3.3 Rates of Change in the Natural and Social Sciences

The difference quotient

\[
\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}
\]

is the average rate of change of \( y \) with respect to \( x \) over the interval \([x_1, x_2]\) and can be interpreted as the slope of the secant line \( PQ \) in the left figure. Its limit as \( \Delta x \to 0 \) is the derivative \( f'(x_i) \), which can therefore be interpreted as the instantaneous rate of change of \( y \) with respect to \( x \), or the slope of the tangent line at \( P(x_1, f(x_1)) \). Using Leibniz notation, we write the process in the form

\[
\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}
\]

Whenever the function \( y = f(x) \) has a specific interpretation in one of the sciences, its derivative will have a specific interpretation as a rate of change.

If \( s = f(t) \) is the position function of a particle that is moving in a straight line, then \( \Delta s / \Delta t \) represents the average velocity over a time period \( \Delta t \), and \( v = ds / dt \) represents the instantaneous velocity.

Example 1: The position of a particle is given by the equation

\[
s = f(t) = t^3 - 6t^2 + 9t
\]

where \( t \) is measured in seconds and \( s \) in meters.

(a) Find the velocity at time \( t \)
(b) What’s the velocity after 2 s? 4 s?
(c) When is the particle at rest?
(d) When is the particle moving in the positive direction?
(e) Draw a diagram to represent the motion of the particle.
(f) Find the total distance traveled by the particle during the first five seconds.

Solution:

(a) The velocity function is the derivative of the position function

\[
s = f(t) = t^3 - 6t^2 + 9t
\]

\[
v(t) = \frac{ds}{dt} = 3t^2 - 12t + 9
\]
(b) The velocity after 2 s means the instantaneous velocity when \( t = 2 \), that is,

\[
\nu(2) = \left. \frac{ds}{dt} \right|_{t=2} = 3(2)^2 - 12(2) + 9 = -3 \text{ m/s}
\]

The velocity after 4 s is

\[
\nu(4) = \left. \frac{ds}{dt} \right|_{t=4} = 3(4)^2 - 12(4) + 9 = 9 \text{ m/s}
\]

(c) The particle is at rest when \( \nu(t) = 0 \), that is,

\[
3t^2 - 12t + 9 = 0 \Rightarrow 3(t^2 - 4 + 3) = 0 \Rightarrow 3(t - 3)(t - 1) = 0 \Rightarrow t = 1, \ t = 3
\]

(d) The particle moves in the positive direction when \( \nu(t) > 0 \) that is,

\[
3t^2 - 12t + 9 > 0 \Rightarrow 3(t - 3)(t - 1) = 0 \Rightarrow t < 1, \ t > 4
\]

(e)

\[
\begin{array}{c}
t = 3 \\
s = 0
\end{array}
\]

\[
\begin{array}{c}
t = 0 \\
s = 0
\end{array}
\]

\[
\begin{array}{c}
t = 1 \\
s = 4
\end{array}
\]

\[
\begin{array}{c}
t = 5
\end{array}
\]

(f) The distance traveled in the first second is

\[
| f(1) - f(0) | = |4 - 0| = 4
\]

From \( t = 1 \) to \( t = 3 \) the distance traveled is

\[
| f(3) - f(1) | = |0 - 4| = 4
\]

From \( t = 3 \) to \( t = 5 \) the distance traveled is

\[
| f(5) - f(3) | = |20 - 0| = 20
\]

The total distance traveled is 4+4+20=28 m.

Example 2: One of the quantities of interest in thermodynamics is compressibility. If a given substance is kept at a constant temperature, then its volume \( V \) depends on its pressure \( P \). We can consider the rate of change of volume with respect to pressure- namely, the derivative \( \frac{dV}{dP} \). As \( P \) increases, \( V \) decreases, so \( \frac{dV}{dP} < 0 \). The compressibility is defined by introducing a minus sign and dividing this derivative by the volume \( V \):

\[
\text{Isothermal compressibility} = \beta = -\frac{1}{V} \frac{dV}{dP}
\]

Thus, \( \beta \) measures how fast, per unit volume, the volume of a substance decreases as the pressure on it increases at constant temperature.
For instance, the volume \( V \) (in cubic meters) of a sample of air at 25\(^\circ\)C was found to be related to the pressure \( P \) (in kilopascals) by the equation

\[
V = \frac{5.3}{P}
\]

The rate of change of \( V \) with respect to \( P \) when \( P = 50 \) kPa is

\[
\left. \frac{dV}{dP} \right|_{P=50} = \frac{-5.3}{P^2} \bigg|_{P=50} = -\frac{5.3}{2500} = -0.00212 \text{ m}^3/\text{kPa}
\]

The isothermal compressibility at that pressure is

\[
\beta = -\frac{1}{V} \left. \frac{dV}{dP} \right|_{P=50} = \frac{0.00212}{5.3} = 0.02 \text{ (m}^3/\text{kPa})/\text{m}^3
\]

Example 3: (a) Find the average rate of change of the area of a circle with respect to its radius \( r \) as \( r \) changes from 1 to 2. (b) Find the instantaneous rate of change when \( r = 1 \)

Solution: (a. \( s = \pi r^2 \)

\[
\frac{s(2) - s(1)}{2 - 1} = \frac{\pi(2)^2 - \pi(1)^2}{1} = 3\pi
\]

(b. method 1.

\[
\lim_{\Delta r \to 0} \frac{s(1+\Delta r) - s(1)}{\Delta r} = \lim_{\Delta r \to 0} \frac{\pi(1+\Delta r)^2 - \pi(1)^2}{\Delta r} = \lim_{\Delta r \to 0} \frac{\pi(1 + 2\Delta r + \Delta r^2) - \pi}{\Delta r} = \lim_{\Delta r \to 0} \frac{2\pi \Delta r + \pi \Delta r^2}{\Delta r} = \lim_{\Delta r \to 0} (2\pi + \pi \Delta r) = 2\pi
\]

Method 2. \( s' = 2\pi r \). \( s'(1) = 2\pi(1) = 2\pi \)

Example 3. The population of a slowly growing bacterial colony after \( t \) hours is given by

\[n = 100 + 24t + 2t^2.\]

Find the growth rate after 2 hours.

Solution. \( \frac{dn}{dt} = 24 + 4t \)

\[
\left. \frac{dn}{dt} \right|_{t=2} = 24 + 4(2) = 32
\]