2.3 Calculating Limits using the Limit Laws

Limit Laws Suppose that \( c \) is a constant and the limits \( \lim_{x \to a} f(x) = A \), and \( \lim_{x \to a} g(x) = B \). Then,

\[
\lim_{x \to a} cf(x) = cA \\
\lim_{x \to a} [f(x) \pm g(x)] = A \pm B \\
\lim_{x \to a} [f(x)g(x)] = AB \\
\lim_{x \to a} [f(x)/g(x)] = A/B \quad \text{if } B \neq 0
\]

\[
\lim_{x \to a} x^n = a^n \quad \text{(} n \text{ is a positive integer)} \\
\lim_{x \to a} \sqrt[n]{x} = \sqrt[n]{a} \quad \text{if } n \text{ is even, we assume that } a \geq 0
\]

From the above laws and rules, for any polynomials and rational functions \( f(x) \), we have

\[
\lim_{x \to a} f(x) = f(a) \quad \text{if } a \text{ is in the domain of } f(x)
\]

Example 1: Find \( \lim_{x \to 0} \frac{x-2}{x^2-4} \)

Solution: \( \lim_{x \to 0} \frac{x-2}{x^2-4} = \lim_{x \to 0} \frac{x-2}{(x-2)(x+2)} = \lim_{x \to 0} \frac{1}{x+2} = \frac{1}{2} \) (0 is in the domain of \( \frac{x-2}{x^2-4} \))

Example 2: Find \( \lim_{x \to 2} \frac{x-2}{x^2-4} \)

Solution: \( \lim_{x \to 2} \frac{x-2}{x^2-4} = \lim_{x \to 2} \frac{x-2}{(x-2)(x+2)} = \lim_{x \to 2} \frac{1}{x+2} = \frac{1}{4} \)

Exercise 1: Find \( \lim_{x \to 4} \sqrt{x-2} \)

Solution:

\[
\lim_{x \to 4} \sqrt{x-2} = \lim_{x \to 4} \sqrt{x-2} \sqrt{x+2} = \lim_{x \to 4} \frac{x-4}{(x-4)(\sqrt{x+2})} = \lim_{x \to 4} \frac{1}{\sqrt{x+2}} = \lim_{x \to 4} \frac{1}{\sqrt{4+2}} = \frac{1}{4}
\]

Example 3: Find \( \lim_{h \to 0} \frac{(h-2)^2-4}{h} \)

Solution: \( \lim_{h \to 0} \frac{(h-2)^2-4}{h} = \lim_{h \to 0} \frac{h^2-4h+4-4}{h} = \lim_{h \to 0} \frac{h^2-4h}{h} = \lim_{h \to 0} (h-4) = -4 \)

Exercise 2: Find \( \lim_{x \to 1} g(x) \) where \( g(x) = \begin{cases} x+2, & x \neq 1 \\ 5, & x = 1 \end{cases} \)

Solution: \( \lim_{x \to 1} g(x) = \lim_{x \to 1} (x+2) = 1 + 2 = 3 \)
**Squeeze Theorem:** If (a) \( g(x) \leq f(x) \leq h(x) \) for \( x \) in an open interval contains \( a \) possibly except \( a \), and (b) \( \lim_{x \to a} g(x) = \lim_{x \to a} h(x) = A \), then we have

\[
\lim_{x \to a} f(x) = A.
\]

Example 4: Show that \( \lim_{x \to 0} x \sin \frac{1}{x} = 0 \)

Solution: Since \( -1 \leq \sin \frac{1}{x} \leq 1 \), so \( -|x| \leq x \sin \frac{1}{x} \leq |x| \), and we know that

\[
\lim_{x \to 0} |x| = 0 \quad \text{and} \quad \lim_{x \to 0} -|x| = 0
\]

Thus, \( \lim_{x \to 0} x \sin \frac{1}{x} = 0 \) by Squeeze Theorem.