5.5 Multiple-Angle and Product-to-Sum Formulas

Let's get double angle formulas first.

$$\cos 2\alpha = \cos(\alpha+\alpha) = \cos\alpha\cos\alpha - \sin\alpha\sin\alpha = \cos^2\alpha - \sin^2\alpha = 1-2\sin^2\alpha = 2\cos^2\alpha - 1$$

$$\sin 2\alpha = \sin(\alpha+\alpha) = \sin\alpha\cos\alpha + \cos\alpha\sin\alpha = 2\sin\alpha\cos\alpha$$

$$\tan 2\alpha = \tan(\alpha+\alpha) = \frac{\tan\alpha+\tan\alpha}{1-\tan\alpha\tan\alpha} = \frac{2\tan\alpha}{1-\tan^2\alpha}$$

Double Angle Formulas:

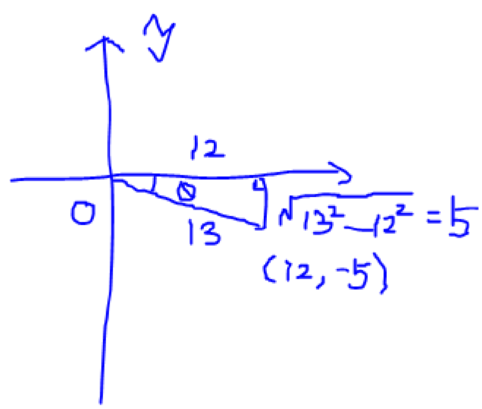
$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = 1-2\sin^2\alpha = 2\cos^2\alpha - 1$$

$$\sin 2\alpha = 2\sin\alpha\cos\alpha$$

$$\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha}$$

Example: Use the following to find $\sin 2\theta, \tan 2\theta$.

$$\cos\theta = \frac{12}{13}, \quad \frac{3\pi}{2} < \theta < 2\pi$$



Thus, $\sin\theta = -\frac{5}{13}, \tan\theta = -\frac{5}{12}$

$\sin 2\theta$

$= 2\sin\theta\cos\theta$

$= 2\left(-\frac{5}{13}\right)\frac{12}{13}$

$= -\frac{120}{169}$

$$\tan 2\theta$$
$$= \frac{2\tan\theta}{1-\tan^2\theta}$$
$$= \frac{2\left(-\frac{5}{12}\right)}{1-\left(-\frac{5}{12}\right)^2}$$
$$= -\frac{120}{119}$$

Example: Write $\cos 3\theta$ in terms of $\cos\theta$

$$\cos 3\theta$$
$$= \cos(2\theta+\theta)$$
$$= \cos 2\theta\cos\theta - \sin 2\theta\sin\theta$$
$$= (2\cos^2\theta - 1)\cos\theta - 2\sin\theta\cos\theta\sin\theta$$
$$= 2\cos^3\theta - \cos\theta - 2\sin^2\theta\cos\theta$$
$$= 2\cos^3\theta - \cos\theta - 2(1-\cos^2\theta)\cos\theta$$
$$= 2\cos^3\theta - \cos\theta - 2\cos\theta + 2\cos^3\theta$$
$$= 4\cos^3\theta - 3\cos\theta$$

Example: Prove $\frac{2\sin 3\theta}{\sin 2\theta} = 4\cos\theta - \sec\theta$

$$\frac{2\sin 3\theta}{\sin 2\theta} = \frac{2\sin(2\theta+\theta)}{2\sin\theta\cos\theta} = \frac{\sin 2\theta\cos\theta + \cos 2\theta\sin\theta}{\sin\theta\cos\theta}$$
$$= \frac{2\sin\theta\cos\theta\cos\theta + (2\cos^2\theta - 1)\sin\theta}{\sin\theta\cos\theta}$$
$$= \frac{2\sin\theta\cos^2\theta + 2\cos^2\theta\sin\theta - \sin\theta}{\sin\theta\cos\theta}$$
$$= \frac{4\sin\theta\cos^2\theta - \sin\theta}{\sin\theta\cos\theta}$$
$$= \frac{4\sin\theta\cos^2\theta}{\sin\theta\cos\theta} - \frac{\sin\theta}{\sin\theta\cos\theta}$$
$$= 4\cos\theta - \frac{1}{\cos\theta}$$
$$= 4\cos\theta - \sec\theta$$

Example: Solve $2\cos x + \sin 2x = 0$

Solution: $2\cos x + \sin 2x = 0$

$2\cos x + 2\sin x\cos x = 0$
$2\cos x(1+\sin x) = 0$
$\cos x = 0 \quad \text{and} \quad 1+\sin x = 0$
$x = \frac{\pi}{2}, \frac{3\pi}{2} \quad \text{and} \quad x = \frac{3\pi}{2}$

So, all solutions (or the general solution) are

$x = \frac{\pi}{2} + 2n\pi \quad \text{and} \quad x = \frac{3\pi}{2} + 2n\pi$

Example: Find all solutions of $2-\sin^2\theta = 2\cos^2\frac{\theta}{2}$ in the interval $[0, 2\pi)$.

