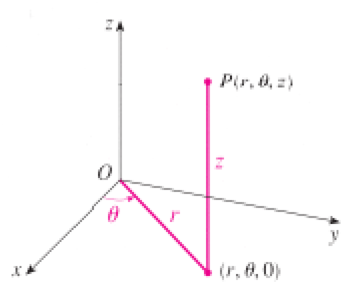
13.9 Cylindrical and Spherical Coordinates

In Section 13.4 we introduced the polar coordinate system in order to give a more convenient description of certain curves and regions. In the three dimensions there are two coordinate systems that are similar to polar coordinates and give convenient descriptions of some commonly occurring surfaces and solids. They will be especially useful in the next section when we use them to compute volumes and triple integrals.

In the **Cylindrical coordinate system**, a point P in three-dimensional space is represented by the ordered triple (r,θ,z) , where r and θ are polar coordinates of the projection of P onto the xy plane to P .



The relationship of the two coordinates are:

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases} \qquad\qquad \begin{cases} r^2 = x^2 + y^2 \\ \tan\theta = \dfrac{y}{x} \\ z = z \end{cases}$$

Example 1: Find the rectangular coordinates with the point with cylindrical coordinates $\left(4,\frac{\pi}{3},2\right)$.

Solution: $x = 4\cos\frac{\pi}{3} = 4\left(\frac{1}{2}\right) = 2,\ y = 4\sin\frac{\pi}{3} = 4\left(\frac{\sqrt{3}}{2}\right) = 2\sqrt{3},\ z = 2$.

Thus, the point is $\left(2,2\sqrt{3},2\right)$ in rectangular coordinates.

Example 2: Find the cylindrical coordinates with the point with rectangular coordinates $(3,-3,7)$.

Solution: $r = \sqrt{3^2 + (-3)^2} = 3\sqrt{2},\quad \tan\theta = \frac{-3}{3} = -1 \rightarrow \theta = \frac{7\pi}{4} + 2n\pi,\quad z = 7$.

Thus, the point is $\left(3\sqrt{2}, \frac{7\pi}{4}, 7\right)$ in cylindrical coordinates.

Example 3: Describe the surfaces whose equation in cylindrical coordinates are given as

a. $r = 1$ (cylinder)

b. $\theta = \frac{\pi}{3}$ (plane)

c. $z = 1$ (plane)

d. $z = r$ $(\Rightarrow z^2 = r^2 \Rightarrow z^2 = x^2 + y^2$ (cone)

The spherical coordinates (ρ, θ, ϕ) of a point P in space are shown in the following Figure 1, where $\rho = |OP|$ is the distance from the origin, θ is the same angle as in cylindrical coordinates, and ϕ is the angle between the positive z axis and the line segment OP. Note that $\rho \geq 0, \quad 0 \leq \phi \leq \pi$.

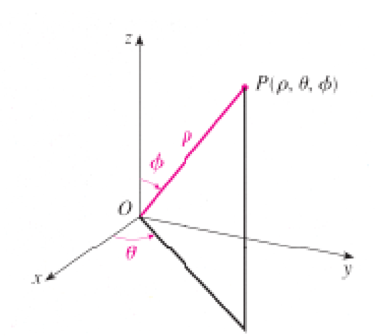


Figure 1

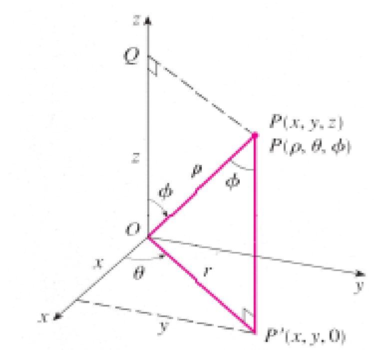


Figure 2

The relationship between spherical coordinates and rectangular coordinates can be seen freom Figure 2:

$$\begin{cases} x = \rho \sin\phi \cos\theta \\ y = \rho \sin\phi \sin\theta \\ z = \rho \cos\phi \end{cases} \qquad x^2 + y^2 + z^2 = \rho^2$$

Example 4: The point $(2, \pi/4, \pi/3)$ is given in spherical coordinates. Find its rectangular coordinates.

