangles, but we won't do this calculation now.

Example: Find the area under the graph of

\[ f(x) = x^3 + 1 \text{ from } x = 0 \text{ to } x = 2. \]

Since it doesn't matter what kind of rectangles we use, let's use the easiest kind: right rectangles.
First, let's get an idea with 4 right rectangles to approximate the area:

Each rectangle has width \( \frac{2-0}{4} = \frac{2}{4} \)

Sum of areas of 4 right rectangles:
\[
\frac{2}{4} (f(\frac{2}{4}) + f(2\cdot\frac{2}{4}) + f(3\cdot\frac{2}{4}) + f(4\cdot\frac{2}{4}))
\]

Sum of areas of \( n \) right rectangles:
\[
S_n = \frac{2}{n} \left( f(\frac{2}{n}) + f(2\cdot\frac{2}{n}) + f(3\cdot\frac{2}{n}) + \ldots + f(n\cdot\frac{2}{n}) \right)
\]

\[
= \frac{2}{n} \left( \frac{8}{n^3} + 1 + 2^3 \frac{8}{n^3} + 3^3 \frac{8}{n^3} + \ldots + n^3 \frac{8}{n^3} + 1 \right)
\]

\[
= \frac{2}{n} \left( \frac{8}{n^3} + 2^3 \frac{8}{n^3} + 3^3 \frac{8}{n^3} + \ldots + n^3 \frac{8}{n^3} \right)
\]

\[
= 2 + \frac{2}{n} \cdot \frac{8}{n^3} \left( 1 + 2^3 + 3^3 + \ldots + n^3 \right)
\]

\[
= 2 + \frac{16}{n^4} \cdot \frac{n}{4} \cdot \frac{n(n+1)}{2} \cdot \frac{n(n+1)(2n+1)}{6}
\]

\[
= 2 + \frac{16}{n^4} \left( \frac{n(n+1)}{2} \right)^2
\]
\[
\begin{align*}
2 + \frac{16}{n^4} \frac{n^4 + 2n^3 + n^2}{4} &= 2 + \frac{16}{n^4} \frac{n^4 + 2n^3 + n^2}{4} \\
&= 2 + \frac{16}{4} \\
&= 6.
\end{align*}
\]
Section 6.3

Definition of the definite integral

Suppose $f$ is a continuous function on $[a, b]$.

We can define

$$
\int_{a}^{b} f(x) \, dx = \lim_{n \to \infty} \frac{b-a}{n} \sum_{i=1}^{n} f\left(a + i \frac{b-a}{n}\right)
$$

How do we understand this definition?

- When $f(x) \geq 0$, the definite integral $\int_{a}^{b} f(x) \, dx$
  is the area under the graph of $f(x)$ from $x = a$ to $x = b$.
  This is because the Riemann Sum is the sum of the areas of $n$ right rectangles on $[a, b]$.

![Diagram of function and rectangles]
Sum of areas of \( n \) right rectangles:

\[
\begin{align*}
&= \frac{b-a}{n} f(a + \frac{b-a}{n}) + \frac{b-a}{n} f(a + 2 \frac{b-a}{n}) \\
&\quad + \ldots + \frac{b-a}{n} f(a + n \frac{b-a}{n}) \\
&= \frac{b-a}{n} \left( f(a + \frac{b-a}{n}) + f(a + 2 \frac{b-a}{n}) + \ldots + f(a + n \frac{b-a}{n}) \right) \\
&= \frac{b-a}{n} \sum_{i=1}^{n} f(a + i \frac{b-a}{n})
\end{align*}
\]

Taking the limit of this as \( n \to \infty \) gives the exact area under the graph.

- When \( f(x) \leq 0 \), the definite integral \( \int_{a}^{b} f(x) \, dx \) is negative of the area between the graph of \( f(x) \) and the \( x \)-axis (because all \( f \)-values in the Riemann Sum are negative).

\[
\int_{a}^{b} f(x) \, dx = -A
\]

\[
\int_{a}^{b} -f(x) \, dx = A
\]

- Integrals of functions over intervals where they are positive and negative can be computed as the difference of the areas:

\[
\int_{a}^{b} f(x) \, dx = A_1 - A_2
\]
NOTE: We defined a definite integral in terms of a Riemann sum using right rectangles. But we could equally well give a definition in terms of left or midpoint rectangles.

