Appendix D: Review Trigonometry

Relationship of Radians and Degrees

Angles can be measured in degrees or in radians (abbreviated as rad). The angle given by a complete revolution contains $360^\circ$, which is the same as $2\pi$ rad. Thus,

$$1^\circ = \frac{\pi}{180} \text{ rad} \quad 1 \text{ rad} = \left(\frac{180}{\pi}\right)^\circ \quad 180^\circ = \pi$$

Example 1: Find the radian measure of $30^\circ$, and express $\frac{\pi}{4}$ rad in degrees.

Solution: $30^\circ = 30\left(\frac{\pi}{180}\right) = \frac{\pi}{6} \text{ rad} \quad \frac{\pi}{4} \text{ rad} = \frac{\pi}{4}\left(\frac{180}{\pi}\right) = 45^\circ$

In calculus, we use radians to measure angles except when otherwise indicated. The following table gives the correspondence between degree and radian measures of some common angles.

<table>
<thead>
<tr>
<th>Degrees</th>
<th>$0^\circ$</th>
<th>$30^\circ$</th>
<th>$45^\circ$</th>
<th>$60^\circ$</th>
<th>$90^\circ$</th>
<th>$120^\circ$</th>
<th>$135^\circ$</th>
<th>$150^\circ$</th>
<th>$180^\circ$</th>
<th>$270^\circ$</th>
<th>$360^\circ$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Radians</td>
<td>$0$</td>
<td>$\frac{\pi}{6}$</td>
<td>$\frac{\pi}{4}$</td>
<td>$\frac{\pi}{3}$</td>
<td>$\frac{2\pi}{3}$</td>
<td>$\frac{3\pi}{4}$</td>
<td>$\frac{5\pi}{6}$</td>
<td>$\pi$</td>
<td>$\frac{3\pi}{2}$</td>
<td>$2\pi$</td>
<td></td>
</tr>
</tbody>
</table>

Arc length of a circle

$$a = r\theta \quad \theta = \frac{a}{r}$$

Example 2: (a) If the radius of a circle is 10 cm, what angle is subtended by an arc of 12 cm?
(b) If a circle has radius 6 cm, what is the length of an arc subtended by a central angle of $\frac{3\pi}{8}$ rad?

Solution: (a) $r = 10, \ a = 12, \ \theta = \frac{a}{r} = \frac{12}{10} = 1.2 \text{ rad}$

(b) $r = 6, \ \theta = \frac{3\pi}{8} \text{ rad, } a = r\theta = 6\left(\frac{3\pi}{8}\right) = \frac{9\pi}{4} \text{ cm}$

The **Standard position** of an angle occurs when we place its vertex at the origin of a coordinate system and its initial side on the positive $x-$ axis as in the following figure.

A **positive angle** is obtained by rotating the initial side counterclockwise until it coincides with the terminal side.

**Negative angles** are obtained by clockwise rotation as in the left figure.

Notice that different angles can have the same terminal side. For instance, $\theta, \theta + 2n\pi$ have same terminal side; $\frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}$ have same terminal side.

The following figures show angles in standard position.
The Trigonometric Functions

\[ \sin \theta = \frac{\text{opp}}{\text{hyp}} \quad \csc \theta = \frac{\text{hyp}}{\text{opp}} \]
\[ \cos \theta = \frac{\text{adj}}{\text{hyp}} \quad \csc \theta = \frac{\text{hyp}}{\text{adj}} \]
\[ \tan \theta = \frac{\text{opp}}{\text{adj}} \quad \cot \theta = \frac{\text{adj}}{\text{opp}} \]

The above definition does not apply to obtuse or negative angles, so for a general angle \( \theta \) in standard position, we let \( P(x, y) \) be any point on the terminal side of \( \theta \), let \( r \) be the distance |OP| as in the following figure.

\[ \sin \theta = \frac{y}{r} \quad \csc \theta = \frac{r}{y} \]
\[ \cos \theta = \frac{x}{r} \quad \csc \theta = \frac{r}{x} \]
\[ \tan \theta = \frac{y}{x} \quad \cot \theta = \frac{x}{y} \]

From the above definition, it is clear that \( \tan \theta, \sec \theta \) are not defined when \( x = 0 \), and \( \cot \theta, \csc \theta \) are not defined when \( y = 0 \). Notice that, the above two definitions of the trigonometric functions are consistent when \( \theta \) is an acute angle.

If \( \theta \) is a number, the convention is that \( \sin \theta \) means the sine of the angle whose radian measure is \( \theta \). For example,

\[ \sin 3 \approx 0.14112, \quad \sin 3^0 \approx 0.05234 \]

For exact trigonometric ratios for certain angles can be read from the triangles in the following figures.

\[ \sin \frac{\pi}{4} = \frac{1}{\sqrt{2}} \quad \sin \frac{\pi}{6} = \frac{1}{2} \quad \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2} \]
\[ \cos \frac{\pi}{4} = \frac{1}{\sqrt{2}} \quad \cos \frac{\pi}{6} = \frac{\sqrt{3}}{2} \quad \cos \frac{\pi}{3} = \frac{1}{2} \]
\[ \tan \frac{\pi}{4} = 1 \quad \tan \frac{\pi}{6} = \frac{1}{\sqrt{3}} \quad \tan \frac{\pi}{3} = \sqrt{3} \]
The signs of the trigonometric functions for angles in each of the four quadrants can be determined from the above definition.

Example 3: Find the exact trigonometric ratios for \( \theta = \frac{2\pi}{3} \)

Solution: From the left figure we see that a point on the terminal line for \( \theta = \frac{2\pi}{3} \) is \( P(-1, \sqrt{3}) \).

