Complex Eigenvalues

In this section, we look at solutions for the homogeneous system

(1) $$x'(t) = Ax(t)$$

where A is a real constant $n \times n$ matrix with complex eigenvalues. If A is a real matrix, the complex eigenvalues must occur in conjugate pairs ($\lambda = \alpha \pm \beta i$). With complex eigenvalues, the corresponding eigenvector may have complex numbers, but we want our solutions to only have real numbers in them. To explore the complex eigenvalues' effect, we begin with examples.

For the case: a matrix has two complex eigenvalues. How to find the solution?
It is similar to the case when the matrix has two different eigenvalues, we need to find the eigenvalues and the corresponding eigenvectors first.
Theorem: Suppose that $\lambda_{1,2} = \alpha \pm \beta i$ are two complex conjugate eigenvalues of A, and $\xi_{1,2} = u \pm wi$ are the corresponding eigenvectors of A. Then

$$e^{\alpha t}(u\cos\beta t - w\sin\beta t) \text{ and } e^{\alpha t}(u\sin\beta t + w\cos\beta t)$$

are two independent solutions to $x'(t) = Ax(t)$.

In the following examples, once you have obtained the eigenvalues and eigenvectors, you can use the above theorem directly (In this case, make sure to remember the formula **correctly**), but I do not use the formula directly since I can not remember it and I can easily derive it. Now, let's see the procedure to find the solution to system (1) when the matrix has two conjugate complex eigenvalues.

Complex Key Operations Review:

$i^2 = 1$

$e^{ai} = \cos a + i\sin a$

$(a+bi)(a-bi) = a^2 + b^2$

$$\frac{1}{a+bi} = \frac{a-bi}{(a+bi)(a-bi)} = \frac{a-bi}{a^2+b^2}$$

Example : Solve the differential equation system: $x' = \begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix} x$. Also describe the behavior of the solutions as $t \to \infty$.

Step 1: Find the eigenvalues.

Solving $\begin{vmatrix} -1-\lambda & -4 \\ 1 & -1-\lambda \end{vmatrix} = 0$, we have $\lambda_{1,2} = -1 \pm 2i$.

Step 2: Find the corresponding eivenvector for only one of the eigenvalue.
When $\lambda = -1 + 2i$.

$$\begin{pmatrix} -1-(-1+2i) & -4 \\ 1 & -1-(-1+2i) \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -2i & -4 \\ 1 & -2i \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \xrightarrow{-\frac{1}{2}iR_1+R_2\to R_2} \begin{pmatrix} -2i & -4 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow -2i\xi_1 - 4\xi_2 = 0$$

Let $\xi_2 = -i,$ we have $\xi_1 = 2,$ we have eigenvector $\xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 2 \\ -i \end{pmatrix}$

Step 3: Simplify

$$e^{\lambda t}\xi = e^{(-1+2i)t}\begin{pmatrix} 2 \\ -i \end{pmatrix} = e^{-t}e^{2ti}\begin{pmatrix} 2 \\ -i \end{pmatrix}$$

$$= e^{-t}(\cos 2t + i\sin 2t)\begin{pmatrix} 2 \\ -i \end{pmatrix}$$

$$= \begin{pmatrix} 2e^{-t}(\cos 2t + i\sin 2t) \\ -ie^{-t}(\cos 2t + i\sin 2t) \end{pmatrix}$$

$$= \begin{pmatrix} 2e^{-t}\cos 2t + i2e^{-t}\sin 2t \\ -ie^{-t}\cos 2t + e^{-t}\sin 2t \end{pmatrix}$$

$$= \begin{pmatrix} 2e^{-t}\cos 2t + i2e^{-t}\sin 2t \\ e^{-t}\sin 2t - ie^{-t}\cos 2t \end{pmatrix}$$

$$= \begin{pmatrix} 2e^{-t}\cos 2t \\ e^{-t}\sin 2t \end{pmatrix} + \begin{pmatrix} i2e^{-t}\sin 2t \\ -ie^{-t}\cos 2t \end{pmatrix}$$

$$= \begin{pmatrix} 2e^{-t}\cos 2t \\ e^{-t}\sin 2t \end{pmatrix} + \begin{pmatrix} 2e^{-t}\sin 2t \\ -e^{-t}\cos 2t \end{pmatrix} i$$

The real part and imaginary part of the above expression are two independent solutions of the differential equation.

Step 4: We have the general solution

$$x = c_1\begin{pmatrix} 2e^{-t}\cos 2t \\ e^{-t}\sin 2t \end{pmatrix} + c_2\begin{pmatrix} 2e^{-t}\sin 2t \\ -e^{-t}\cos 2t \end{pmatrix}$$

Remark: We can see that $x \to 0,$ as $t \to \infty.$

Example : Solve the differential equation system: $x' = \begin{pmatrix} 1 & 2 \\ -5 & -1 \end{pmatrix}x, x(0) = \begin{pmatrix} -2 \\ -5 \end{pmatrix}.$

Step 1: Find the eigenvalues.

Solving $\begin{vmatrix} 1-\lambda & 2 \\ -5 & -1-\lambda \end{vmatrix} = 0$, we have $\lambda_{1,2} = \pm 3i.$

Step 2: Find the corresponding eivenvector for only one of the eigenvalue.

When $\lambda = 3i.$ We have one eigenvector $\xi = \begin{pmatrix} 1+3i \\ -5 \end{pmatrix}.$

Step 3: Simplify

$$e^{\lambda t}\xi = e^{3ti}\begin{pmatrix}1+3i\\-5\end{pmatrix} = \begin{pmatrix}e^{3ti}(1+3i)\\-5e^{3ti}\end{pmatrix}$$

$$= \begin{pmatrix}(\cos 3t + i\sin 3t)(1+3i)\\-5(\cos 3t + i\sin 3t)\end{pmatrix}$$

$$= \begin{pmatrix}\cos 3t - 3\sin 3t + i(\sin 3t + 3\cos 3t)\\-5\cos 3t - i5\sin 3t\end{pmatrix}$$

$$= \begin{pmatrix}\cos 3t - 3\sin 3t\\-5\cos 3t\end{pmatrix} + \begin{pmatrix}\sin 3t + 3\cos 3t\\-5\sin 3t\end{pmatrix} i$$

Step 4: We have the general solution

$$x = c_1\begin{pmatrix}\cos 3t - 3\sin 3t\\-5\cos 3t\end{pmatrix} + c_2\begin{pmatrix}\sin 3t + 3\cos 3t\\-5\sin 3t\end{pmatrix}$$

Step 5: Using the initial conditions, we have

$$c_1\begin{pmatrix}1\\-5\end{pmatrix} + c_2\begin{pmatrix}3\\0\end{pmatrix} = \begin{pmatrix}-2\\-5\end{pmatrix}$$

$$\Downarrow$$

$$\begin{cases} c_1 = 1\\ c_2 = -1\end{cases}$$

Step 5: We have the solution to initial values:

$$x = \begin{pmatrix}\cos 3t - 3\sin 3t\\-5\cos 3t\end{pmatrix} - \begin{pmatrix}\sin 3t + 3\cos 3t\\-5\sin 3t\end{pmatrix}$$

$$= \begin{pmatrix}-2\cos 3t - 4\sin 3t\\-5\cos 3t + 5\sin 3t\end{pmatrix}$$