4.5 Exponential Growth and Decay

In many natural phenomena, quantities grow or decay at a rate proportional to their size. Let \( y(t) \) be the value of a quantity at time \( t \). If the rate of change of \( y \) with respect to \( t \) is proportional to its size \( y(t) \) at any time, then we have the following differential equation (equation involves derivative)

\[
\frac{dy}{dt} = ky
\]

where \( k \) is a constant. The solution to the above differential equation is

\[
y(t) = y(0)e^{kt}
\]

The solution to differential equation (1), \( y(t) \) grow exponentially when \( k > 0 \) or the solution is decay exponentially when \( k < 0 \) (suppose \( y(0) > 0 \)) as \( t \) becomes greater.
Example 1: A bacteria culture starts with 2000 bacteria and the population triples every half-hour. Assuming that the population grows at a rate proportional to its size.

a. Find an expression for the number of bacteria after $t$ hours.

b. Find the number of bacteria after 10 min.

c. When will the population reach 10,000.

Solution. Let $y(t)$ be the number of bacteria at time $t$.

$$y'(t) = ky(t) \quad (k \text{ is a constant})$$

$$y(t) = y(0)e^{kt}$$

Since $y(0.5) = 3y(0)$ (triple), and $y(0.5) = y(0)e^{0.5k}$, we have

$$y(0)e^{0.5k} = 3y(0)$$

$$e^{0.5k} = 3$$

$$0.5k = \ln 3$$

$$k = 2\ln 3 \approx 2.1973$$

a. $y(t) = y(0)e^{kt} = 2000e^{2.1973t}$

b. $y(10/60) = 2000e^{2.1973(10/60)} \approx 2885$

c. $y(t) = 10000$

$$2000e^{2.1973t} = 10000$$

$$e^{2.1973t} = 5$$

$$2.1973t = \ln 5$$

$$t = \frac{\ln 5}{2.1973} \approx 0.7325 \text{ (hours)}$$

Example 2: Polonium-210 has a half-life of 140 days. This means that the rate of decay is proportional to the amount present, and half of the given quantity will disintegrate in 140 days.

a. If a sample has a mass of 200 mg, find a formula for the mass that remains after $t$ days.

b. When will the mass be reduced to 10 mg.

Solution: Let $y(t)$ be the quantity remains after $t$ days.

$$y'(t) = ky(t) \quad (k \text{ is a constant})$$

$$y(t) = y(0)e^{kt} = 200e^{kt}$$

Since $y(140) = \frac{1}{2} y(0) = 100$ (triple), and $y(140) = y(0)e^{140k}$, we have
\[200e^{140k} = 100\]
\[e^{140k} = \frac{1}{2}\]
\[140k = \ln 0.5\]
\[k = \frac{\ln 0.5}{140} \approx -0.005\]

a. \(y(t) = y(0)e^{kt} = 200e^{-0.005t}\)

b. \(y(t) = 10\)
\[200e^{-0.005t} = 10\]
\[e^{-0.005t} = \frac{10}{200}\]
\[-0.005t = \ln \left( \frac{1}{20} \right)\]
\[t = -\frac{\ln \left( \frac{1}{20} \right)}{-0.005} \approx 605 \text{ (days)}\]

Example 3: Newton’s Law of Cooling states that the rate of cooling of an object is proportional to the temperature difference between the object and its surroundings, provided that this difference is not very large. On a hot day a thermometer is taken outside from an air conditioned room where the temperature is 21 degrees. The outdoor temperature is 37 degrees. After one minute it reads 27 degrees. How many degrees does it read after 3 minutes?

Solution. Let \(y(t)\) be the degrees, the thermometer reads after \(t\) minutes taken outside, so we have \(y(0) = 21\), \(y(1) = 27\), and

(1) \(y'(t) = ky(t) - 37\)

Let \(u(t) = y(t) - 37\), so \(u'(t) = y'(t)\), plugging into formula (1), we have
\[u'(t) = ku(t)\]
\[u(t) = u(0)e^{kt}\]
\[y(t) - 37 = (y(0) - 37)e^{kt}\]
\[y(t) = 37 + (y(0) - 37)e^{kt} = 37 - 16e^{kt}\]

Since \(y(1) = 27\), we have
\[37 - 16e^k = 27\]
\[16e^k = 10\]
\[ e^k = \frac{10}{16} \]
\[ k = \ln \left( \frac{10}{16} \right) \approx -0.47 \]
\[ y(t) = 37 - 16e^{-0.47t} \]
\[ y(3) = 37 - 16e^{-0.47(3)} \approx 33.1 \]

**Interest formula** for investment. If an amount \( A \) is invested at an interest rate \( i \) and interest is compounded \( n \) times a year. Then after \( t \) years, the value of the investment is

\[ A(1 + \frac{i}{n})^{nt} \]

If it is continuously compounded, after \( t \) years, the value of the investment is

\[ Ae^{it} \]

Example 5: If $200 is borrowed at 14% interest, find the amounts due at the end of 2 years if the interest is compounded (a) annually (once a year) (b) quarterly (4 times a year) (c) monthly (12 times a year) (e) continuously.

Solution. \( i = 14\% = 0.14 \), \( t = 2 \).

a. \[ 200 \left( 1 + \frac{0.14}{1} \right)^{1 \times 2} = 200(1.14)^2 \approx 259.2 \]

b. \[ 200 \left( 1 + \frac{0.14}{4} \right)^{4 \times 2} \approx 263.4 \]

c. \[ 200 \left( 1 + \frac{0.14}{12} \right)^{12 \times 2} \approx 264.2 \]

d. \[ 200e^{0.14 \times 2} \approx 264.6 \]