Chapter 8
Basic concepts of reliability analysis by probability methods

8.1 Introduction

This chapter provides the theoretical background for the reliability analysis used in other chapters, Chapter 2 in particular. Some basic concepts of probability theory are discussed as these are essential to the understanding and development of quantitative reliability analysis methods. Definitions of terms commonly used in system reliability analysis are also included. The three methods discussed are the cut-set, the state-space, and the network reduction methods.

8.2 Definitions

The following terms, defined in Chapter 1, are commonly used in system reliability analysis: component, failure, failure rate, mean time between failures (MTBF), mean time to repair (MTTR), and system. Additional definitions more specifically related to power distribution systems are given in 1.4. See Section 1.4

8.3 Basic probability theory

This subchuse discusses some of the basic concepts of probability theory. An appreciation of these ideas is essential to the understanding and development of reliability analysis methods.

8.3.1 Sample space

Sample space is the set of all possible outcomes of a phenomenon. For example, consider a system of three distribution links. Assuming that each link exists either in the operating or “up” state or in the failed or “down” state, the sample space is

\[ S = \{ (1U, 2U, 3U), (1D, 2U, 3U), (1U, 2D, 3U), (1D, 2D, 3U), (1D, 2U, 3D), (1U, 2D, 3D), (1D, 2D, 3D) \} \]

Here, 1U, 1D denote that the component i is up or down, respectively. The possible outcomes of a system are also called “system states,” and the set of all possible system states is called “system-state space.”

8.3.2 Event

In the example of three distribution links, the descriptions (1D, 2D, 3U), (1D, 2U, 3D), (1U, 2D, 3D), and (1D, 2D, 3D) define an event in which two or three lines are in the failed state. Assuming that a minimum of two lines is needed for successful system operation, this set of
states also defines the system failure. The event \( A \) is, therefore, a set of system states, and the event \( A \) is said to have occurred if the system is in a state that is a member of set \( A \).

### 8.3.3 Probability

A simple and useful way of looking at the probability of an occurrence of the event is by using a large number of observations.

Consider, for example, that a system is energized at time \( t = 0 \), and the state of the system is noted at time \( t \). This is said to be one observation. Now, if this process is repeated \( N \) times and the system is observed in the failed state \( N_f \) times, the probability of the system being in a failed state at time \( t \) is

\[
P_f(t) = \frac{N_f}{N}
\]

\( N \to \infty \) \hspace{1cm} (8-1)

### 8.3.4 Combinatorial properties of event probabilities

Certain combinatorial properties of event probabilities that are useful in reliability analysis are discussed in this subclause.

#### 8.3.4.1 Addition rule of probabilities

Two events, \( A_1 \) and \( A_2 \), are mutually exclusive if they cannot occur together. For events \( A_1 \) and \( A_2 \) that are not mutually exclusive (that is, events which can happen together)

\[
P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)
\]

where

\[
P(A_1 \cup A_2) \quad \text{is the probability of } A_1 \text{ or } A_2, \text{ or both happening; and}
\]

\[
P(A_1 \cap A_2) \quad \text{is the probability of } A_1 \text{ and } A_2 \text{ happening together.}
\]

When \( A_1 \) and \( A_2 \) are mutually exclusive, they cannot happen together; that is, \( P(A_1 \cap A_2) = 0 \), therefore Equation (8-2) reduces to

\[
P(A_1 \cup A_2) = P(A_1) + P(A_2)
\]

\hspace{1cm} (8-3)

#### 8.3.4.2 Multiplication rule of probabilities

If the probability of occurrence of event \( A_1 \) is affected by the occurrence of \( A_2 \), then \( A_1 \) and \( A_2 \) are not independent events.
The conditional probability of event $A_1$, given that event $A_2$ has already occurred, is denoted by $P(A_1 \mid A_2)$ and

$$P(A_1 \cap A_2) = P(A_1 \mid A_2) \cdot P(A_2)$$ (8-4)

This formula is also used to calculate the conditional probability

$$P(A_1 \mid A_2) = \frac{P(A_1 \cap A_2)}{P(A_2)}$$ (8-5)

When, however, events $A_1$ and $A_2$ are independent, that is the occurrence of $A_2$ does not affect the occurrence of $A_1$

$$P(A_1 \cap A_2) = P(A_1) \cdot P(A_2)$$ (8-6)

### 8.3.4.3 Complementation

$\overline{A}_1$ is used to denote the complement of event $A_1$. The component $\overline{A}_1$ is the set of states that are not members of $A_1$. For example, if $A_1$ denotes states indicating system failure, then the states not representing system failure make $\overline{A}_1$.

$$P(\overline{A}_1) = 1 - P(A_1)$$ (8-7)

### 8.3.5 Random variable

A random variable can be defined as "a quantity that assumes values in accordance with probabilistic laws." A discrete random variable assumes discrete values, whereas a random variable that assumes values from a continuous interval is termed a "continuous random variable." For example, the state of a system is a discrete random variable, and the time between two successive failures is a continuous random variable.