How to evaluate a definite integral

I. Directly from the definition, using Riemann Sums.

Example: Find \( \int_0^7 x^3 \, dx \).

\[
\int_0^7 x^3 \, dx = \lim_{n \to \infty} \frac{7-0}{n} \sum_{i=1}^{n} \left( 0 + \frac{7-0}{n} \right)^3 \\
= \lim_{n \to \infty} \frac{7}{n} \sum_{i=1}^{n} \frac{7^3}{n^3} i^3 \\
= \lim_{n \to \infty} \frac{74}{n^4} \sum_{i=1}^{n} i^3 \\
= \lim_{n \to \infty} \frac{74}{n^4} \left( \frac{n(n+1)}{2} \right)^2 \\
= \lim_{n \to \infty} \frac{74}{n^4} \frac{n^4 + 2n^3 + n^2}{4} \\
= \lim_{n \to \infty} \frac{74}{n^4} \frac{n^4(1 + \frac{2}{n} + \frac{1}{n^2})}{4} \\
= \frac{74}{4}.
\]
II. By understanding the shape of the graph of \( f(x) \).

Examples:

1. Find \( \int_0^3 x - 1 \, dx \)

   Understand graph \( y = x - 1 \). A line!

   \[
   \int_0^3 x - 1 \, dx = -A_1 + A_2
   \]

   \[
   = -\frac{1}{2} + \frac{1}{2} \cdot 4
   \]

   \[
   = \frac{3}{2}
   \]

2. Find \( \int_0^1 \sqrt{1 - x^2} \, dx \)

   Understand graph \( y = \sqrt{1 - x^2} \)

   \[
   \Rightarrow \quad y^2 = 1 - x^2
   \]

   \[
   y^2 + x^2 = 1 \quad \text{(part of)} \quad \text{circle, radius 1, center (0,0)}
   \]

   \[
   \int_0^1 \sqrt{1 - x^2} \, dx = \frac{\pi \cdot 1^2}{4}
   \]
3. Find \( \int_{-1}^{1} \sin(x^3) \, dx \)

Notice that \( f(x) = \sin(x^3) \) is an odd function (\( f(-x) = -f(x) \)). Odd functions have graphs that are like...

\[
\int_{-1}^{1} \sin(x^3) \, dx = -A + A = 0
\]

"net area is zero"

III. Use the Fundamental Theorem of Calculus, which we’ll learn in the next section.

---

**Properties:**

- If \( c \) is a constant, \( \int_{a}^{b} c \, dx = c \, (b-a) \)

- \( \int_{a}^{b} f(x) \, dx = c \int_{a}^{b} f(x) \, dx \)

- \( \int_{a}^{b} f(x) + g(x) \, dx = \int_{a}^{b} f(x) \, dx + \int_{a}^{b} g(x) \, dx \)

- \( \int_{a}^{b} f(x) - g(x) \, dx = \int_{a}^{b} f(x) \, dx - \int_{a}^{b} g(x) \, dx \)
If \( f(x) \leq g(x) \) on \([a, b]\),
\[
\int_a^b f(x) \, dx \leq \int_a^b g(x) \, dx
\]

If \( a < c < b \),
\[
\int_a^b f(x) \, dx = \int_a^c f(x) \, dx + \int_c^b f(x) \, dx
\]

\[A_1 + A_2 = A_1 + A_2\]

\[
\left| \int_a^b f(x) \, dx \right| \leq \int_a^b |f(x)| \, dx
\]

\(\int_a^b f(x) \, dx\) involves some cancellation of positive and negative areas.

\(\int_a^b |f(x)| \, dx\) has no cancellation in area.
Definition

If \( a < b \),

\[
\int_{b}^{a} f(x) \, dx = - \int_{a}^{b} f(x) \, dx
\]

Example: Find

\[
\int_{0}^{1} x^3 \, dx - \int_{\frac{1}{2}}^{1} x^3 \, dx.
\]

This equals

\[
\int_{0}^{1} x^3 \, dx + \int_{1}^{\frac{7}{4}} x^3 \, dx = \frac{7^4}{4}
\]

(see previous example)
Section 6.4

The Fundamental Theorem of Calculus

Suppose that $f$ is a continuous function on $[a, b]$ and $F$ is an antiderivative of $f$. Then

$$
\int_{a}^{b} f(x) \, dx = F(b) - F(a)
$$

Examples:

1. Find $\int_{0}^{\frac{\pi}{2}} \cos x \, dx$.

   $F(x) = \sin x$ is an antiderivative of $f(x) = \cos x$.

   So 
   $$
   \left. \int_{0}^{\frac{\pi}{2}} \cos x \, dx = \sin x \right|_{0}^{\frac{\pi}{2}} = \sin\left(\frac{\pi}{2}\right) - \sin(0) = 1.
   $$

What if we used a different antiderivative?

