5.2 Maximum and Minimum Values

**Definition:** A function \( f \) has an absolute maximum (or global maximum) at \( c \) if \( f(c) \geq f(x) \) for all \( x \) in \( D \), where \( D \) is the domain of \( f \). The number \( f(c) \) is called the maximum value of \( f \) on \( D \). Similarly, \( f \) has an absolute minimum (or global minimum) at \( c \) if \( f(c) \leq f(x) \) for all \( x \) in \( D \), where \( D \) is the domain of \( f \). The number \( f(c) \) is called the minimum value of \( f \) on \( D \). The maximum and minimum values of \( f \) are called the extreme values of \( f \).

In the following figure, \( f \) has absolute maximum at \( d \) with the maximum value \( f(d) \), and has absolute minimum at \( a \) with the minimum value \( f(a) \).

![Graph of a function with maximum and minimum values marked](image)

In the above figure, if we consider only values of \( x \) near \( b \) (for instance, if we restrict our attention to the interval \((a, c)\)), then \( f(b) \) is the largest of those values of \( f(x) \) and is called local maximum value of \( f \). Likewise, \( f(c) \) is called a local minimum value of \( f \) because \( f(c) \leq f(x) \) for \( x \) near \( c \) [in the interval \((b, d)\), for instance]. The function also has a local minimum at \( e \). In general, we have following.

**Definition:** A function \( f \) has a local maximum (or relative maximum) at \( c \) if \( f(c) \geq f(x) \) when \( x \) is near \( c \). \( f \) has a local minimum (or relative minimum) at \( c \) if \( f(c) \leq f(x) \) when \( x \) is near \( c \).

![Graphs showing local maximum and minimum](image)

Minimum value 0  
No minimum 
absolute maximum, minimum: 37, -27

We have seen that some functions have extreme values, whereas others do not. The following theorem gives conditions under which a function is guaranteed to possess extreme values.
The Extreme Value Theorem: If \( f \) is continuous on a closed interval \([a,b]\), then \( f \) attains an absolute maximum value \( f(c) \) and an absolute minimum value \( f(d) \) at some numbers \( c \) and \( d \) in \([a,b]\).

Note that an extreme value can be taken on more than once (the above right figure).

The following figures show that a function need not possess extreme values if either hypothesis (continuity or closed interval) is omitted from the Extreme Value Theorem.

Fermat's Theorem: If \( f \) has a local maximum or minimum at \( c \), and if \( f'(c) \) exists, then \( f'(c) = 0 \)

Note that when \( f'(c) = 0 \), \( f \) doesn’t necessarily have a maximum or minimum at \( c \). (In other words, the converse of Fermat’s Theorem is false in general.) See the left figure below.
If \( f(x) = x^3 \), then \( f'(0) = 0 \), but \( f \) has no minimum or maximum.

If \( f(x) = |x| \), then \( f(0) = 0 \) is a minimum value, but \( f'(0) \) does not exist.

We should bear in mind that there may be an extreme value where \( f'(c) \) does not exist. For instance, the function \( f(x) = |x| \) has its (local and absolute) minimum value at 0 (see the above right figure), thus that value can not be found by setting \( f'(x) = 0 \).

Fermat’s Theorem does suggest that we should at least start looking for extreme values of \( f \) at the numbers \( c \) where \( f'(c) = 0 \) or \( f'(c) \) does not exist. Such numbers are given a special name.

Definition: A **critical number** of a function \( f \) is a number \( c \) in the domain of \( f \) such that either \( f'(c) = 0 \) or \( f'(c) \) does not exist.

**Example 1.** Find the critical numbers of \( f(x) = x^{3/5} (4 - x) \).

**Solution.**

\[
f'(x) = \frac{3}{5} x^{-2/5} (4 - x) + x^{3/5} (-1) = \frac{12 - 8x}{5x^{2/5}}
\]

When \( x = \frac{3}{2} \), \( f'(\frac{2}{3}) = 0 \), and \( x = 0 \), \( f'(x) \) does not exist. Thus, the critical numbers are \( \frac{3}{2} \) and 0.

**Theorem:** If \( f \) has a local extremum at \( c \), then \( c \) is a critical number of \( f \).

To find an absolute maximum or minimum of a continuous function on a closed interval, we note that either it is a local extremum [in which case it occurs at a critical number or it occurs at an endpoint of the interval. Thus, the following three-step procedure always works.

**The closed interval method.** To find the absolute maximum and minimum values of a continuous function \( f \) on a closed interval \([a, b]\).

1. Find the values of \( f \) at the critical numbers of \( f \) in \((a, b)\).
2. Find the values of \( f \) at the endpoints of the interval.
3. The largest of the values from steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

**Example 2.** Find the absolute maximum and minimum of \( f(x) = x^3 + 6x^2 + 9x - 1 \) on \([-2, 1]\).

**Solution.**
Step 1. $f'(x) = 3x^2 + 12x + 9$

Let $f'(x) = 0$, we have

$$3x^2 + 12x + 9 = 0 \Rightarrow 3(x^2 + 4x + 3) = 0 \Rightarrow 3(x+1)(x+3) = 0 \Rightarrow x = -1, -3$$

$x = -3$ is not in the interval $[-2, 1]$, so omit it.

Step 2. $f(-1) = (-1)^3 + 6(-1)^2 + 9(-1) - 1 = -5$

Step 3. $f(-2) = (-2)^3 + 6(-2)^2 + 9(-2) - 1 = -3$

$f(1) = (1)^3 + 6(1)^2 + 9(1) - 1 = 15$

Thus, the absolute maximum is 15, absolute minimum is -5.

Exercise: Find the absolute maximum and minimum of $f(x) = \frac{x}{x^2 - 2x - 3}$ on $[0, 1]$.

Answer:

Step 1.

$$f'(x) = \frac{x^2 - 2x - 3 - x(2x - 2)}{(x^2 - 2x - 3)^2} = -\frac{x^2 + 3}{(x^2 - 2x - 3)^2}$$

Let $f'(x) = 0 \Rightarrow x^2 + 3 = 0 \Rightarrow$ no solution.

$x^2 - 2x - 3 = 0$ means $f'(x)$ does not exist.

$(x - 3)(x + 1) = 0 \Rightarrow x = 3, x = -1$

Thus the critical points are $x = 3, x = -1$

Step 2. No critical points in the interval $[0, 1]$.

Step 3. $f(0) = \frac{0}{0^2 - 2(0) - 3} = 0$

$$f(1) = \frac{1}{1^2 - 2(1) - 3} = -\frac{1}{4}$$

Thus, absolute maximum is 0, absolute minimum is $-\frac{1}{4}$ on $[0, 1]$. 