### 8.3.6 Probability distribution function

Probability distribution function describes the variability of a random variable. For a discrete random variable $X$, assuming values $x_i$, the probability density function is defined by

$$P_X(x) = P(X = x)$$ (8-8)

The probability density function for a discrete random variable is also called the "probability mass function" and has the following properties:

a) $P_X(x) = 0$ unless $x$ is one of the values $x_0, x_1, x_2, \ldots$

b) $0 \leq P_X(x_i) \leq 1$

c) $\sum_i P_X(x_i) = 1$

Another useful function is the cumulative distribution function. It is defined by

$$F_X(x) = P(X \leq x) = \sum P_X(x_i), x_i \leq x$$ (8-9)
The probability density function \( f_X(x) \) [or simply \( f(x) \)] for a continuous random variable is defined so that

\[
P(a \leq X \leq b) = \int_a^b f(y) \, dy
\]  
(8-10)

If, for example, \( X \) denotes the time to failure, Equation (8-10) gives the probability that the failure will occur in the interval \((a,b)\). The corresponding probability distribution function for a continuous random variable is

\[
F(x) = P(-\infty \leq X \leq x) = \int_{-\infty}^x f(y) \, dy
\]  
(8-11)

The function \( f(x) \) has certain specific properties (see Singh and Billinton [B3]1) including the following:

\[
\int_{-\infty}^{\infty} f(x) \, dx = 1
\]  
(8-12)

8.3.7 Expectation

The probabilistic behavior of a random variable is completely defined by the probability density function. It is often, however, desirable to have a single value characterizing the random variable. One such value is the expectation. It is defined by

\[
E(X) = \sum_i x_i P_X(x_i) \quad \text{for a discrete random variable,}
\]

\[
= \int_{-\infty}^{\infty} x f(x) \, dx \quad \text{for a continuous random variable.}
\]

The expectation of \( X \) is also called the "mean value of \( X \)" and has a special relationship to the average value of \( X \) in that, if the random variable \( X \) is observed many times and the arithmetic average of \( X \) is calculated, it will approach the mean value as the number of observations increases.

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1The numbers in brackets preceded by the letter B correspond to those of the bibliography in 8.6.
8.3.8 Exponential distribution

There are several special probability distribution functions (see Singh and Billinton [B3]); but the one of particular interest in reliability analysis is the exponential distribution, having the probability density function of

\[ f(x) = \lambda e^{-\lambda x} \]  \hspace{1cm} (8-13)

where \( \lambda \) is a positive constant. The mean value of the random variable \( X \), with exponential distribution is

\[ d = \int_0^\infty \lambda e^{-\lambda x} \, dx = \frac{1}{\lambda} \]  \hspace{1cm} (8-14)

Also the probability distribution is

\[ F(x) = \int_0^x \lambda e^{-\lambda y} \, dy = 1 - e^{-\lambda x} \]  \hspace{1cm} (8-15)

If the time between failures obeys the exponential distribution, the mean time between failures is \( d = \frac{1}{\lambda} \), where \( \lambda \) denotes the failure rate of the component. It should be noted that the failure rate for exponential distribution and only the exponential distribution is constant.

8.4 Reliability measures

The term “reliability” is generally used to indicate the ability of a system to continue to perform its intended function. Several measures of reliability are described in the literature, and some of the meaningful indexes for repairable systems, especially power distribution systems, are described in this subclause.

a) Unavailability. Unavailability is the “steady-state probability that a component or system is out of service due to failures or scheduled outages.” If only the failed state is considered, this term is called “forced unavailability.”

b) Availability. Availability is the “steady-state probability that a component or system is in service.” Numerically, availability is the complement of unavailability, that is

\[ \text{Availability} = 1 - \text{unavailability} \]

c) Frequency of system failure. This index can be defined as the “mean number of system failures per unit time.”

d) Expected failure duration. This index can be defined as the “expected or long-term average duration of a single failure event.”
8.5 Reliability evaluation methods

Numerical values for reliability measures can be obtained either by analytical methods or through digital simulation. Only the analytical techniques are discussed here (a discussion of the simulation approach can be found in (Singh and Billinton [B3]). The three methods described in this chapter are the state-space, network reduction, and cut-set methods. The state-space method is very general but becomes cumbersome for relatively large systems. The network reduction method is applicable when the system consists of series and parallel subsystems. The cut-set method is becoming increasingly popular in the reliability analysis of transmission and distribution networks and has been primarily used in this book. The state-space and network reduction methods are discussed in this chapter for reference and for the potential benefit to the users of this book.

8.5.1 Minimal cut-set method

The cut-set method can be applied to systems with simple as well as complex configurations and is a very suitable technique for the reliability analysis of power distribution systems. A cut-set is a “set of components whose failure alone will cause system failure,” and a minimal cut-set has no proper subset of components whose failure alone will cause system failure. The components of a minimal cut-set are in parallel since all of them must fail in order to cause system failure and various minimal cut-sets are in series as any one minimal cut-set can cause system failure.