$F(x) = \sin x + 7$ is an antiderivative of $f(x) = \cos x$.

So

$$
\left. \int_{0}^{\frac{\pi}{2}} \cos x \, dx = \sin x + 7 \right|_{0}^{\frac{\pi}{2}} = \sin\left(\frac{\pi}{2}\right) + 7 - (\sin(0) + 7) = 1
$$

The 7's cancel out and we get the same answer.
2. Find \( \int_{1}^{3} |x-2| \, dx \)

**Method 1: Graphical**

Adding up the areas of the two triangles, we get \( \int_{1}^{3} |x-2| \, dx = \frac{1}{2} + \frac{1}{2} = 1 \).

**Method 2: Fundamental Theorem of Calculus.**

\[ f(x) = |x-2| \text{ changes behavior at } x=2. \]

\[ f(x) = |x-2| = \begin{cases} x-2 & x \geq 2 \\ -(x-2) & x < 2 \end{cases} \]

\[ \int_{1}^{3} |x-2| \, dx = \int_{1}^{2} |x-2| \, dx + \int_{2}^{3} |x-2| \, dx \]

\[ = \int_{1}^{2} -(x-2) \, dx + \int_{2}^{3} (x-2) \, dx \]

\[ = \left[ -\frac{x^2}{2} + 2x \right]_{1}^{2} + \left[ \frac{x^2}{2} - 2x \right]_{2}^{3} \]

\[ = \left[ -\frac{3}{2} + 4 \right] + \left[ \frac{9}{2} - 6 \right] - \left( \frac{1}{2} - 2 \right) \]

\[ = \frac{1}{2} + \frac{1}{2} = 1 \]
Another version of the Fundamental Theorem of Calculus

Let \( f \) be a continuous function on \([a, b]\).

Define \( g(x) = \int_a^x f(u) \, du \) for \( a \leq x \leq b \).

\( g(x) \) measures the "net area" under the graph of \( f(u) \)
\( u = a \) to \( u = x \), so it depends on \( x \), so it's a function of \( x \).

\[ g'(x) = f(x) \]

Proof using 1st version of the Fundamental Theorem of Calculus:

If \( F \) is an antiderivative of \( f \),
\[ g(x) = \int_a^x f(u) \, du = F(x) - F(a) \]

Differentiating both sides,
\[ g'(x) = F'(x) = f(x) \]

(\( F \) is an antiderivative of \( f \))

\[ \frac{d}{dx} \left( \int_1^x \sin(t^2) \, dt \right) = \sin(x^3) \]
2. Find \( \frac{d}{dx} \int_{x}^{1} \sin(t^3) dt \).

\[
\int_{x}^{1} \sin(t^3) dt = -\int_{1}^{x} \sin(t^3) dt
\]

So \( \frac{d}{dx} \int_{x}^{1} \sin(t^3) dt = \frac{d}{dx} \left( -\int_{1}^{x} \sin(t^3) dt \right) = -\sin(x^3) \).

3. \( \frac{d}{dx} \int_{2}^{x^2} \sin(t^3) dt = \sin((x^2)^3) \cdot 2x \)

\[
\frac{d}{dx} g(x^2) = g'(x^2) \cdot 2x \quad \text{Chain Rule}
\]

4. \( \frac{d}{dx} \int_{3}^{\cos x} e^t \ dt = e^{\cos x} \cdot (-\sin x) \)

---

The indefinite integral

Because of the relation between antiderivatives & integrals, the notation \( \int f(x) \, dx \) is used to denote the most general antiderivative of \( f \).

**Definition** Suppose that \( F \) is an antiderivative of \( f \).

\[
\int f(x) \, dx = F(x) + C
\]

**Examples:**

1. \( \int \frac{-1}{\sqrt{1-x^2}} \, dx = \cos^{-1}(x) + C \)
\[ 3. \quad \int \sec^2 x \, dx = \tan x + C \]

\[ 3. \quad \int \sqrt{x^2 + (x+1)^2} \, dx \]

\[ = \int x^{\frac{1}{2}} (x^2 + 2x + 1) \, dx \]

\[ = \int x^{\frac{5}{2}} + 2x^{\frac{3}{2}} + x^{\frac{1}{2}} \, dx \]

\[ = \frac{2}{7} x^{\frac{7}{2}} + 2 \cdot \frac{2}{5} x^{\frac{5}{2}} + \frac{2}{3} x^{\frac{3}{2}} + C \]