6.1 Sigma Notation

Definition: If $a_m, a_{m+1}, \ldots, a_n$ are real numbers and m and n are integers such that $m \le n$, then

$$\sum_{i=m}^{n} a_i = a_m + a_{m+1} + a_{m+2} + \cdots + a_{n-1} + a_n$$

Example 1

a. $\sum_{i=1}^{4} i^2$

b. $\sum_{i=5}^{n} i$

c. $\sum_{j=0}^{3} 2^j$

d. $\sum_{k=2}^{5} \frac{2}{k}$

e. $\sum_{k=2}^{6} \frac{2}{5}$

f. $\sum_{i=0}^{3} \frac{i+2}{i^2+1}$

Example 2

a. Write the sum $2^4 + 3^4 + 4^4 + 5^4 + 6^4$ in sigma notation.

b. Write the sum $\frac{2}{5} + \frac{4}{6} + \frac{8}{7} + \frac{16}{8} + \frac{32}{9}$ in sigma notation.

c. Write the sum $\frac{2}{5} + \frac{4}{10} + \frac{8}{17} + \frac{16}{26} + \cdots + \frac{2^{n-1}}{n^2+1}$ in sigma notation.

Theorem: If c is any constant (that is, it does not depend on i), then

a. $\sum_{i=m}^{n} ca_i = c\sum_{i=m}^{n} a_i$

b. $\sum_{i=m}^{n} (a_i + b_i) = \sum_{i=m}^{n} a_i + \sum_{i=m}^{n} b_i$

c. $\sum_{i=m}^{n} (a_i - b_i) = \sum_{i=m}^{n} a_i - \sum_{i=m}^{n} b_i$

Example 3: Find $\sum_{i=1}^{n} 1$

Example 4: Prove the formula for the sum of the first n positive integers:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

Example 5: Prove the formula for the sum of the squares of the first n positive integers:

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

Theorem. Let c be a constant and n a positive integer. Then

a. $\sum_{i=1}^{n} 1 = n$

b. $\sum_{i=1}^{n} c = cn$

c. $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

d. $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

e. $\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$

Example 6: Evaluate $\sum_{i=1}^{n} i(4i^2 - 3)$

Example 7. Find $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n}\left[\left(\frac{i}{n}\right)^2 + 1\right]$