A simple approach for the identification of minimal cut-sets is described in Chapter 2, but more formal algorithms are also available in the literature (see Singh and Billinton [B3]). Once the minimal cut-sets have been obtained, the reliability measures can be obtained by the application of suitable formulas (see Shooman [B1] and Singh [B2]). Assuming component independence and denoting the probability of failure of components in cut-set \( C_i \) by \( P(C_i) \), the probability (unavailability) and the frequency of system failure for \( m \) minimal cut-sets are given by

\[
P_f = P(\bar{C}_1 \cup \bar{C}_2 \cup \bar{C}_3 \cup ... \cup \bar{C}_m)
\]

\[
= P(\bar{C}_1) + P(\bar{C}_2) + ... + P(\bar{C}_m) \sum_{j=1}^{m} \text{terms} - \sum_{j=1}^{m} \sum_{i=1}^{j} P(\bar{C}_i \cap (\bar{C}_j)) + ...
\]

\[
+ [P(\bar{C}_1 \cap \bar{C}_j)] \sum_{i=j}^{m} \text{terms}
\]

\[
\vdots
\]

\[
(-1)^{m-1} P(\bar{C}_1 \cap \bar{C}_2 \cap ... \cap \bar{C}_m) \sum_{i=m}^{m} \text{terms}
\]

\[(8-16)\]
where $\overline{C}_1 \cap \overline{C}_2$, for example, denotes the failure of components of both the minimal cut-sets 1 and 2 and, therefore, $P(\overline{C}_1 \cap \overline{C}_2)$ means the probability of failure of all the components contained in $\overline{C}_1$ and $\overline{C}_2$, that is

$$P(\overline{C}_1 \cap \overline{C}_2) = \prod P_{id} \text{ and } i \in (\overline{C}_1 \cup \overline{C}_2)$$

where

- $P_{id}$ is the probability of component $i$ being in the failed state
  $$= r_i / (d_i + r_i)$$
  $$= \lambda_i / (\lambda_i + \mu_i)$$
- $d_i$ is the MTBF of component $i$.
- $\lambda_i$ is the failure rate of component $i$.
  $$= 1 / d_i$$
- $r_i$ is the MTTR of component $i$.
- $\mu_i$ is the repair rate of component $i$.
  $$= 1 / r_i$$
- $\Pi$ is the product.

The frequency of failure is given by

$$f_j = P(\overline{C}_1)W_1 + P(\overline{C}_2)W_2 + ... + P(\overline{C}_m)W_m - [P(\overline{C}_1 \cap \overline{C}_2)W_{1,2} + P(\overline{C}_1 \cap \overline{C}_3)W_{1,3}$$

$$+ ... + P(\overline{C}_i \cap \overline{C}_j)W_{i,j}, i \neq j$$

$$... + (-1)^{m-1}P(\overline{C}_1 \cap \overline{C}_2 \cap ... \cap \overline{C}_m)W_{1,2,...,m}$$

(8-17)

where

$$W_{i,j} = \sum_{k \in \overline{C}_i \cup \overline{C}_j} \mu_k$$

$k \in \overline{C}_i \cup \overline{C}_j$

The mean failure duration is given by

$$d_f = P_f / f_f$$

When the mean time between the failure of components is much larger than the mean time to repair (or in other words, the component availabilities approach unity), Equation (8-16) and (8-17) can be approximated (see Singh [B2]) by simpler equations:
\[ P_f = \sum_{i=1}^{n} P(C_i) = \sum_{i=1}^{n} P_{cs_i} \]  
\[ (8-18) \]

and

\[ f_f = \sum_{i=1}^{n} P(C_i)W_i = \sum_{i=1}^{n} f_{cs_i} \]  
\[ (8-19) \]

where \( P_{cs_i} \) and \( f_{cs_i} \) are the probability and frequency of cut-set event \( i \), respectively.

Also,

\[ d_f = P_f f_f = \sum_{i=1}^{n} P_{cs_i} f_{cs_i} = \sum_{i=1}^{n} f_{cs_i} r_{cs_i} = \sum_{i=1}^{n} f_{cs_i} \]  
\[ (8-20) \]

where:

\( d_f \) is the system mean failure duration; and
\( r_{cs_i} \) is the mean duration of cut-set event \( i \).

