3.5 Chain Rule

Review Formulas:

\[
\frac{d}{dx}(c) = 0 \\
\frac{d}{dx}(x^n) = nx^{n-1} \\
\frac{d}{dx}(\sin x) = \cos x \\
\frac{d}{dx}(\cos x) = -\sin x \\
\frac{d}{dx}(\tan x) = \sec^2 x \\
\frac{d}{dx}(\cot x) = -\csc^2 x \\
\frac{d}{dx}(\sec x) = \sec x \tan x \\
\frac{d}{dx}(\csc x) = -\csc x \cot x
\]

Question: What’s \( F'(x) \) if \( F(x) = (x^2 + 1)^2 \) ?

Answer: \[
\frac{d}{dx} F(x) = \frac{d}{dx} (x^2 + 1)^2 = \frac{d}{dx} (x^4 + 2x^2 + 1) = 4x^3 + 4x
\]

What’s the \( F'(x) \) if \( F(x) = (x^2 + 1)^{100} \) ?

\( F'(x) \neq 100(x^2 + 1)^{99} \) for sure. Fortunately, we can use chain rule to work out this question.

**Chain Rule** If the derivative \( g'(x) \) and \( f'(g(x)) \) both exist, and \( F = f \circ g \) is the composite function defined by \( F(x) = f(g(x)) \), then \( F'(x) \) exists and is given by the product

\[
F'(x) = f'(g(x))g'(x)
\]

In Leibniz notation, if \( y = f(u) \) and \( u = g(x) \) are both differentiable functions, then

\[
\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}
\]

For \( F(x) = (x^2 + 1)^{100} \), let \( y = u^{100} \) and \( u = x^2 + 1 \), then we have

\[
F'(x) = \frac{dy}{du} \frac{du}{dx} = 100u^{99}(2x) = 100(x^2 + 1)^{99}(2x) = 200x(x^2 + 1)^{99}
\]

**Example 1:** \( y = (4x^2 - 6)^7 \), find \( y' \).

**Solution:** Let \( u = 4x^2 - 6 \), so \( y = u^7 \)

\[
\frac{dy}{du} = 7u^6 = 7(4x^2 - 6)^6, \quad \frac{du}{dx} = 8x
\]

\[
\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} = 7(4x^2 - 6)^6(8x) = 56x(4x^2 - 6)^6
\]
Example 2. \( y = (x^2 - 3x + 1)^3 \), find \( y'(1) \).

Solution. Let \( u = x^2 - 3x + 1 \), \( y = u^3 \)

\[
\frac{du}{dx} = 2x - 3, \quad \frac{dy}{du} = 5u^4 = 5(x^2 - 3x + 1)^3 \\
\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} = 5(x^2 - 3x + 1)^4(2x - 3) \\
y'(1) = 5(1 - 3 + 1)^4(2 - 3) = -5
\]

Example 3. \( y = \cos(x^3) \), find \( y' \)

Solution. Let \( y = \cos u \), \( u = x^3 \)

\[
\frac{dy}{du} = -\sin u = -\sin x^3, \quad \frac{du}{dx} = 3x^2 \\
y' = -3x^2 \sin x^3
\]

Example 4. \( y = \sin(\tan \sqrt{x}) \), find \( y' \)

Solution.

\[
y' = \cos(\tan \sqrt{x})(\tan \sqrt{x})' = \cos(\tan \sqrt{x}) \sec^2 \sqrt{x}(\sqrt{x})' = \frac{1}{2\sqrt{x}} \cos(\tan \sqrt{x}) \sec^2 \sqrt{x}
\]

Example 5. Given \( f'(1) = 2, f'(4) = 3, f(1) = 4, f(4) = 5 \), \( g'(1) = 6, g'(4) = 7, g(1) = 4, g(4) = 9 \), and \( h(x) = f(g(x)) \). Find \( h'(1) \).

Solution.

\[
h'(x) = f'(g(x))g'(x) \\
h'(1) = f'(g(1))g'(1) = f'(4)g'(1) = 3(6) = 18
\]

Exercise: Find the tangent line to curve \( y = \sqrt{x^2 - 1} \) at the point \((\sqrt{5}, 2)\).

Answer:

\[
y' = \frac{d}{dx}(\sqrt{x^2 - 1}) = \frac{d}{dx}(x^2 - 1)^{1/2} = \frac{1}{2}(x^2 - 1)^{-1/2} \frac{d}{dx}(x^2 - 1) = \frac{1}{2}(x^2 - 1)^{-1/2} (2x) = x(x^2 - 1)^{-1/2} \\
y'(\sqrt{5}) = \sqrt{5} \left[ (\sqrt{5})^2 - 1 \right]^{-1/2} = \sqrt{5}(5 - 1)^{-1/2} = \frac{\sqrt{5}}{2}
\]

Tangent line equation is \( y - 2 = \frac{\sqrt{5}}{2} (x - \sqrt{5}) \).