14.4 Green’s Theorem

Green’s Theorem gives the relationship between a line integral around a simple closed curve \( C \) and a double integral over the plane region \( D \) bounded by \( C \). The positive orientation of a simple closed curve \( C \) refers to a single counterclockwise traversal of \( C \), in other words, when you walk along the curve \( C \), the enclosed region is on your left side.

**Green’s Theorem:** Let \( C \) be a positively oriented, piecewise-smooth, simple closed curve in the plane and let \( D \) be the region bounded by \( C \). If \( P \) and \( Q \) have continuous partial derivatives on an open region that contains \( D \), then

\[
\int_C P\,dx + Q\,dy = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)\,dA
\]

**Example 1:** Evaluate \( \int_C x\,dx + xy\,dy \), where \( C \) is the triangular curve consisting of the line segment from \((0,0)\) to \((1,0)\), from \((1,0)\) to \((0,1)\), and from \((0,1)\) to \((0,0)\).

**Solution:** Let \( P(x, y) = x^4, Q(x, y) = xy \).

\[
\int_C x^4\,dx + xy\,dy = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)\,dA = \iint_D (y - 0)\,dA
\]

\[
= \int_0^1 \int_0^{1-x} y\,dy\,dx = \cdots = \frac{1}{6}
\]

**Example 2:** Evaluate \( \oint_C (3y - e^{\sin x})\,dx + (7x + \sqrt{y^7 + 1})\,dy \), where \( C \) is the circle \( x^2 + y^2 = 9 \).

**Solution:** Let \( P(x, y) = 3y - e^{\sin x}, Q(x, y) = 7x + \sqrt{y^7 + 1} \).
\[ \oint_{C} (3y - e^{\sin x})dx + (7x + \sqrt{y^3} + 1)dy = \iint_{D} \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA = \iint_{D} (7-3) \, dA = \int_{0}^{2\pi} \int_{0}^{3} 4r \, dr \, d\theta = 36\pi \]

Question 1: Evaluate \( \oint_{C} y^2dx + 3xydy \), where \( C \) is the boundary of the semi-annular region \( D \) in the upper half-plane between the circles \( x^2 + y^2 = 1 \) and \( x^2 + y^2 = 4 \).

**Solution:** Let \( P(x, y) = y^2 \), \( Q(x, y) = 3xy \)

\[ \oint_{C} y^2dx + 3xydy = \iint_{D} \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA = \iint_{D} (3y - 2y) \, dA = \int_{0}^{2\pi} \int_{1}^{2} (r \sin \theta) \, r \, dr \, d\theta = \frac{14}{3} \]

Let \( D \) be the region bounded by closed curve \( C \), we have the following formulas:

**Area of region** \( D = \oint_{C} x \, dy = -\oint_{C} y \, dx = \frac{1}{2} \left( \oint_{C} x \, dy - y \, dx \right) \)

**Remark:** We can easily prove the above formulas by Green’s Theorem.

**Example 3:** Find the area enclosed by the ellipse \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \)

**Solution:** The ellipse has the parametric equation: \( x = a \cos \theta, \ y = b \sin \theta, \ 0 \leq \theta \leq 2\pi \).

Area = \( \frac{1}{2} \left( \oint_{C} x \, dy - y \, dx \right) = \frac{1}{2} \int_{0}^{2\pi} a \cos \theta (b \cos \theta \, d\theta) - b \sin \theta (-a \sin \theta \, d\theta) = \frac{1}{2} \int_{0}^{2\pi} abd \theta = \pi ab \)