The application of Equations (8-19) and (8-20) to power distribution systems is discussed in Chapter 2. The components in a minimal cut-set behave like a parallel system, and \( f_{cs_i} \) (assuming \( n \) components in \( C_i \)) can be computed as follows:

\[ f_{cs_i} = \prod_{j=1}^{n} P_{jd} \sum_{j=1}^{n} \mu_j \]  
\[ (8-21) \]

and

\[ r_{cs_i} = 1/\sum_{j=1}^{n} \mu_j \]  
\[ (8-22) \]

For example, for a cut-set having three components 1, 2, and 3:

\[ f_{cs} = \frac{\lambda_1 \lambda_2 \lambda_3 (\mu_1 + \mu_2 + \mu_3)}{(\lambda_1 + \mu_1)(\lambda_2 + \mu_2)(\lambda_3 + \mu_3)} \]

\[ = \lambda_1 \lambda_2 \lambda_3 (r_1 r_2 + r_2 r_3 + r_3 r_1), \text{ assuming } \lambda_i \ll \mu_i \]

and
8.5.2 State-space method

The state-space method is a very general approach and can be used when the components are independent as well as for systems involving dependent failure and repair modes. The different steps of this approach are illustrated using a simple example of a component in series with two parallel components, as shown in Figure 8-1.

![Figure 8-1 — One component in series with two components in parallel](image)

a) **Enumerate the possible system states.** Assuming each component can exist either in the up or operating state (U) or in the down or failed state (D) and that the components are independent, there are eight possible system states. These states are numbered 1 through 8 in Figure 8-2, and the description of the component states is indicated in each system state.

b) **Determine interstate transition rates.** The transition rate from \( s_i \) (that is, state \( i \)) to \( s_j \) is the mean rate of the system passing from \( s_i \) to \( s_j \). For example, in Figure 8-2 the system can transit from \( s_1 \) to \( s_2 \) by the failure of component 1 and the repair of component 1, will put the system back into \( s_1 \), Therefore, the transition rate from \( s_1 \) to \( s_2 \) is \( \lambda_1 \), and the transition rate from \( s_2 \) to \( s_1 \) is \( \mu_1 \).

c) **Determine state probabilities.** When the components can be assumed to be independent, state probabilities can be found by the product rule as indicated in Equation (8-6). When, however, statistical dependence is involved, a set of simultaneous equations needs to be solved to obtain state probabilities (see Singh and Billinton [B33]). Only the independent case is discussed here and for this, say the probability of being in state 2 can be determined by

\[
P_2 = P_{1u} P_{2u} P_{3u}
\]  

(8-23)
Figure 8-2—State transition diagram for the system shown in Figure 8-1

where

\[ P_{iu} \] is the probability of component \( i \) being in “up” (operating) state
\[ = \frac{d_i}{(d_i + r_i)} = \frac{\mu_i}{(\lambda_i + \mu_i)} \]

and

\[ P_{id} \] is the probability of component \( i \) being in “down” (failed) state
\[ = \frac{r_i}{(d_i + r_i)} = \frac{\lambda_i}{(\lambda_i + \mu_i)} \]

d) Determine Reliability Measures. The states contributing the failure, or success, or any other event of interest are identified. For the system shown in Figure 8-1, if the links 2 and 3 are fully redundant, system failure can occur if either component 1 fails, or components 2 and 3 fail, or if all components fail. The state space \( S \) is shown in Figure 8-2 is

\[ S = \{1, 2, 3, 4, 5, 6, 7, 8\} \]
The subset $A$ (representing failure) can be identified as:

$$A = \{2, 5, 6, 7, 8\}$$

and the subset representing the success states is

$$S - A = \{1, 3, 4\}$$

Unavailability or the probability of the system being in the down state is now given by

$$P_f = \sum_{i \in A} P_i$$  \hspace{1cm} (8-24)

where $i \in A$ indicates that summation is over all states contained in subset $A$.

Applied to our example

$$P_f = P_2 + P_3 + P_6 + P_7 + P_8$$

where $P_i$ can be found by the product rule (see Equation 8-23).

The frequency of system failure, that is, the frequency of encountering subset $A$, can be computed by the following relationship:

$$f_f = \sum_{i \in A} P_i \sum_{j \in A} \lambda_{ij}$$  \hspace{1cm} (8-25)

where $\lambda_{ij}$ equals the transition rate from state $i$ to state $j$.

$$f_f = P_2 \lambda_1 + P_3 (\lambda_1 + \lambda_3) + P_6 (\lambda_1 + \lambda_2)$$

The mean failure duration can be obtained from $P_f$ and $f_f$ using

$$d_f = P_f / f_f$$  \hspace{1cm} (8-26)

In the preceding analysis, it was assumed that the failure of a component does not alter the probability of failure of the remaining components. If, however, it is assumed that after the system failure, no further component failure will take place, the state transition diagram in Figure 8-2 will be modified as shown in Figure 8-3. Once component 1 fails or components 2 and 3 fail, no further failure is possible. The probabilities in this case cannot be calculated by simple multiplication; they can be computed by solving a set of linear equations (see Singh and Billinton [B3]). Once the state probabilities have been calculated, the remaining procedure is the same.
8.5.3 Network reduction method

The network reduction method is useful for systems consisting of series and parallel subsystems. This method consists of successively reducing the series and parallel structures by equivalent components. Knowledge of the series and parallel reduction formulas is essential for the application of this technique.