Solution:

$$\begin{cases} x = \rho \sin\phi \cos\theta = 2\sin\frac{\pi}{3}\cos\frac{\pi}{4} = \sqrt{\frac{3}{2}} = \frac{\sqrt{6}}{2} \\ y = \rho \sin\phi \sin\theta = 2\sin\frac{\pi}{3}\sin\frac{\pi}{4} = \sqrt{\frac{3}{2}} = \frac{\sqrt{6}}{2} \\ z = \rho\cos\phi = 2\cos\frac{\pi}{3} = 1 \end{cases}$$

Thus, the point $(2, \pi/4, \pi/3)$ is $\left(\sqrt{\frac{3}{2}}, \sqrt{\frac{3}{2}}, 1\right)$ in rectangular coordinates.

Example 5: The point $(0, 2\sqrt{3}, -2)$ is given in rectangular coordinates. Find its spherical coordinates.

Solution: $\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{0 + 12 + 4} = 4$

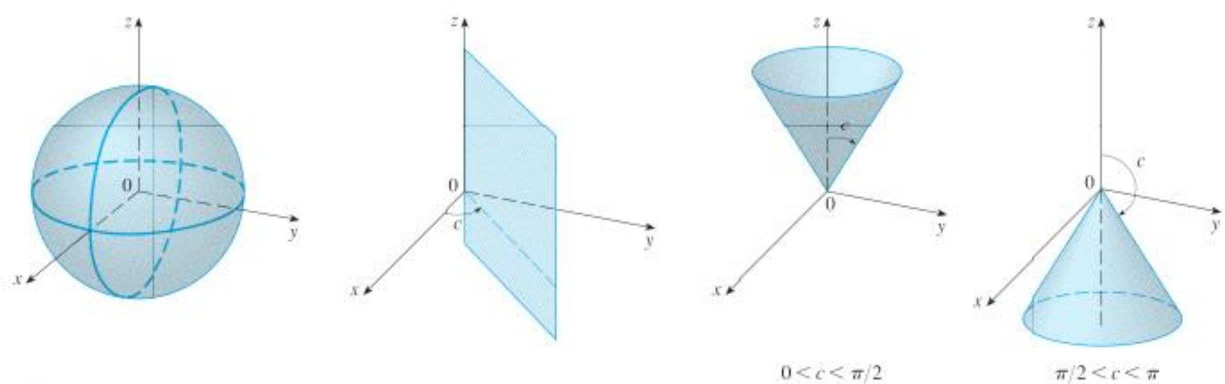
$$\cos\phi = \frac{z}{\rho} = \frac{-2}{4} = -\frac{1}{2} \Rightarrow \phi = \frac{2\pi}{3}$$

$$\cos\theta = \frac{x}{\rho\sin\phi} = 0 \Rightarrow \theta = \frac{\pi}{2}$$

Thus, the sperical coordinates of the given point are $\left(4, \frac{2\pi}{3}, \frac{\pi}{2}\right)$.

Example 6: Describe the surface for equations:

a. $\rho = c$ (sphere)
b. $\theta = c$ (plane)
c. $\phi = c$ (cone)



Example 7: Find a rectangular equation for the surface whose spherical equation is $\rho = \sin\theta\sin\phi$.

Solution:

$$\rho = \sin\theta\sin\phi \rightarrow \rho^2 = \rho\sin\theta\sin\phi \Rightarrow x^2 + y^2 + z^2 = y \Rightarrow x^2 + \left(y - \frac{1}{2}\right)^2 + z^2 = \left(\frac{1}{2}\right)^2$$

Thus, this is a sphere with center at $\left(0, \frac{1}{2}, 0\right)$ and radius is $\frac{1}{2}$