Therefore, taking \( x = -1, \ y = \sqrt{3}, \ r = 2 \), we have

\[
\begin{align*}
\sin \frac{2\pi}{3} &= \frac{\sqrt{3}}{2}, \quad \cos \frac{2\pi}{3} = -\frac{1}{2}, \quad \tan \frac{2\pi}{3} = -\sqrt{3} \\
\csc \frac{2\pi}{3} &= \frac{2}{\sqrt{3}}, \quad \sec \frac{2\pi}{3} = -2, \quad \cot \frac{2\pi}{3} = -\frac{1}{\sqrt{3}}
\end{align*}
\]

The following table gives some values of \( \sin \theta \), \( \cos \theta \) by the method of the above example.

<table>
<thead>
<tr>
<th>( \theta )</th>
<th>( \theta )</th>
<th>( \frac{\pi}{6} )</th>
<th>( \frac{\pi}{4} )</th>
<th>( \frac{\pi}{3} )</th>
<th>( \frac{\pi}{2} )</th>
<th>( \frac{2\pi}{3} )</th>
<th>( \frac{3\pi}{4} )</th>
<th>( \frac{5\pi}{6} )</th>
<th>( \pi )</th>
<th>( \frac{3\pi}{2} )</th>
<th>( 2\pi )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sin \theta )</td>
<td>0</td>
<td>1</td>
<td>( \frac{\sqrt{2}}{2} )</td>
<td>( \frac{\sqrt{3}}{2} )</td>
<td>1</td>
<td>( \frac{\sqrt{3}}{2} )</td>
<td>( \frac{\sqrt{2}}{2} )</td>
<td>( \frac{1}{2} )</td>
<td>( \frac{1}{2} )</td>
<td>( \frac{1}{2} )</td>
<td>0</td>
</tr>
<tr>
<td>( \cos \theta )</td>
<td>1</td>
<td>( \frac{\sqrt{3}}{2} )</td>
<td>( \frac{\sqrt{2}}{2} )</td>
<td>0</td>
<td>( -\frac{1}{2} )</td>
<td>( -\frac{\sqrt{2}}{2} )</td>
<td>( -\frac{\sqrt{3}}{2} )</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>
Example 4. If \( \cos \theta = \frac{2}{5} \) and \( 0 < \theta < \frac{\pi}{2} \), find the other five trigonometric functions of \( \theta \).

Solution: Since \( \cos \theta = \frac{2}{5} \), we can label the hypotenuse as having length 5 and the adjacent side as having length 2 in the left figure. If the opposite side has length \( x \), then the Pythagorean Theorem gives \( x^2 + 2^2 = 5^2 \Rightarrow x = \sqrt{21} \). Thus, we have

\[
\sin \theta = \frac{\sqrt{21}}{5}, \quad \tan \theta = \frac{\sqrt{21}}{2}, \quad \csc \theta = \frac{5}{\sqrt{21}}, \quad \sec \theta = \frac{5}{2}, \quad \cot \theta = \frac{2}{\sqrt{21}}
\]

Example 5 Use a calculator to approximate the value of \( x \) in the following figure.

Solution: From the diagram, we see that

\[
\tan 40^\circ = \frac{16}{x}
\]

\[
x = \frac{16}{\tan 40^\circ} \approx 19.07
\]

Trigonometric Identities

\[
csc \theta = \frac{1}{\sin \theta}, \quad \sec \theta = \frac{1}{\cos \theta}, \quad \cot \theta = \frac{1}{\tan \theta}, \quad \tan \theta = \frac{\sin \theta}{\cos \theta}, \quad \cot \theta = \frac{\cos \theta}{\sin \theta}
\]

\[
\sin^2 \theta + \cos^2 \theta = 1, \quad 1 + \tan^2 \theta = \sec^2 \theta, \quad 1 + \cot^2 \theta = \csc^2 \theta
\]

\[
\sin(-\theta) = -\sin \theta, \quad \cos(-\theta) = \cos \theta
\]

\[
\sin(\theta + 2\pi) = \sin \theta, \quad \cos(\theta + 2\pi) = \cos \theta
\]

\[
\tan(\theta + \pi) = \tan \theta, \quad \cot(\theta + \pi) = \cot \theta
\]

\[
\sin 2\theta = 2 \sin \theta \cos \theta
\]

\[
\cos 2\theta = \cos^2 \theta - \sin^2 \theta = 2 \cos^2 \theta - 1 = 1 - 2 \sin^2 \theta
\]
\[
\sin(x - y) = \sin x \cos y - \cos x \sin y \\
\cos(x - y) = \cos x \cos y + \sin x \sin y \\
\sin(x + y) = \sin x \cos y + \cos x \sin y \\
\cos(x + y) = \cos x \cos y - \sin x \sin y
\]

\[
\sin x \cos y = \frac{1}{2} [\sin(x + y) + \sin(x - y)] \\
\cos x \cos y = \frac{1}{2} [\cos(x + y) + \cos(x - y)] \\
\sin x \sin y = -\frac{1}{2} [\cos(x + y) - \cos(x - y)]
\]

Example 6. Find all values of \(x\) in the interval \([0, 2\pi]\) such that \(\sin x = \sin 2x\)

Solution: \(\sin x = \sin 2x \Rightarrow \sin x = 2 \sin x \cos x \Rightarrow \sin x (1 - 2 \cos x) = 0\)

There are two possibilities for \(x\):

\[
\begin{align*}
\sin x &= 0 \\
1 - 2 \cos x &= 0 \\
\downarrow & \quad \downarrow \\
x &= 0, \pi, 2\pi \\
& \quad x = \frac{\pi}{3}, \frac{5\pi}{3}
\end{align*}
\]

Graphs of the Trigonometric Functions.