8.5.4 Series system

The components are said to be in series when the failure of any one component causes system failure. It should be noted that the components do not have to be physically connected in series; it is the effect of failure that is important. Two types of series systems are discussed in 8.5.4.1 and 8.5.4.2.

8.5.4.1 Independent components

For the series system of independent components, the failure and repair rate the equivalent component are given by

\[ \lambda_{eq} = \sum_{i=1}^{n} \lambda_i \]  

(8-27)
\[ \mu_3 = \lambda_3 \left( \prod_{i=1}^{n} \left( 1 + \frac{\lambda_i}{\mu_i} \right) - 1 \right) \]  \hspace{1cm} (8-28)

where \( \lambda_3 \) and \( \mu_3 \) are the equivalent failure and repair rates of the series system and
\[ \prod_{i=1}^{n} \] denotes the product of values 1 through \( n \) (\( n \) being the number of components).

Assuming the \( \lambda_i \) is much smaller than \( \mu_i \) (which, in other words, means that the MTBF is much larger than the MTTR), the quantities involving the products of \( \lambda_i \) can be neglected. Equation (8-27) reduces to
\[ r_3 = \frac{1}{\mu_3} = \sum_{i=1}^{n} r_i \lambda_i / \lambda_3 \]  \hspace{1cm} (8-29)

### 8.5.4.2 Components involving dependence

When it is assumed that after the system failure no more components will fail, the equivalent failure and repair parameters are
\[ \lambda_s = \sum_{i=1}^{n} \lambda_i \] and \[ r_s = \sum_{i=1}^{n} r_i \lambda_i / \lambda_3 \]  \hspace{1cm} (8-30)

It can be seen from Equations (8-28) and (8-29) that, for component MTBF to be much larger than MTTR, the \( r_3 \) for the dependent and independent cases should be practically equal.

### 8.5.5 Parallel system

Two components are considered in parallel when either can ensure system success. The equivalent failure and repair rates of a parallel system of two components are given by
\[ \lambda_p = \frac{\lambda_1 \lambda_2 r_1 + r_2}{1 + \lambda_1 r_1 + \lambda_2 r_2} \]  \hspace{1cm} (8-31)

and
\[ \mu_p = \mu_1 + \mu_2 \]  \hspace{1cm} (8-32)

If \( \lambda_1 \), \( r_1 \) and \( \lambda_2 \), \( r_2 \) are much smaller than 1, then Equation (8-30) can be written as
\[ \lambda_p = \lambda_1 \lambda_2 (r_1 + r_2) \]  \hspace{1cm} (8-33)
8.6 Bibliography


Economic Dispatch

- What is the most economic way to produce the needed power?
- How can we operate the power system in the most economic way, while satisfying security constraints?
- $\text{Generator outputs, } P_i$
- generator cost $C_i(P_i)$
- total cost $\sum_{i=1}^{n} C_i(P_i)$
- What we want is to minimize the cost subject to a set of constraints.
- So how do we define these $C_i(P_i)$s?
- Economic operation can be divided into two main problems:
  1. Economic dispatch says that:
     - In a certain set of generators in service
     - If we want to minimize cost & over that set (of generators in service)
  2. Unit commitment:
     - A more challenging problem to solve:
     - In which generators do we keep online (to ensure secure operation at minimal cost)
     - Start-up costs (associated with getting system up & running, pre-ramping, personnel, maintenance etc)
     - Once units are online, they do not affect the economic dispatch.
once the unit is committed, economic dispatch is used
due to the old way of doing things.

- Old way – vertically integrated utilities
  - utilities owned generation
  - transmission
  - distribution

- New Way – electricity markets
  - more players involved in the generation.
  - acquire offers of energy production from all the possible generators, then to effectively compare their cost and to choose the most appropriate or “cheapest” units to commit.

- so the general concept of economic dispatch is still valid for new markets

- cost curves are specified differently in the two ways.

- we will only cover economic dispatch here
  - but we will mention along the way the differences in doing things in the old and new ways.
Typical fuel-cost curve

\[ C_i(P_j) \text{ (\$/hr)} \]

\[ P_{3i} \text{ (MW)} \rightarrow \text{ convex junction} \]

- Starting point is heat-rate curve (it says that for every million of Btu/hr I need a certain Btu/IP)

- Heat rate is a ratio between the amount of energy input and the energy extracted

- Heat energy input rate (which says that if I want to calculate the total heat energy going in/hr I need

\[ F_i(P_j) = P_{3i} \cdot H_i(P_j) \text{ (MBtu/hr)} \]

- If I knew the cost of natural gas that produces X MBtu of energy, I can get the cost function
If the fuel cost is \( \frac{K}{\text{MBtu}} \)

then

\[ C_i(P_{gi}) = K F_i(P_{gi}) \]

so it all starts from least rate curve.

