1.3 Vector Functions

Sometimes, it is convenient to present a curve in the plane by describing its $x$- and $y$-components as functions of some other variable, say $t$. A curve of the type $x = x(t), y = y(t)$ is called a **parametric curve** and the variable $t$ is called a **parameter**. For example, $x = t, y = t^2$ is a parametric curve with a parameter $t$ (what's this curve? You can convert to $y = x^2$, so it is a parabola).

**Vector Functions**

For each value of the parameter $t$, we may view the point $(x(t), y(t))$ on a parametric curve as the endpoint of the vector

$$(1) \quad \mathbf{r}(t) = \langle x(t), y(t) \rangle = x(t)i + y(t)j$$

which begins at the origin and ends at the point $(x(t), y(t))$. A vector is called a position vector if this vector’s starting point is origin.

The following figure shows a position vector for the vector function $\mathbf{r}(t) = \langle 500t, 500\sqrt{3t} - 16t^2 \rangle$.

![Position Vector](image.png)

A function such as $\mathbf{r}$ in (1), whose range is a set of vectors, is called a vector function (or vector-valued function) of $t$.

**Example 1**: Sketch the curve defined by the vector function $\mathbf{r}(t) = \langle t^2 - 2t, \ t + 1 \rangle$

**Solution**: The endpoint of the vector $\mathbf{r}(t)$ is $(x, y)$ where $x = t^2 - 2t, \ y = t + 1$. Each value of $t$ gives a point on the curve, as shown in the following table. In the following figure, we plot the points $(x, y)$ determined by several values of the parameter and join them to produce a curve.
Any curve of the form \( y = f(x) \) can be put into parametric form by letting \( x = t \), and then \( y = f(t) \).

**Vector Equation of a Line**

A line is determined by a point and a direction. The direction of a line in the plane is determined by a specifying a slope or by specifying a vector that is parallel to the line.

Let \( P_0 \) be a point in \( L \) and \( \vec{v} \) be a vector parallel to the line \( L \).

Let \( P \) be any arbitrary point on \( L \), and let \( \vec{r}_0 \) and \( \vec{r} \) be the position vector of \( P_0 \) and \( P \), let \( \vec{a} = \vec{P_0P} \). Since \( \vec{a} = \vec{PP_0} \parallel \vec{v} \), we have \( \vec{a} = t \vec{v} \). By Triangle Law, we have \( \vec{r} = \vec{r}_0 + \vec{a} = \vec{r}_0 + t \vec{v} \).

**Vector Equation of a Line:**

\[
\vec{r} = \vec{r}_0 + t \vec{v}
\]

If \( \vec{r}_0 = < x_0, y_0 > \), \( \vec{v} = < a, b > \), then the vector equation of a line can be written parametrically as follows:

\[
< x, y > = < x_0 + at, y_0 + bt > \quad \text{or} \quad x = x_0 + at, \quad y = y_0 + bt
\]

**Example 2:** Find parametric equations of the line that passes through the points \((3, 1)\) and \((2, -2)\).

**Solution:** Let \( \vec{v} = < 2 - 3, -2 - 1 > = < -1, -3 > \), \( P_0 = < 3, 1 > \), then we have the parametric equation is

\[
x = 3 - t, \quad y = 1 - 3t
\]

**Remark:** You can choose \( \vec{v} = < 3 - 2, -1 - (-2) > = < 1, 3 > \), and \( P_0 = < 2, -2 > \).

**Question:** Example 2: Find parametric equations of the line \( l \) that passes through the point \((1,2)\) and parallel to \( x = 3 - 3t, \quad y = 1 - 4t \).

**Answer:** Since line \( x = 3 - 3t, \quad y = 1 - 4t \) parallel to \( \vec{v} = < -3, -4 > \), so line \( l \) parallel to \( \vec{v} = < -3, -4 > \). Thus, the line equation for line \( l \) is \( x = 1 - 3t, \quad y = 2 - 4t \).
Converting between Cartesian equations and parametrized equations.

Example 3: For parametric curve, \( x = 1 + \cos \theta, y = 2 + \sin \theta \) with parameter \( \theta \), eliminate the parameter to find the Cartesian equation of this curve, what’s this curve?

Solution:

\[
x = 1 + \cos \theta, \quad y = 2 + \sin \theta \implies x - 1 = \cos \theta, \quad y - 2 = \sin \theta \implies (x - 1)^2 + (y - 2)^2 = \cos^2 \theta + \sin^2 \theta = 1
\]

So, this curve is a circle with center (1, 2) and radius is 1.

Example 3: Identify the curve given in parametric form by

\[
x = \sin \theta, \quad y = \cos^2 \theta \quad 0 \leq \theta \leq 2\pi
\]

Solution:

\[
x^2 + y = \sin^2 \theta + \cos^2 \theta = 1 \Rightarrow y = 1 - x^2
\]

It’s parabola.

Example 4: Identify the Cartesian equation of the parametrized line given in Example 2.

Solution:

\[
x = 3 - t \implies t = 3 - x \quad y = 1 - 3t \implies y - 1 = -3(3 - x) \implies 3x - y - 8 = 0
\]