3.8 Higher Derivatives

Given \( f(x) = x^5 \)

\[ f'(x) = 5x^4 \]

\( (f'(x))' = (5x^4)' = 20x^3 \) we call this the second derivative of \( f(x) \), write as \( f''(x) \)

\( (f''(x))' = (20x^3)' = 60x^2 \) we call this the third derivative of \( f(x) \), write as \( f'''(x) \)

\( (f'''(x))' = (60x^2)' = 120x \) we call this the fourth derivative of \( f(x) \), write as \( f^{(4)}(x) \)

Notations for higher derivatives: If \( y = f(x) \), then

\[ y'' = f''(x) = \frac{d}{dx} \left( \frac{dy}{dx} \right) = \frac{d^2 y}{dx^2} = D^2 f(x) = D_x^2 f(x) \]

\[ y''' = f'''(x) = \frac{d}{dx} \left( \frac{d^2 y}{dx^2} \right) = \frac{d^3 y}{dx^3} = D^3 f(x) = D_x^3 f(x) \]

\[ y^{(4)} = f^{(4)}(x) = \frac{d}{dx} \left( \frac{d^3 y}{dx^3} \right) = \frac{d^4 y}{dx^4} = D^4 f(x) = D_x^4 f(x) \]

\[ y^{(n)} = f^{(n)}(x) = \frac{d}{dx} \left( \frac{d^{(n-1)} y}{dx^{(n-1)}} \right) = \frac{d^n y}{dx^n} = D^n f(x) = D_x^n f(x) \]

Example 1. Given \( f(x) = x^3 - 20x + 3 \). Find \( f'''(x), f^{(4)} \)

Solution: \( f(x) = x^3 - 20x + 3 \)

\[ f'(x) = 3x^2 - 20 \]

\[ f''(x) = 6x \]

\[ f'''(x) = 6 \]

\[ f^{(4)}(x) = 0 \]

Exercise: Given \( f(x) = x^{50} \). Find \( f^{(51)}(x) \) \quad (answer: \( f^{(51)}(x) = 0 \))

Example 2. Given \( \mathbf{r}(t) = \langle t^3 - 2t + 1, t^2 + 2 \rangle \). Find \( \mathbf{r}''(t) \)

Solution: \( \mathbf{r}'(t) = \langle 3t^2 - 2, 2t \rangle \)

\[ \mathbf{r}''(t) = \langle 6t, 2 \rangle \]

Properties: (1) \( s = s(t) \) is the position function of an object that moves in a straight line, we have \( a(t) = v'(t) = s''(t) \), where \( a(t), \ v(t) \) are the acceleration and speed, respectively.

(2) \( \mathbf{r}(t) \) represent the position of an object in a plane, we have \( \mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t) \), where \( \mathbf{a}(t), \ \mathbf{v}(t) \) are the acceleration and velocity, respectively.
Example 2. Find the acceleration of a position vector $\vec{r}(t) = <\sin t, \cos 2t>$ at time $t = \frac{\pi}{6}$.

Solution: $\vec{r}'(t) = <\cos t, -2\sin 2t>$

$$\vec{a}(t) = \vec{r}''(t) = <-\sin t, -4\cos 2t>$$

At time $t = \frac{\pi}{6}$, the acceleration is

$$\vec{a}(\frac{\pi}{6}) = \vec{r}''(\frac{\pi}{6}) = <\sin \frac{\pi}{6}, -4\cos \left(2 \cdot \frac{\pi}{6}\right)> = \left<-\frac{1}{2}, -4\cos \frac{\pi}{3}\right> = \left<-\frac{1}{2}, -2\right>$$

Example 3. If $f(x) = \frac{1}{x}$, find $f^{(n)}(x)$.

Solution. $f(x) = \frac{1}{x} = x^{-1}$

$$f'(x) = -x^{-2} = -\frac{1}{x^2}$$

$$f''(x) = (-2)(-1)x^{-3} = 2x^{-3} = \frac{2}{x^3}$$

$$f'''(x) = (-3)(-2)(-1)x^{-4} = -3 \cdot 2 \cdot 1 \cdot x^{-4}$$

$$f^{(4)}(x) = (-4)(-3)(-2)(-1)x^{-5} = 4 \cdot 3 \cdot 2 \cdot 1 \cdot x^{-5}$$

$$f^{(5)}(x) = (-5)(-4)(-3)(-2)(-1)x^{-6} = -5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot x^{-6}$$

$$f^{(n)}(x) = (-n)(-n-1)\cdots(-2)(-1)x^{-(n+1)} = (-1)^n n \cdot (n-1)\cdots 2 \cdot 1 \cdot x^{-(n+1)} = (-1)^n n!x^{-(n+1)}$$

Example 4. Find $D^{101}\cos x$

Solution.

$$D\cos x = -\sin x \quad D^2\cos x = -\cos x \quad D^3\cos x = \sin x \quad D^4\cos x = \cos x \quad D^5\cos x = -\sin x$$

We see that the successive derivatives occur in a cycle of length 4, and in particular, $D^{4k}\cos x = \cos x$, so $D^{100}\cos x = \cos x$.

Thus, $D^{101}\cos x = -\sin x$
Example 5. Find \( y'' \) if \( x^4 + y^4 = 9 \).

Solution.

\[
\frac{d}{dx} (x^4 + y^4) = \frac{d}{dx} (9) \Rightarrow 4x^3 + \frac{d}{dx} (y^4) = 0 \Rightarrow
\]

\[
4x^3 + \frac{dy}{dx} (y^4) = 0 \Rightarrow 4x^3 + 4y^3 \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{x^3}{y^3}.
\]

\[
y'' = \frac{d}{dx} \left( -\frac{x^3}{y^3} \right) = -\frac{y^3 \frac{d}{dx} (x^3) - x^3 \frac{d}{dx} (y^3)}{(y^3)^2} = -\frac{y^3 3x^2 - x^3 3y^2 \frac{dy}{dx}}{y^6}
\]

\[
y'' = -\frac{y^3 3x^2 - x^3 3y^2 \left( -\frac{x^3}{y^3} \right)}{y^6} = -\frac{3x^2 y^3 + 3x^6}{y^6} = -\frac{3x^2 y^4 + 3x^6}{y^7} = -\frac{3x^2 (y^4 + x^4)}{y^7} = -\frac{27x^2}{y^7} \quad \text{(since } y^4 + x^4 = 9)\]

Exercise: Find a second degree polynomial \( P \) such that \( P(2) = 5 \), \( P'(2) = 5 \), \( P''(2) = 2 \).

Answer: Let \( P(x) = ax^2 + bx + c \), \( P'(x) = 2ax + b, P''(x) = 2a \). Thus,

\[
P(2) = a(2)^2 + b(2) + c = 4a + 2b + c \quad P'(2) = 2ax + b, P''(2) = 2a.
\]

Thus, we have

\[
\begin{align*}
4a + 2b + c &= 5 \\
4a + b &= 5 \\
2a &= a
\end{align*} \Rightarrow
\begin{align*}
a &= 1 \\
b &= 1 \\
c &= -1
\end{align*}
\]

Thus, the polynomial is \( P(x) = x^2 + x - 1 \).