-at low MW the least rate curve can be approximated by a \( \frac{1}{x} \) \( x \rightarrow \infty \) at \( P_{gi} \rightarrow 0 \)

-at high MW the least rate curve can be approximated by a linear function then generalize.

so we can write the least rate curve equation as:

\[ H_i(P_{gi}) = \frac{\alpha'}{P_{gi}} + \beta' + \gamma' P_{gi} \]

\( \alpha' \) intercept, \( \beta' \) slope

\( \gamma' \), dominant part if \( P_{gi} \) is large.

\( \alpha' \), dominant part if \( P_{gi} \) is small.

To get cost, we multiply our model by \( P_{gi} \) and add k

The corresponding fuel-cost curve is \( \alpha' = \alpha \ldots \)

\[ C_i(P_{gi}) = \alpha + \beta P_{gi} + \gamma P_{gi}^2 \text{ $/hr} \]

where \( \alpha, \beta, \gamma \) are all true and we end up with the convex function shape.
Given the system has \( m \) generators that are committed (online)

(Not) you can "choose" to buy units online

and offline like small circuits, if a unit is committed, it is "committed" because it may take hours and a good deal of $$$ to buy units online + offline.

\[ \forall \text{ all the loads } S_i, \text{ are given} \]

(We will vary the generation, but need to make sure we always satisfy the load)

- Determine the \( P_{g_i} \) and \( |V_i| \), \( i = 1, \ldots, m \) to minimize the total cost.

\[ C_T = \sum_{i=1}^{m} C_i (P_{g_i}) \]

subject to all the power flow equations and the "security" inequality constraints.

\[ P_{g_i}^{\min} < P_{g_i} < P_{g_i}^{\max} \]
Transmission flows \( |P_{ij}| \leq P_{ij}^{\max} \) on all lines

- ie can't increase power flow on the line as much as you want, because of line thermal limits, sags \( \rightarrow \) obstruction with trees etc.

\[ |V_i|_{\min} \leq |V_i| \leq |V_i|_{\max} \]

- regulated voltage limits by authorities

The problem is also known as:

Optimal Power Flow:

resolved using a range of methods that fall under the concept of:

- non-linear programming
- techniques derived from Newton's
- successive linear programs (LP)

- take non-linear \( \rightarrow \) linearize \( \rightarrow \) iterate
- interior point methods (prime choice for large systems)

- upper limit \( P_{ij}^{\max} \) thermal limits on the turbine generator unit (mainly mech. limits pipes, bungs, oil, gas, conveyer of coal, boilers size of steam turbines etc etc... )
Lower limit: is set by boiler dynamics and other thermodynamic considerations.

Fuel → Boiler → Steam

- "Flame out" may occur - ie. you don't have enough fuel into the furnace to sustain the flame.
- This is very dangerous - Fuel is coming into the furnace but no burning flame. So at some point enough fuel may accumulate and an explosion may occur.

  - ex. a 300 MW unit may have a lower limit around 120 MW.

Notes: we are optimizing (P-V) to reduce cost.

  - This is where this comes from in power flow.

  - This is the full version of the problem, but we will simplify this A LOT.

- We will make some approximations to make the problem a linear problem.
Approximation

- Little coupling between $P/Q$ and $G/V$
- If we don't need to coordinate $P$ variation with $V$ variation.
  - We can look at the $P_{gi}$ optimization without considering the $V_i$.
- We will be able to reduce many of the power flow equations AND ignore some of the inequality constraints.
- what we will do in class is a linear simplified version of the real problem.

- in class so we will neglect:

  1. Transmission line constraints (called assumptions)
  2. Line losses (for now)

- Assumptions: P-Q coupling strong

- neglecting cross couplings

- we are going to assume that P-Q is what really affects economic dispatch, this is not the case in the real world where reactive power does have an effect, i.e. there is cross coupling.

- Simplified Problem:

\[
\text{minimize} \sum_{i=1}^{m} C_i (P_i) \\
\text{such that} \sum_{i=1}^{m} P_i = P_d = \sum_{i=1}^{n} P_i
\]

- effectively a simplified power flow equation that neglects ALL system topology issues.

- as if all generation and loads are at the same bus.
This is because we neglected the constraints which would require knowledge of detailed power flow in all lines.

where

\[ P_{g_i}^{\text{min}} \leq P_{g_i} \leq P_{g_i}^{\text{max}} \quad i = 1, \ldots, m \]

\( P_{g_i} \) are physical constraints on what we can produce.

This is really a first approximation to the problem.

- What we want is to extract intuition into economic operation.

Solution involves incremental costs (ICs).

\[ IC_i = \frac{dC_i(P_{g_i})}{dP_{g_i}} \]

- Slope of the fuel cost curve.

\( C_i \) has units \$/hr.

So \( IC_i \) has units \$/hr/MW = \$/MWhr.

- It tells us how the cost changes for the next MW change in generation.