Solution:

$2-\sin^2\theta = 2\cos^2\frac{\theta}{2}$
$2-\sin^2\theta = 1+\cos\theta$
$2-(1-\cos^2\theta) = 1+\cos\theta$
$\cos^2\theta - \cos\theta = 0$
$\cos\theta(\cos\theta - 1) = 0$
$\cos\theta = 0 \quad \text{and} \quad \cos\theta - 1 = 0$
$\theta = \frac{\pi}{2}, \frac{3\pi}{2} \quad \text{and} \quad \theta = 0$

We can use the sum and difference formulas discussed in the preceding section to easily obtain the following.

Product to Sum Formulas:

$$\sin x\sin y = \frac{1}{2}[\cos(x-y) - \cos(x+y)]$$
$$\cos x\cos y = \frac{1}{2}[\cos(x-y) + \cos(x+y)]$$
$$\sin x\cos y = \frac{1}{2}[\sin(x+y) + \sin(x-y)]$$
$$\cos x\sin y = \frac{1}{2}[\sin(x+y) - \sin(x-y)]$$

Product to Sum formulas are used in calculus to evaluate integrals involving the products of sines and cosines of two different angles.

Example: Rewrite the product $\sin 2\theta\cos 6\theta$ as a sum or difference.

$\sin 2\theta \cos 6\theta$

$= \frac{1}{2}[\sin(2\theta + 6\theta) + \sin(2\theta - 6\theta)]$

$= \frac{1}{2}[\sin 8\theta + \sin(-4\theta)]$

$= \frac{1}{2}\sin 8\theta - \frac{1}{2}\sin 4\theta$

Sometimes, it is useful to reverse the procedure and write a sum of trigonometric functions as a product. We can easily obtain the following formulas:

$$\sin x + \sin y = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

$$\sin x - \sin y = 2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$

$$\cos x + \cos y = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

$$\cos x - \cos y = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$

Example: Find the exact value of $\sin 195^0 - \sin 105^0$

Solution:

$\sin 195^0 - \sin 105^0$

$= 2\cos\left(\frac{195^0 + 105^0}{2}\right)\sin\left(\frac{195^0 - 105^0}{2}\right)$

$= 2\cos 150^0 \sin 45^0$

$= 2\frac{-\sqrt{3}}{2}\frac{\sqrt{2}}{2}$

$= -\frac{\sqrt{6}}{2}$

Example: Solve $\sin 3\theta + \sin 5\theta = 0$

$\sin 3\theta + \sin 5\theta = 0$

$2\sin\left(\frac{3\theta + 5\theta}{2}\right)\cos\left(\frac{3\theta - 5\theta}{2}\right) = 0$

$\sin 4\theta \cos(-\theta) = 0$

$\sin 4\theta \cos\theta = 0$

$\sin 4\theta = 0$ and $\cos\theta = 0$

$4\theta = 0, \pi$ and $\theta = \dfrac{\pi}{2}, \dfrac{3\pi}{2}$

All solutions are

$\theta = \dfrac{n\pi}{2}, \dfrac{\pi}{4} + \dfrac{n\pi}{2}$ and $\theta = \dfrac{\pi}{2} + 2n\pi, \dfrac{3\pi}{2} + 2n\pi$

(You can find out they are same as $\theta = \dfrac{n\pi}{4}$)

Example: Verifying the identity: $\dfrac{\sin 3x - \sin x}{\cos x + \cos 3x} = \tan x$

$$\frac{\sin 3x - \sin x}{\cos x + \cos 3x}$$

$$= \frac{2\cos\left(\frac{3x + x}{2}\right)\sin\left(\frac{3x - x}{2}\right)}{2\cos\left(\frac{x + 3x}{2}\right)\cos\left(\frac{x - 3x}{2}\right)}$$

$$= \frac{\cos 2x \sin x}{\cos 2x \cos(-x)}$$

$$= \frac{\sin x}{\cos x}$$

$$= \tan x$$