- Increase in cost per increase in MW.
The fuel cost curve is usually quadratic
\[ C_i(P_{g_i}) = \alpha + \beta P_{g_i} + \gamma P_{g_i}^2 \]
- generically the coefficients are positive (differentiate)

\[ \frac{dC_i}{dP_{g_i}} = \beta + 2\gamma P_{g_i} \]
- linear with positive coefficients.

**Economic Dispatch Ignoring \( P_g \) Limits**

- to minimize total cost ensuring power balance

**Optimal dispatch rule:**

- operate every generator at the same
  incremental cost.

Intuitively, consider two generators with

\[ IC_1 > IC_2 \]

- if next MW from generator 1 will cost more than the
  next MW from generator 2

If we reduce \( P_{g1} \) by \( x \) MW saves \( 2C_1 \) $/MWh

If we increase \( P_{g2} \) costs \( 2C_2 \) $/MWh

ie transfer of 1 MW from generator 1 to gen. 2 we

save \( IC_1 \) and spend \( IC_2 \)

net savings \( IC_1 - IC_2 > 0 \)

so we would keep doing that until \( IC_1 < IC_2 \)

that is why we want a strategy where \( IC_1 \approx IC_2 \)
Another way to look at this is mathematically.
We need to introduce what is called Lagrangean multipliers.

Minimize
\[ C_T = C_1(P_{g_1}) + \ldots + C_m(P_{g_m}) \]

subject to
\[ P_{g_1} + P_{g_2} + \ldots + P_{g_m} = P_0 \]

Solve using Lagrangean multipliers.

Replace the cost function \( C_T \) by the augmented problem
\[ \tilde{C}_T = C_f(\mathbf{P}) + \sum_{i=1}^{m} \lambda_i (P_{g_i} - P_0) \]

Lagrangean multiplier.

- We know that the minimum of a function can be found by differentiating it and setting it to zero.
  This is a local minima doesn't guarantee a minimum in general, but we can use other
  properties about the problem to make sure it is a minimum.

- Minima are given by points where all partial derivatives are zero to give the stationary points.

What are we going to differentiate w.r.t. to ??

The \( P_{g_1} \rightarrow P_{g_1}, P_{g_2}, \ldots \)

In the augmented problem we also have \( \lambda \) as a variable.
\[ \frac{\partial \tilde{C}_T}{\partial P_{g_i}} = 0 \quad i = 1, \ldots, m \]

\[ \frac{\partial \tilde{C}_T}{\partial \lambda} = 0 \]

Solving gives

\[ \frac{\partial \tilde{C}_T}{\partial P_{g_i}} = \frac{\partial C_T}{\partial P_{g_i}} - \lambda \]

\[ = \frac{d}{d P_{g_i}} C_i(P_{g_i}) - \lambda \]

\[ \text{I.C.} \]

\[ = I C_i - \lambda = 0 \quad \text{for } i = 1, \ldots, m \]

\[ \frac{\partial \tilde{C}_T}{\partial \lambda} = - \left( \sum_{i=1}^{m} P_{g_i} - P_D \right) = 0 \]

\[ \Rightarrow \sum_{i=1}^{m} P_{g_i} - P_D = 0 \quad \text{this is mostly} \]

\[ \text{but the original} \]

\[ \text{constraint we had} \]

\[ \text{but we also got another relationship} \]

\[ I C_i = \lambda \quad i = 1, \ldots, m \]

\[ \text{This is an optimal dispatch rule} \]

\[ \text{all gen. at same inc. cost which is equal to } \lambda. \]
- If the cost curves have the coefficients, then they are convex junctions, and the stationary point is a minimum.

- If the IC. curves are monotonic (they are here be they are linear) then the solution is unique.

  Now we can do a global optimization & get a minimum.
Lagrangian Multiplier, $\lambda$

- $\lambda$ is referred to as the system incremental cost rate, the cost increase that would result from a 1 MW increase in demand at any particular load.

- It relates the increased fuel cost rate ($\$/hr) to increased demand (MW).

- Let's say we have a given demand $P_0^*$, the optimal $P_0^*$, corresponding to cost $C_T^*$ (assuming we have done the economic dispatch) (this is our base case).

- If the load increases (incrementally with the new cost $C_T$.)

$$P_0 = P_0^* + \Delta P_0$$

find the new cost $C_T$.

Solution:

$$C_T = C_T^* + \Delta C_T$$

$$C_T = \sum_{i=1}^{n} C_i \left( P_{gi} \right)$$

If differentiable:

$$\Rightarrow \Delta C_T = \sum_{i=1}^{n} \Delta C_i \left( P_{gi} \right) \Delta P_{gi}$$

This relates small changes in generation to small changes in cost.
but \[
\frac{dC_i(P_{gi})}{dP_{gi}} = \lambda
\]

because it comes from a solution to the economic dispatch problem.

\[\Rightarrow \quad \Delta C_T = \lambda \sum_{i=1}^{n} \Delta P_{gi} = \lambda \Delta P_D\]

\[\Rightarrow \quad \Delta C_T = \lambda \Delta P_D\]

this is why \( \lambda \) is called system incremental cost, i.e., as the load across the whole system changes, \( \lambda \) will see a change in the cost across the whole system that is related through \( \lambda \).

\( \lambda \) is the additional cost of supplying the next MWh.

\[\therefore \quad \text{Two generators operated on economic dispatch:}\]

\[C_1(P_{g1}) = 900 + 45 P_{g1} + 0.01 P_{g1}^2 \quad \text{$/hr}$\]

\[C_2(P_{g2}) = 2500 + 43 P_{g2} + 0.003 P_{g2}^2 \quad \text{$/hr}$\]

differentiating we get I.C. curves.

\[I_{C_1} = 45 + 0.02 \quad \text{P}_{g1} \quad \text{$/MWh}$\]

\[I_{C_2} = 43 + 0.006 \quad \text{P}_{g2} \]
optimal scheduling \( \Rightarrow J_{C_1} = J_{C_2} \)

Looking at \( C_1 + C_2 \) we see that gen 2 is much more expensive for low levels of generation than gen 0 is.

This actually disappears in the optimal dispatch problem, since it only considers the TC curves.

From TC curves we say that it's better to load gen 2 because it has lower TC.

\[ \text{Economic Dispatch} \quad \text{VS} \quad \text{Unit Commitment} \]

Economic dispatch assumes that we have to use these generators, whereas committed whether we like it or not.

So we have to live with the fixed costs. The best we can do to alter the generation to reduce the remaining part of the cost function.

(Good idea)

\[ J_{C_1} \]

\[ J_{C_2} \]

\[ \lambda \]

\[ P_{g_1}, \ P_{g_2} \]

\[ P_0 \]

A value \( P_{g_1} + P_{g_2} = P_0 \)
we want to satisfy total demand = 700 MW

\[ IC_1 = IC_2 \]

\[ \Rightarrow 45 + 0.02 P_{g1} = 43 + 0.006 P_{g2} \]

\[ \Rightarrow P_{g1} + P_{g2} = 700 \]

2 equations + 2 unknowns

\[ P_{g1} = 84.6 \text{ MW} \]
\[ P_{g2} = 615.4 \text{ MW} \]

\[ IC_1 = IC_2 = \$ 46.69 \text{ MWh} \]

what if we have a more general trend of selling nonlinear

\[ P_{g1} + P_{g2} + P_{g3} = P_D \]

- if sum is less than \( P_D \) \( \Rightarrow \) increase \( V \) and \( n \), vice versa.

- note an algorithm that relies on the monotonic nature of these curves can be designed.
Iterative Process:

1) Pick a value for $n$
2) Find correspondingly $P_1(n)$, $P_2(n)$ ....
3) If $E[P_1(n)] - P_D < 0$, increase $n$ and go to 2.
4) If $E[P_1(n)] - P_D > 0$, decrease $n$ and go to 2.
5) $n = 0$, stop.

- The electricity markets dictate the participants to bid in curve $f_{LC}$.

- Note we have simplified this as:

  1. no losses
  2. no congestion

- What if we have limits on the generators?

  If $P_2 < 600$ MW, the dispatch by cannot be implemented.

  * Now $P_2 = 600$ MW
  can no longer achieve relatively in $I.C.$
  
  we know that $P_1 + P_2 = 700$

  $\therefore P_1 + 600 = 700$

  $\therefore P_1 = 100$ MW

  Then $LC_1 = 45 + 0.02(P_{100})$

  $= 47.04/kWh$

  If the next MW of demand would cost $47.04/kWh vs 46.67/kWh
If by constraining $P_{j2}$ we operate in a sub-optimal system, this will occur any time a generator in the system hits its generation limit.

- At $P_{j3}$ generator 3 cannot be economically dispatched and sits at its generation limit, while generators 1 and 2 can be economically dispatched.

- At $P_{j4}$ only generator 1 is responding to load changes and beyond $P_{j4}$ there is no longer equality in any of the incremental costs. Generator 1 is called "marginal unit."
- If $P_d$ increases, $\lambda$ increases to provide more generation.

- Eventually reach $I_2$ where $P_g_3$ reaches a limit.

- Further load increase must be met by $P_g_1 + P_g_2$ with equal $I_c$ ($I_c = I_c_2 \neq I_c_3$)

- Process continues as $P_d$ continues to increase, until $P_g_2$ also hits its upper limit.

- The sensitivity of unit cost rate $C_T$ to increases in $P_d$ (total demand) is still given by $\lambda$.

$$\Delta C_T = \lambda \sum \Delta P_{g_1} = \lambda \Delta P_d$$

- Where the summation is over the units that are not at their limits, and $\lambda$ is the common $I_c$ of those units.

- $\lambda$ is still called the system incremental cost.

- Extend the example by including a minimum of 50 MW for every generator. How does the generator look like?